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THE DUAL OF THE HENSTOCK-KURZWEIL SPACE

Abstract

We prove that if T is a continuous linear functional on the space $\mathcal{D}$ of Henstock-Kurzweil integrable functions on $[a_1, b_1] \times \cdots \times [a_m, b_m]$, then there exists a function g of strong bounded variation on $[a_1, b_1] \times \cdots \times [a_m, b_m]$ such that

$$T(f) = (HK) \underset{[a_1,b_1]\times\cdots\times[a_m,b_m]}{\int\cdots\int} f(x_1, \ldots, x_m) g(x_1, \ldots, x_m)\, dx_1 \ldots dx_m\,.$$

1 Introduction

A well known theorem of Zygmund-Alexiewicz (see [10], [5] or [11], [1]) says that T is a continuous linear functional on the space of Henstock-Kurzweil integrable functions on $[a, b]$ if and only if there exists a function $g : [a, b] \mapsto \mathbb{R}^1$ of essentially bounded variation such that

$$T(f) = (HK) \int_a^b f(x) g(x)\, dx\,.$$

In the multidimensional case, Kurzweil [4] proved that if $g : [a_1, b_1] \times \cdots \times [a_m, b_m] \mapsto \mathbb{R}^1$ is a function of strong bounded variation, then

$$T(f) = (HK) \underset{[a_1,b_1]\times\cdots\times[a_m,b_m]}{\int\cdots\int} f(x_1, \ldots, x_m) g(x_1, \ldots, x_m) dx_1 \ldots dx_m\,. \quad (1)$$

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is a continuous linear functional on the space $\mathcal{D}$ of Henstock-Kurzweil integrable functions on $[a_1, b_1] \times \cdots \times [a_m, b_m]$. This led Mikusiński and Ostaszewski [9] to ask whether (1) gives the general form of a continuous linear functional on $\mathcal{D}$?

In this paper, we answer their question in the affirmative by using the theory of LH integral.

To simplify notation we give the proofs only in the two-dimensional case.

2 Definitions and Remarks

Definition 2.1 *Let $E = [a, b] \times [c, d]$ be a rectangle in two-dimensional Euclidean space $\mathbb{R}^2$. A division D of E is a collection $D = \{(I_1, (\xi_1, \eta_1)), \cdots, (I_p, (\xi_p, \eta_p))\}$ where $I_1, \ldots, I_p$ are nonoverlapping rectangles, $(\xi_1, \eta_1), \ldots, (\xi_p, \eta_p)$ are points, $\cup_{i=1}^{p} I_i = E$, and $(\xi_i, \eta_i) \in I_i$ for $i = 1, 2, \ldots, p$. For brevity, we write $D = \{(I, (\xi, \eta))\}$ where I denotes a typical rectangle in D and (ξ, η) is the associated point of I. If δ is a positive function on E, then a division D of E is called δ-fine whenever $d(I_i) < \delta(\xi_i, \eta_i)$ for $i = 1, 2, \ldots, p$, where $d(I_i)$ denotes the length of the diagonal line of I_i.*

Definition 2.2 *(see [5], [3]). A function f defined on a rectangle E is said to be Henstock-Kurzweil integrable to A if for every $\epsilon > 0$ there is a positive function δ on E such that for any δ-fine division $D = \{(I, (\xi, \eta))\}$ of E, we have*

$$\left| \Big((D) \sum f(\xi, \eta)|I| \Big) - A \right| < \epsilon\,.$$

Here $|I|$ is the area (or measure) of I and $(D) \sum f(\xi, \eta)|I|$ the sum of $f(\xi, \eta)|I|$ for all $(I, (\xi, \eta)) \in D$.

Definition 2.3 *Let $\{X_n\}$ be a sequence of closed subsets of a rectangle $E = [a, b] \times [c, d]$ with $X_n \subset X_{n+1}$ for all n, and $\cup_{n=1}^{\infty} X_n = E$. A function f defined on E is said to fulfill the condition (L) on $\{X_n\}$ if f is Lebesgue integrable on each X_n and*

$$(L) \int\!\!\int_{X_n \cap ([a,x] \times [c,y])} f(s, t)\, ds\, dt$$

converges uniformly on E. Also, f is said to fulfill the condition (H) on $\{X_n\}$ if for each n there exists $\delta_n(\xi, \eta) > 0$ satisfying $S((\xi, \eta), \delta_n(\xi, \eta)) \subset E \setminus X_n$ when $(\xi, \eta) \in E \setminus X_n$ such that $\lim_{n \to \infty} \tau_n = 0$, where $S((\xi, \eta), \delta_n(\xi, \eta))$ is an open circular disc with center (ξ, η) and radius $\delta_n(\xi, \eta)$,

$$\tau_n(x, y) = \sup \left| (D) \sum_{(\xi, \eta) \notin X_n} f(\xi, \eta)|I| \right|,$$

(the supremum being taken over all δ_n-fine divisions $D = (I, (\xi, \eta))$ of $[a, x] \times [c, y]$ and the sum is over $(I, (\xi, \eta))$ in D with $(\xi, \eta) \notin X_n$), and

$$\tau_n = \sup_{(x,y)\in E} \tau_n(x, y)\,.$$

Definition 2.4 *A function f is said to be LH integrable on $E = [a, b] \times [c, d]$ if there exists a sequence of closed subsets X_n of E with $X_n \subset X_{n+1}$ for all n and $\cup_{n=1}^{\infty} X_n = E$ such that f fulfills both the condition (L) and the condition (H) on $\{X_n\}$. The (LH) integral of f on E is given by*

$$(LH)\int\int_E f(x, y)\,dx\,dy \;=\; \lim_{n\to\infty}(L)\int\int_{X_n} f(x, y)\,dx\,dy\,.$$

Write $F(x, y)$ for the LH primitive of $f(x, y)$ on E.

Remark 2.5 *a) Obviously, if f is Lebesgue integrable on a rectangle E, then f is LH integrable on E.*

b) In the one-dimensional case, the LH integral is equivalent to the Henstock-Kurzweil integral (see [8]).

Definition 2.6 *(see [7]). Let F be a function defined on $E = [a, b] \times [c, d]$, $I = [\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \subset E$.*

- *We define $F(I) = F(\alpha_1, \alpha_2) + F(\beta_1, \beta_2) - F(\alpha_1, \beta_2) - F(\beta_1, \alpha_2)$. Then $F(I)$ is called the value of F on the rectangle I.*
- *Let $X \subset E$. A function F defined on E is said to be $AC^{**}(X)$ if for every $\epsilon > 0$ there are a $\delta(x, y) > 0$ and a $\eta > 0$ such that for any two δ-fine partial divisions of E with the associated points in X, namely $D_1 = \{(I_1, (x_1, y_1)\}$ and $D_2 = \{I_2, (x_2, y_2)\}$ with $x_1, x_2 \in X$ satisfying $(D_1 \setminus D_2) \sum |I| < \eta$ we have $|(D_1 \setminus D_2) \sum F(I)| < \epsilon$.*
- *A function F defined on E is said to be ACG^{**} if $E = \cup_{i=1}^{\infty} X_i$ so that each X_i is closed in E and F is $AC^{**}(X_i)$ for each i.*

Definition 2.7 *(see [6]). Let G be an open set in E. An elementary set I is called a nonabsolute subset of G if there exists $\delta(x, y) > 0$ for $(x, y) \in E$ such that I is the complement of a δ-fine cover of $E \setminus G$. A δ-fine cover of $E \setminus G$ is the union of the rectangles $I_1, I_2, \ldots, I_k$ such that $\{(I_i, (x_i, y_i))\}$ is δ-fine with $(x_i, y_i) \in E \setminus G$ and the union contains $E \setminus G$. We say that I is a nonabsolute subset of G involving δ.*

Definition 2.8 *Let $\mathcal{D}$ be the space of all LH integrable functions on E. We define a norm in $\mathcal{D}$ as follows:*

$$\|f\|_{\mathcal{D}} = \sup\left\{\left|\int\int_{[a,x]\times[c,y]} f(s,t)\,ds\,dt\right| \,:\, (x,y)\in E\right\}.$$

As usual, we regard two functions f and g as identical if $f(x,y) = g(x,y)$ almost everywhere on E. Then $\mathcal{D}$ is a normed linear space and we call it the LH space.

Remark 2.9 *It follows from the definition of LH integration that the L space (the family of all Lebesgue integrable functions on E), which is a subspace of $\mathcal{D}$, is dense in space $\mathcal{D}$.*

3 Equivalence of Integrals

In the one-dimensional Euclidean space, by means of a category argument and by using the Harnack extension, we proved that the LH integral and the Henstock-Kurzweil integral are equivalent [8]. In [6] Lee reformulated Harnack extension for the Henstock-Kurzweil integral in $\mathbb{R}^n$. To prove that the LH integral and the Henstock-Kurzweil integral are equivalent in $\mathbb{R}^2$, we need to reformulate the Harnack extension for the LH integral in $\mathbb{R}^2$.

The following Harnack extension differs slightly from that given in [6].

Lemma 3.1 (Harnack extension) *If the following conditions are satisfied:*

(i) f is Lebesgue integrable on a closed subset X of E;

(ii) f is LH integrable on every elementary subset I of $E\setminus X$;

(iii) there is a function F_0 on E such that for every $\epsilon > 0$ there exists $\delta(x,y) > 0$ such that for any nonabsolute subset Q of $E\setminus X$ involving δ we have

$$\left|(LH)\int\int_{([a,x]\times[c,y])\cap Q} f(s,t)\,ds\,dt - F_0(x,y)\right| < \epsilon \text{ for all } (x,y)\in E\,,$$

then f is LH integrable on E and

$$(LH)\int\int_E f(x,y)\,dx\,dy = (L)\int\int_X f(x,y)\,dx\,dy + F_0(E)\,.$$

PROOF. For each positive integer n, choose an open subset O_n such that $O_n \supset X$, $|O_n - X| < 1/n$ and $O_n \supset O_{n+1}$. In view of (iii), there exists $\delta_n(\xi,\eta) > 0$. (We may assume that $\delta_n(\xi,\eta)$ satisfies $S((\xi,\eta),\delta_n(\xi,\eta)) \subset E \setminus X$ when $(\xi,\eta) \in E \setminus X$ and $S((\xi,\eta),\delta_n(\xi,\eta)) \subset O_n$ when $(\xi,\eta) \in X$.) such that for any nonabsolute subset Q_n of $E \setminus X$ involving δ_n we have

$$\left| \int\!\!\int_{([a,x]\times[c,y])\cap Q_n} f(s,t)\,ds\,dt - F_0(x,y) \right| < \frac{1}{2n} \text{ for all } (x,y) \in E\,, \tag{2}$$

We choose a sequence $\{Q_n\}$ of nonabsolute subsets of $E \setminus X$ such that (2) hold and $Q_n \subset Q_{n+1}$ for each n. Note that Q_n is the union of finitely many open rectangles and $|Q_n| > |E \setminus X| - 1/n$. The rest of the proof follows the same way as that of Lemma 4 of [8], only note that put $X_0 = \cap_{n=1}^{\infty}(E \setminus Q_n)$. □

We therefore have the following assertion.

Theorem 3.2 *If f is Henstock-Kurzweil integrable on E, then it is LH integrable there and*

$$(LH)\int\!\!\int_E f(x,y)\,dx\,dy = (HK)\int\!\!\int_E f(x,y)\,dx\,dy\,.$$

PROOF. We shall use a standard category argument (see [8]). Let F be the Henstock-Kurzweil primitive of f on E. We say a point (x,y) is regular if there is a rectangle $I \subset E$ containing (x,y) as an interior point such that f is LH integrable on I with F as its LH primitive on I. Because F is ACG^{**} on E (see [7]) and by the Baire category theorem, f is Lebesgue and therefore LH integrable on some rectangle in E. In other words, the set of regular points is nonempty. Let P be the set of all non regular points in E. Then P is closed and we shall prove that indeed P is empty. Suppose P is not empty. Again, in view of the Baire category theorem, there is a portion P_0 of P such that F is $AC^{**}(P_0)$. Let $J_0 = [a_0,b_0] \times [c_0,d_0]$ be the smallest closed rectangle containing P_0. Then f is Lebesgue integrable on P_0. Now, put

$$F_0(x,y) = (HK)\int\!\!\int_{(J_0\setminus P_0)\cap([a_0,x]\times[c_0,y])} f(s,t)\,ds\,dt\,.$$

Obviously, F_0 is still $AC^{**}(P_0)$ and therefore for every $\epsilon > o$ there exist a $\delta_1(x,y) > 0$ and a $\eta > 0$ such that for any two δ_1-fine partial divisions of J_0 with the associated points in P_0, namely, $D_1 = \{(I_1,(x_1,y_1))\}$ and $D_2 = \{(I_2,(x_2,y_2))\}$ with $(x_1,y_1),(x_2,y_2) \in P_0$ satisfying $(D_1 \setminus D_2)\sum |I| < \eta$ we have

$$|(D_1 \setminus D_2)\sum F_0(I)| < \frac{\epsilon}{2}\,. \tag{3}$$

Since $f - f\chi_{P_0}$ is Henstock-Kurzweil integrable on J_0 with the primitive F_0, it follows from Henstock Lemma [3] that for a given $\epsilon > 0$ there is $\delta_2(x,y) > 0$ (we may assume $S((x,y),\delta_2(x,y)) \subset J_0 \setminus P_0$ when $(x,y) \in J_0 \setminus P_0$) such that for any δ_2-fine partial division $D = \{(I,(x,y))\}$ of J_0 with $(x,y) \in P_0$ we have

$$\Big|(D)\sum((f(x,y) - f\chi_{P_0}(x,y))|I| - F_0(I))\Big| < \frac{\epsilon}{2}$$

i.e.,

$$\Big|(D)\sum F_0(I)\Big| < \frac{\epsilon}{2}\,. \tag{4}$$

Let $\{I_{ij}\}$ be an rectangular net division of J_0, where $a_0 = x_0 < x_1 < \cdots < x_m = b_0$, $c_0 = y_0 < y_1 < \cdots < y_n = d_0$,

$$\sup_{\substack{1\le i\le m\\ 1\le j\le n}} \{(x_i - x_{i-1}),(y_j - y_{j-1})\} < \frac{\eta}{2[(b_0 - a_0) + (d_0 - c_0)]}\,,$$

and $I_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ for $i = 1,2,\ldots,m$; $j = 1,2,\ldots,n$.

Define $\delta(x,y)$ as follows:

If $x_{i-1} < x < x_i$, $y_{j-1} < y < y_j$, let

$$2\delta(x,y) = \min\{\delta_1(x,y),\delta_2(x,y),(x - x_{i-1}),(x_i - x),(y - y_{j-1}),(y_j - y)\}.$$

If $x = x_i$, $y_{j-1} < y < y_j$, let

$$2\delta(x,y) = \min\{\delta_1(x,y),\delta_2(x,y),(x - x_{i-1}),(x_{i+1} - x),(y - y_{j-1}),(y_j - y)\}.$$

If $x_{i-1} < x < x_i$, $y = y_j$, let

$$2\delta(x,y) = \min\{\delta_1(x,y),\delta_2(x,y),(x - x_{i-1}),(x_i - x),(y - y_{j-1}),(y_{j+1} - y)\}.$$

If $x = x_i$, $y = y_j$, let

$$2\delta(x,y) = \min\{\delta_1(x,y),\delta_2(x,y),(x - x_{i-1}),(x_{i+1} - x),(y - y_{j-1}),(y_{j+1} - y)\}.$$

Then for any δ-fine division D of J_0, write $D = D^{'} \cup D^{''}$, where $D^{'}$ denotes the partial division of D for which the associated points in P_0 and $D^{''}$ otherwise. By (3) and (4) we obtain

$$\Big|(D^{'})\sum F_0(I \cap ([a_0,x] \times [c_0,y]))\Big| < \epsilon \ \text{ for all } \ (x,y) \in J_0\,. \tag{5}$$

Put $Q = \sum_{I\in D''} I$. Thus (5) implies

$$\left|\int\!\!\int_{Q\cap([a_0,x]\times[c_0,y])} f(s,t)\,ds\,dt - F_0(x,y)\right| < \epsilon \ \text{ for all } \ (x,y) \in E\,.$$

It follows from Lemma 3.1 that the function f is LH integrable on J_0 and we have

$$(LH)\int\int_{J_0} f(x,y)dxdy =$$

$$(L)\int\int_{P_0} f(x,y)dxdy + F_0(J_0) = (HK)\int\int_{J_0} f(x,y)dxdy$$

which is a contradiction. Hence the proof is complete. □

Theorem 3.3 *If f is LH integrable on E, then it is Henstock-Kurzweil integrable there.*

PROOF. The proof follows as that in [8] (p. 524). □

Thus, from Theorem 3.2 and Theorem 3.3 we get that the Henstock-Kurzweil integral and the LH integral are equivalent in $\mathbb{R}^2$. In addition the LH space $\mathcal{D}$ can also be said to be the Henstock-Kurzweil space.

4 The General Form of a Continuous Linear Functional on the Space $\mathcal{D}$

Definition 4.1 *Let $E = [a,b] \times [c,d]$ be a rectangle in $\mathbb{R}^2$.*

- *A function $g : E \mapsto \mathbb{R}^1$ is said to be of bounded variation if $\sup_{i=1}^n |g(I_i)| < +\infty$, where the supremum is taken over all partitions of E into a finite collection of nonoverlapping nondegenerate closed rectangles I_i, $i = 1, 2, \ldots, n$. Let us denote $\sup \sum_{i=1}^n |g(I_i)|$ by $V(g(x,y); E)$.*
- *A function $g : E \mapsto \mathbb{R}^1$ is said to be of strong bounded variation if g is of bounded variation on E, and for every $x \in [a,b]$, $g(x,\cdot)$ is of bounded variation, for every $y \in [c,d]$, $g(\cdot,y)$ is of bounded variation.*

Remark 4.2 *a) In Definition 4.1, "all partitions $\{I_i\}_{1\le i\le n}$ of E" can be replaced by "all rectangular net partitions $\{I_{ij}\}_{1\le i\le m, 1\le j\le n}$ of E, where $a = x_0 < x_1 < \cdots < x_m = b$, $c = y_0 < y_1 < \cdots < y_n = d$ and $I_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ for $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$".*

b) In Definition 4.1, the condition "for every $x \in [a,b]$, $g(x,\cdot)$ is of bounded variation, for every $y \in [c,d]$, $g(\cdot,y)$ is of bounded variation" can be replaced by the condition "for some $x \in [a,b]$, $g(x,\cdot)$ is of bounded variation and for some $y \in [c,d]$, $g(\cdot,y)$ is of bounded variation".

Definition 4.3 *A function G is said to satisfy the Lipschitz condition on a rectangle E if there is a constant $L > 0$ such that $|G(I)| < L|I|$ for any sub-rectangle I of E where $G(I)$ is the value of G on I.*

Definition 4.4 *A function G is said to be of strong bounded slope variation on a rectangle E if the following conditions are satisfied:*

1. *There is a constant $M > 0$ such that*

$$\sum_{\substack{1\le i\le m-1\\ 1\le j\le n-1}} \left| \frac{G(I_{ij})}{|I_{ij}|} + \frac{G(I_{i+1,j+1})}{|I_{i+1,j+1}|} - \frac{G(I_{i,j+1})}{|I_{i,j+1}|} - \frac{G(I_{i+1,j})}{|I_{i+1,j}|} \right| \le M$$

   for all rectangular net partitions $\{I_{ij}\}_{1\le i\le m, 1\le j\le n}$ of E, where $E = [a,b]\times[c,d]$, $a = x_0 < x_1 < \cdots < x_m = b$, $c = y_0 < y_1 < \cdots < y_n = d$, $I_{ij} = [x_{i-1}, x_i]\times[y_{j-1}, y_j]$ for $i = 1,2,\ldots,m$; $j = 1,2,\ldots,n$, and $G(I_{ij})$ is the value of G on I_{ij};

2. *There is a M_1 such that for all $\{I_i\}_{1\le i\le m}$ we have*

$$\sum_{i=1}^{m-1} \left| \frac{G(I_i)}{|I_i|} - \frac{G(I_{i+1})}{|I_{i+1}|} \right| \le M_1$$

   where $I_i = [x_{i-1}, x_i]\times[y_1, y_2]$, $a = x_0 < x_1 < \cdots < x_m = b$, $c \le y_1 \le y_2 \le d$, and $G(I_i)$ is the value of G on I_i;

3. *There is a M_2 such that for all $\{J_j\}_{1\le j\le n}$ we have*

$$\sum_{i=1}^{n-1} \left| \frac{G(J_j)}{|J_j|} - \frac{G(J_{j+1})}{|I_{j+1}|} \right| \le M_2$$

   where $J_j = [x_1, x_2]\times[y_{j-1}, y_j]$, $a \le x_1 < x_2 \le b$, $c = y_0 < y_1 < \cdots < y_n = d$, and $G(J_j)$ is the value of G on J_j;

Lemma 4.5 *Let Q be a subset of E and the measure of Q zero. Let g be a function on $E\setminus Q$. If there exists a constant $M > 0$ such that*

$$\sum_{\substack{1\le i\le m\\ 1\le j\le n}} |g(I_{ij})| \le M \quad \text{for any} \quad \{I_{ij}\} \text{ of } E$$

(where $a = x_0 < x_1 < \cdots < x_m = b$, $c = y_0 < y_1 < \cdots < y_n = d$, $I_{ij} = [x_{i-1}, x_i]\times[y_{j-1}, y_j]$, $i = 1,2,\ldots,m$; $j = 1,2,\ldots,n$, and $(x_i, y_i) \notin Q$), then there is a function h of bounded variation on E such that $g(x,y) = h(x,y)$ for almost all $(x,y)\in E$.

PROOF. For the sake of brevity we assume that $g(x,y) = 0$ for all $(x,y) \in E_1$, where $E_1 = ([a,b] \times [c,c']) \cup ([a,a'] \times [c',d])$ and $a < a' < b$, $c < c' < d$.
Step 1. First, we define h on E. For each $(x,y) \in E$, take $a = x_0 < x_1 < \cdots < x_m < \cdots < x$; $c = y_0 < y_1 < \cdots < y_n < \cdots < y$, $(x_n, y_n) \to (x,y)$ and $(x_i, y_i) \notin Q$ for $i = 1,2,\ldots$; $j = 1,2,\ldots$. Because

$$\sum_{\substack{1 \le i \le \infty \\ 1 \le j \le \infty}} |g([x_{i-1}, x_i] \times [y_{j-1}, y_j])|$$

converges, $g(x_0, y_n) = g(x_n, y_0) = 0$, and

$$g(x_n, y_n) = \sum_{\substack{1 \le i \le n \\ 1 \le j \le n}} |g([x_{i-1}, x_i] \times [y_{j-1}, y_j])|,$$

thus $\lim_{n\to\infty} g(x_n, y_n)$ exists. Let $h(x,y) = \lim_{n\to\infty} g(x_n, y_n)$. We can show that $h(x,y)$ is well-defined. In fact, if (x'_n, y'_n) is another sequence of such points (where $a = x'_0 < x'_1 < \cdots < x'_m < \cdots < x$; $c = y'_0 < y'_1 < \cdots < y'_n < \cdots < y$, $(x'_n, y'_n) \to (x,y)$ and $(x'_i, y'_j) \notin Q$ for $i = 1,2,\ldots$; $j = 1,2,\ldots$), then we choose sub-sequences $\{(x_{n_k}, y_{n_k})\}$ and $\{(x'_{n_k}, y'_{n_k})\}$ from $\{(x_n, y_n)\}$ and $\{(x'_n, y'_n)\}$ respectively such that $x_{n_k} < x'_{n_k} < x_{n_{k+1}}$, $y_{n_k} < y'_{n_k} < y_{n_{k+1}}$ for $k = 1,2,\ldots$. We can easily see that $\lim_{k\to\infty} g(x_{n_k}, y_{n_k}) = \lim_{k\to\infty} g(x'_{n_k}, y'_{n_k})$. Therefore $h(x,y)$ is well-defined on E.
Step 2. We show that h is equal to g almost everywhere. Put $N = \{(x,y) \in E : h(x,y) \ne g(x,y)\}$. Consequently, $Q \subset N \subset E$, and the two-dimensional measure of N is zero. (Suppose that the two-dimensional measure of N isn't zero; it follows from Fubini theorem that there is a straight line $\ell : \{(x,y) : y = \overline{y} + \frac{d-c}{b-a}x\}$ such that the one-dimensional measure of $Q \cap \ell$ is zero, and the one-dimensional measure of $N \cap \ell$ isn't zero. Let g_0 be the restriction of g to the straight line ℓ. Then the one-dimensional measure of the relative discontinuities of g_0 (We can regard g_0 as an one variable function.) on $(E \setminus Q) \cap \ell$ isn't zero. On the other hand, since g_0 is a function of bounded variation on the set $((E \setminus Q) \cap \ell) \setminus Z$, (Where $Z \subset \ell$, and the one-dimensional measure of Z is zero.) the set of all relative discontinuities of g_0 on $((E \setminus Q) \cap \ell) \setminus Z$ is at most countable. This is a contradiction.) Therefore $h(x,y) = g(x,y)$ for almost all $(x,y) \in E$.
Step 3. We show that h is of bounded variation on E. For any rectangular net partition $\{I_{ij}\}_{1\le i\le m, 1\le j\le n}$ of E, where $a = x_0 < x_1 < \cdots < x_m = b$; $c = y_0 < y_1 < \cdots < y_n = d$, $I_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$. It follows from Step 1 that we can choose $\{(x_i^{(k)}, y_j^{(k)})\}$ satisfying $x_{i-1} < x_i^{(1)} < x_i^{(2)} < \cdots < x_i^{(k)} < \cdots < x_i$, $y_{j-1} < y_j^{(1)} < y_j^{(2)} < \cdots < y_j^{(k)} < \cdots < y_j$, $\lim_{k\to\infty}(x_i^{(k)}, y_j^{(k)}) \to (x_i, y_j)$, and

$(x_i^{(0)}, y_j^{(k)}) = (a, y_j^{(k)})$, $(x_i^{(k)}, y_j^{(0)}) = (x_i^{(k)}, c)$ such that

$$\lim_{k\to\infty} g(x_i^{(k)}, y_j^{(k)}) = h(x_i, y_j) \quad (i = 1, 2, \dots, m; \quad j = 1, 2, \dots, n)\,.$$

Hence

$$\begin{aligned}\sum_{\substack{1\le i\le m\\ 1\le j\le n}} |h(I_{ij})| &= \sum_{\substack{1\le i\le m\\ 1\le j\le n}} |h(x_{i-1}, y_{j-1}) + h(x_i, y_j) - h(x_{i-1}, y_j) - h(x_i, y_{j-1})| \\ &= \sum_{\substack{1\le i\le m\\ 1\le j\le n}} \Big|\lim_{k\to\infty} g(x_{i-1}^{(k)}, y_{j-1}^{(k)}) + \lim_{k\to\infty} g(x_i^{(k)}, y_j^{(k)}) \\ &\qquad - \lim_{k\to\infty} g(x_{i-1}^{(k)}, y_j^{(k)}) - \lim_{k\to\infty} g(x_i^{(k)}, y_{j-1}^{(k)})\Big| = \end{aligned}$$

$$\lim_{k\to\infty}\left(\sum_{\substack{1\le i\le m\\ 1\le j\le n}} \Big|g(x_{i-1}^{(k)}, y_{j-1}^{(k)}) + g(x_i^{(k)}, y_j^{(k)}) - g(x_{i-1}^{(k)}, y_j^{(k)}) - g(x_i^{(k)}, y_{j-1}^{(k)})\Big|\right) \le M$$

Thus h is of bounded variation on E. □

Theorem 4.6 *If function G satisfies the Lipschitz condition and is of strong bounded slope variation on rectangle E, then G is the primitive of a function of strong bounded variation on E.*

PROOF. Suppose G satisfies the Lipschitz condition on $E = [a, b] \times [c, d]$. Let I_i, $i = 1, 2, \dots, n$, be a finite sequence of non-overlapping rectangles. Then $\sum_{i=1}^n |G(I_i)| \le \sum_{i=1}^n L|I_i| = L \sum_{i=1}^n |I_i|$. From this, we obtain that G is of bounded variation on E, and therefore $D(G(x, y))$ exists at almost all $(x, y) \in E$, where the derivative $D(G(x, y))$ is a regular derivation (see [2], p. 103). Write

$$g(x, y) = D(G(x, y)) \text{ for } (x, y) \in E \setminus Q\,,$$

where Q is the set of all points at which $D(G(x, y))$ doesn't exist.

First , for any rectangular net partition of E with vertices of all rectangles belonging to $E \setminus Q$, without loss of generality, we may assume that all the points of intersection of rectangular lines are $\{x_{2i-1}, y_{2j-1})\}_{1\le i\le m, 1\le j\le n}$, and $(x_{2i-1}, y_{2j-1}) \in E \setminus Q$ for $i = 1, 2, \dots, m$; $j = 1, 2, \dots, n$, where $a = x_1 < x_3 < \cdots < x_{2m-1} = b$, $c = y_1 < y_3 < \cdots < x_{2n-1} = d$.

At each point $(x, y) \in E \setminus Q$, we have

$$\frac{G(I)}{|I|} \;\to\; g(x, y) \quad \text{as} \quad d(I) \to 0\,,$$

where $(x, y) \in I$ and $d(I)$ denotes the length of the diagonal of I. Then given $\epsilon > 0$ there is a $\sigma > 0$ such that for any regular rectangle I (i.e., the ratio of the shortest and the longest sides of I is some fixed number, say between $1/\lambda$ and λ), $(x_{2i-1}, y_{2j-1}) \in I$ and $d(I) < \sigma$, we have

$$\left|\frac{G(I)}{|I|} - g(x_{2i-1}, y_{2j-1})\right| < \frac{\epsilon}{4mn}, \quad i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n.$$

Now, divide E finer by adding straight lines

$\{(x, y) \mid x = x_1^{''},\ c \le y \le d\}$,

$\{(x, y) \mid x = x_3^{'},\ c \le y \le d\}$,

$\{(x, y) \mid x = x_3^{''},\ c \le y \le d\}$,

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots .$

$\{(x, y) \mid x = x_{2m-3}^{''},\ c \le y \le d\}$,

$\{(x, y) \mid x = x_{2m-1}^{'},\ c \le y \le d\}$,

$\{(x, y) \mid a \le x \le b,\ y = y_1^{''}\}$,

$\{(x, y) \mid a \le x \le b,\ y = y_3^{'}\}$,

$\{(x, y) \mid a \le x \le b,\ y = y_3^{''}\}$,

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots .$

$\{(x, y) \mid a \le x \le b,\ y = y_{2n-3}^{''}\}$,

$\{(x, y) \mid a \le x \le b,\ y = y_{2n-1}^{'}\}$

to the above rectangular net partition, in which $x_{2i-1}^{''} < x_{2i+1}^{'}$, $y_{2j-1}^{''} < y_{2j+1}^{'}$ and $x_{2i-1}^{''} - x_{2i-1} = x_{2i+1} - x_{2i+1}^{'} = y_{2j-1}^{''} - y_{2j-1} = y_{2j+1} - y_{2j+1}^{'} = \sigma/4$, $i = 1, 2, \ldots, m-1$; $j = 1, 2, \ldots, n-1$. Write

$$\begin{aligned} I_{2i-1,2j-1} &= [x_{2i-1}^{'}, x_{2i-1}^{''}] \times [y_{2j-1}^{'}, y_{2j-1}^{''}], \\ I_{2i,2j-1} &= [x_{2i-1}^{''}, x_{2i+1}^{'}] \times [y_{2j-1}^{'}, y_{2j-1}^{''}], \\ I_{2i-1,2j} &= [x_{2i-1}^{'}, x_{2i-1}^{''}] \times [y_{2j-1}^{''}, y_{2j+1}^{'}], \\ I_{2i,2j} &= [x_{2i-1}^{''}, x_{2i+1}^{'}] \times [y_{2j-1}^{''}, y_{2j+1}^{'}]. \end{aligned}$$

(Note that $x_{2i-1}^{'}$ is replaced by x_{2i-1} and $y_{2j-1}^{'}$ by y_{2j-1} when $i = 1$, and $x_{2i-1}^{''}$ is replaced by x_{2i-1} and $y_{2j-1}^{''}$ by y_{2j-1} when $i = m$.) We get a rectangular net partition of E. Hence

$$\sum_{\substack{1\le i\le m-1\\1\le j\le n-1}} |g(x_{2i-1},y_{2j-1})+g(x_{2i+1},y_{2j+1})-g(x_{2i-1},y_{2j+1})-g(x_{2i+1},y_{2j-1})|$$

$$\le \sum_{\substack{1\le i\le m-1\\1\le j\le n-1}} \left|\frac{G(I_{2i-1,2j-1})}{|I_{2i-1,2j-1}|}+\frac{G(I_{2i+1,2j+1})}{|I_{2i+1,2j+1}|}+\frac{G(I_{2i-1,2j+1})}{|I_{2i-1,2j+1}|}+\frac{G(I_{2i+1,2j-1})}{|I_{2i+1,2j-1}|}\right|$$

$$+ \sum_{\substack{1\le i\le m-1\\1\le j\le n-1}} \frac{\epsilon}{mn}$$

$$\le \sum_{\substack{1\le i\le m-1\\1\le j\le n-1}} \left(\left|\frac{G(I_{2i-1,2j-1})}{|I_{2i-1,2j-1}|}+\frac{G(I_{2i,2j})}{|I_{2i,2j}|}-\frac{G(I_{2i-1,2j})}{|I_{2i-1,2j}|}-\frac{G(I_{2i,2j-1})}{|I_{2i,2j-1}|}\right|\right.$$

$$+ \left|\frac{G(I_{2i-1,2j})}{|I_{2i-1,2j}|}+\frac{G(I_{2i,2j+1})}{|I_{2i,2j+1}|}-\frac{G(I_{2i-1,2j+1})}{|I_{2i-1,2j+1}|}-\frac{G(I_{2i,2j})}{|I_{2i,2j}|}\right|$$

$$+\left|\frac{G(I_{2i,2j-1})}{|I_{2i,2j-1}|}+\frac{G(I_{2i+1,2j})}{|I_{2i+1,2j}|}-\frac{G(I_{2i,2j})}{|I_{2i,2j}|}-\frac{G(I_{2i+1,2j-1})}{|I_{2i+1,2j-1}|}\right|$$

$$\left.+\left|\frac{G(I_{2i,2j})}{|I_{2i,2j}|}+\frac{G(I_{2i+1,2j+1})}{|I_{2i+1,2j+1}|}-\frac{G(I_{2i,2j+1})}{|I_{2i,2j+1}|}-\frac{G(I_{2i+1,2j})}{|I_{2i+1,2j}|}\right|\right)+\epsilon =$$

$$\sum_{\substack{1\le i\le 2m-2\\1\le j\le 2n-2}} \left|\frac{G(I_{ij})}{|I_{ij}|}+\frac{G(I_{i+1,j+1})}{|I_{i+1,j+1}|}-\frac{G(I_{i,j+1})}{|I_{i,j+1}|}-\frac{G(I_{i+1,j})}{|I_{i+1,j}|}\right|+\epsilon\le M+\epsilon\,.$$

It follows from Lemma 4.5 that there is a function h of bounded variation on E such that g is equal to h almost everywhere.

Next we show that for each $y\in[c,d]$, $h(\,\cdot\,,y)$ is of bounded variation, and so is $h(x,\cdot\,)$ for each $x\in[a,b]$. We can choose a straight line $\{(x,y): a\le x\le b,$ $y=\overline{y}\in(c,d)\}$, at which $D(G(x,y))$ exists almost everywhere. Without loss of generality, take $a=x_1<x_3<\cdots<x_{2m-1}=b$, and $(x_{2i-1},\overline{y})\in E\setminus Q$ for $i=1,2,\ldots,m$. Then given $\epsilon>0$, there is a $\sigma>0$ such that for any regular rectangle I, $(x_{2i-1},\overline{y})\in I$ and $d(I)<\sigma$, we have

$$\left|\frac{G(I)}{|I|}-g(x_{2i-1},\overline{y})\right|<\frac{\epsilon}{2m}\,,\quad i=1,2,\ldots,m\,.$$

Let $x''_{2i-1}<x'_{2i+1}$, $y_1<\overline{y}<y_2$, $x''_{2i-1}-x_{2i-1}=x_{2i+1}-x'_{2i+1}=\overline{y}-y_1=y_2-\overline{y}=\sigma/4$, $i=1,2,\ldots,m-1$. Write $I_{2i-1}=[x'_{2i-1},x''_{2i-1}]\times[y_1,y_2]$, $I_{2i}=[x''_{2i-1},x'_{2i+1}]\times[y_1,y_2]$ (note that x'_{2i-1} is replaced by x_{2i-1} when $i=1$, and x''_{2i-1} is replaced by x_{2i-1} when $i=m$). Hence

$$\sum_{i=1}^{m-1}|g(x_{2i-1},\overline{y})-g(x_{2i+1},\overline{y})|\le\sum_{i=1}^{m-1}\left|\frac{G(I_{2i-1})}{|I_{2i-1}|}-\frac{G(I_{2i+1})}{|I_{2i+1}|}\right|+\sum_{i=1}^{m-1}\frac{\epsilon}{m}$$

$$\leq \sum_{i=1}^{m-1}\left(\left|\frac{G(I_{2i-1})}{|I_{2i-1}|}-\frac{G(I_{2i})}{|I_{2i}|}\right|+\left|\frac{G(I_{2i})}{|I_{2i}|}-\frac{G(I_{2i+1})}{|I_{2i+1}|}\right|\right)+\epsilon$$
$$=\left(\sum_{i=1}^{2m-2}\left|\frac{G(I_i)}{|I_i|}-\frac{G(I_{i+1})}{|I_{i+1}|}\right|\right)+\epsilon \leq M_1+\epsilon .$$

As in Lemma 4.5, we can prove that

$$\lim_{x_n \nearrow x} g(x_n,\overline{y}) = h(x,\overline{y}) \quad \text{for each} \quad x\in[a,b]\,,$$
$$h(x,\overline{y}) = g(x,\overline{y}) \quad \text{for almost all} \quad x\in[a,b]\,,$$

and further $h(\,\cdot\,,\overline{y})$ is of bounded variation. By Remark 4.2, b) it follows that $h(\,\cdot\,,y)$ is of bounded variation for each $y\in[c,d]$. Similarly $h(x,\,\cdot\,)$ is of bounded variation for each $x\in[a,b]$. Therefore h is a function of strong bounded variation on E, and G is the primitive of h. □

In [4], Kurzweil proved that if g is of strong bounded variation on E, then $T(f)=\int\int_E f(x,y)g(x,y)\,dx\,dy$ defines a continuous linear functional on the LH space $\mathcal{D}$. Conversely, we have the following assertion.

Theorem 4.7 *If T is a continuous linear functional on the LH space $\mathcal{D}$, then*

$$T(f)=\int\int_E f(x,y)g(x,y)\,dx\,dy$$

for all $f\in\mathcal{D}$ and for some g of strong bounded variation on rectangle the $E=[a,b]\times[c,d]$.

PROOF. Put $G(x,y)=T(\chi_{[a,x]\times[c,y]})$, where $\chi_{[a,x]\times[c,y]}$ denotes the characteristic function of $[a,x]\times[c,y]$.

First, take a rectangular net partition $\{I_{ij}\}_{1\leq i\leq m, 1\leq j\leq n}$ of E. Then by the linearity of T we obtain

$$\sum_{\substack{1\leq i\leq m-1\\ 1\leq j\leq n-1}}\left|\frac{G(I_{ij})}{|I_{ij}|}+\frac{G(I_{i+1,j+1})}{|I_{i+1,j+1}|}-\frac{G(I_{i,j+1})}{|I_{i,j+1}|}-\frac{G(I_{i+1,j})}{|I_{i+1,j}|}\right| = \sum_{\substack{1\leq i\leq m-1\\ 1\leq j\leq n-1}}|T(\phi_{ij})|$$

where

$$\phi_{ij}=\frac{1}{|I_{ij}|}\chi_{I_{ij}}+\frac{1}{|I_{i+1,j+1}|}\chi_{I_{i+1,j+1}}-\frac{1}{|I_{i,j+1}|}\chi_{I_{i,j+1}}-\frac{1}{|I_{i+1,j}|}\chi_{I_{i+1,j}}\,.$$

Further by the boundedness of T we obtain

$$\sum_{\substack{1\le i\le m-1\\1\le j\le n-1}} |T(\phi_{ij})| = T\left(\sum_{\substack{1\le i\le m-1\\1\le j\le n-1}} \epsilon_{ij}\phi_{ij}\right) \le \|T\| \left(\| \sum_{\substack{1\le i\le m-1\\1\le j\le n-1}} \epsilon_{ij}\phi_{ij}\|\right) \le 4\|T\|$$

where ϵ denotes $+1$ or -1 as the case may be.

Next, for any $\{I_i\}_{1\le i\le m}$ where $I_i = [x_{i-1}, x_i] \times [y_1, y_2]$, $a = x_0 < x_1 < \cdots < x_m = b$ and $c \le y_1 < y_2 \le d$,

$$\sum_{i=1}^{m-1} \left| \frac{G(I_i)}{|I_i|} - \frac{G(I_{i+1})}{|I_{i+1}|} \right| = \sum_{i=1}^{m-1} |T(\phi_i)|$$

where $\phi_i = \frac{1}{|I_i|}\chi_{I_i} - \frac{1}{|I_{i+1}|}\chi_{I_{i+1}}$. Further by the boundedness of T we obtain

$$\sum_{i=1}^{m-1} |T(\phi_i)| = T\left(\sum_{i=1}^{m-1} \epsilon_i\phi_i\right) \le \|T\| \left(\| \sum_{i=1}^{m-1} \epsilon_i\phi_i\|\right) \le 2\|T\|\,.$$

Similarly, for all $\{J_i\}_{1\le j\le n}$ we have

$$\sum_{j=1}^{n-1} \left| \frac{G(J_j)}{|J_j|} - \frac{G(J_{j+1})}{|J_{j+1}|} \right| \le 2\|T\|\,,$$

where $J_j = [x_1, x_2] \times [y_{j-1}, y_j]$, $a \le x_1 < x_2 \le b$, $c = y_0 < y_1 < \cdots < y_n = d$. Consequently, G is of strong bounded slope variation on E. Put $I = [x_1, x_2] \times [y_1, y_2]$, where $a \le x_1 < x_2 \le b$, $c \le y_1 < y_2 \le d$. By the linearity of T, we obtain

$$\begin{aligned} G(I) &= T\left(\chi_{[a,x_1]\times[c,y_1]} + \chi_{[a,x_2]\times[c,y_2]} - \chi_{[a,x_2]\times[c,y_1]} - \chi_{[a,x_1]\times[c,y_2]}\right) \\ &= T\left(\chi_{[x_1,x_2]\times[y_1,y_2]}\right)\,. \end{aligned}$$

Therefore $|G(I)| \le \|T\| \cdot |I|$. That is, G satisfies the Lipschitz condition on E. It follows from Theorem 4.6 that G is the primitive of a function g which is of strong bounded variation on E. Therefore the representation holds for step functions and so does the representation for a Lebesgue integrable function.

Let f be LH integrable on E. In view of the definition of the LH integral, there is a sequence of closed subsets $\{X_n\}$ of E such that f fulfills both condition (L) and condition (H) on $\{X_n\}$. Write

$$f_n(x,y) = \begin{cases} f(x,y) & , \text{ when } (x,y) \in X_n \\ 0 & , \text{ when } (x,y) \in E \setminus X_n\,. \end{cases}$$

Then f_n, $n = 1, 2, \ldots$, are Lebesgue integrable on E, and the primitives F_n of f_n converge uniformly on E. It follows from the integration by parts formula proved by Kurzweil in [4] that

$$\int\!\!\int_E (f(x,y) - f_n(x,y))\, g(x,y)\, dx\, dy = \int\!\!\int_E (F(x,y) - F_n(x,y))\, dg(x,y)$$

$$\begin{aligned}
&- \int_a^b (F(t,d) - F_n(t,d))\, dg(t,d) + \int_a^b (F(t,c) - F_n(t,c))\, dg(t,c) \\
&- \int_c^d (F(b,t) - F_n(b,t)\, dg(b,t) + \int_c^d (F(a,t) - F_n(a,t))\, dg(a,t) \\
&+ (F(b,d) - F_n(b,d))g(b,d) - (F(b,c) - F_n(b,c))g(b,c) \\
&- (F(a,d) - F_n(a,d))g(a,d) + (F(a,c) - F_n(a,c))g(a,c)\,.
\end{aligned}$$

Thus

$$\begin{aligned}
&\left| \int\!\!\int_E (f(x,y) - f_n(x,y))\, g(x,y)\, dx\, dy \right| \\
\le &\left(\max_{(x,y)\in E} |F(x,y) - F_n(x,y)| \right) V(g(x,y); E) \\
&+ \left(\max_{a\le t\le b} |F(t,d) - F_n(t,d)| \right) V(g(t,d); [a,b]) \\
&+ \left(\max_{a\le t\le b} |F(t,c) - F_n(t,c)| \right) V(g(t,c); [a,b]) \\
&+ \left(\max_{c\le t\le d} |F(b,t) - F_n(b,t)| \right) V(g(b,t); [c,d]) \\
&+ \left(\max_{c\le t\le d} |F(a,t) - F_n(a,t)| \right) V(g(a,t); [c,d]) \\
&+ |F(b,d) - F_n(b,d)|\, |g(b,d)| + |F(b,c) - F_n(b,c)|\, |g(b,c)| \\
&+ |F(a,d) - F_n(a,d)|\, |g(a,d)| + |F(a,c) - F_n(a,c)|\, |g(a,c)|\,.
\end{aligned}$$

We note that g is bounded on E and

$$\lim_{n\to\infty} \left(\max_{(x,y)\in E} |F(x,y) - F_n(x,y) \right) = 0\,.$$

Hence

$$\lim_{n\to\infty} \int\!\!\int_E f_n(x,y) g(x,y)\, dx\, dy = \int\!\!\int_E f(x,y) g(x,y)\, dx\, dy\,.$$

It follows that $T(f) = \lim_{n\to\infty} T(f_n) = \lim_{n\to\infty} \int\int_E f_n(x,y)g(x,y)\,dx\,dy = \int\int_E f(x,y)g(x,y)\,dx\,dy$ and the proof is complete. □

Remark 4.8 *A function of strong bounded variation is a multiplier for the multi-dimensional Henstock-Kurzweil integral, but a function of bounded variation need not be (see Example 4.9).*

Example 4.9 *Let $E = [0,1] \times [0,1]$,*

$$g(x,y) = \begin{cases} \frac{1}{x} & , \quad \textit{when} \quad (x,y) \in (0,1] \times [0,1] \\ 0 & , \quad \textit{when} \quad (x,y) \in \{(x,y) \,|\, x = 0,\, y \in [0,1]\}\,, \end{cases}$$

and let $f(x,y) \equiv 1$. Obviously, g is of bounded variation on E and the variation is zero, but fg is not Henstock-Kurzweil integrable on E.

In conclusion, from Theorem 4.7, Remark 4.8 and [4] we get that T is a continuous linear functional on the space $\mathcal{D}$ of Henstock-Kurzweil integrable functions on $[a_1,b_1] \times \cdots \times [a_m,b_m]$ if and only if there exists a function g of strong bounded variation on $[a_1,b_1] \times \cdots \times [a_m,b_m]$ such that

$$T(f) = (HK) \underset{[a_1,b_1]\times\cdots\times[a_m,b_m]}{\int\cdots\int} f(x_1,\ldots,x_m)g(x_1,\ldots,x_m)\,dx_1\ldots dx_m\,.$$

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