RESEARCH

Real Analysis Exchange
Vol. 22(2), 1996-97, pp. 766–774

E. D'Aniello, Dipartimento di Matematica e Applicazioni R. Caccioppoli, Universitá Federico II, Naples, Italy, e-mail: daniel@matna2.dma.unina.it current address: Department of Mathematics, Wesleyan University, Middletown, CT 06459, USA

A. Hirshberg, Department of Mathematics, Wesleyan University, Middletown, CT 06459, USA, e-mail: ahirshberg@wesleyan.edu,

K. P. S. Bhaskara Rao, Division of Statistics and Mathematics, Indian Statistical Institute, Bangalore 560059, India, e-mail: isibang!kpsbrao@iisc.ernet.in,

R. M. Shortt, Department of Mathematics, Wesleyan University, Middletown, CT 06459, USA e-mail: rshortt@wesleyan.edu

BOUNDED COMMON EXTENSIONS OF VECTOR MEASURES

Abstract

Let $\mathcal{A}$ and $\mathcal{B}$ be fields of subsets of a set Ω, let $\mathbf{X}$ be a normed space with the Hahn-Banach extension property and let $\mu : \mathcal{A} \to \mathbf{X}$ and $\nu : \mathcal{B} \to \mathbf{X}$ be consistent, bounded, vector measures. We give necessary and sufficient conditions for μ and ν to have a bounded common extension to $\mathcal{A} \vee \mathcal{B}$, generalizing already known results for real valued charges.

1 Introduction

Let $\mathcal{A}$ be a field of subsets of a set Ω. We denote by $F(\Omega, \mathcal{A}) = F(\mathcal{A})$ the linear space spanned by indicator functions I_A of sets $A \in \mathcal{A}$. The functions in $F(\Omega, \mathcal{A})$ have finite range and are therefore bounded. If $f \in F(\Omega, \mathcal{A})$, then $\|f\|$ is the supremum norm of f.

Let X be a Banach space. A *finitely additive vector measure* is a set function $\mu : \mathcal{A} \to \mathbf{X}$ such that $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$, whenever $A_1, A_2 \in \mathcal{A}$

Key Words: Finitely additive vector measure, bounded vector measure, common extension
Mathematical Reviews subject classification: Primary 28B99; Secondary 28B05
Received by the editors February 6, 1997

are disjoint. The set function μ is also called simply a *vector measure.*

The *variation* of μ is the extended nonnegative function $|\mu|$ whose value on a set $A \in \mathcal{A}$ is given by

$$|\mu|(A) = \sup_{\pi} \sum_{E \in \pi} \|\mu(E)\|,$$

where the supremum is taken over all partitions π of A into a finite number of pairwise disjoint members of $\mathcal{A}$.

If $|\mu|(\Omega) < \infty$, then μ will be called a *measure of bounded variation.*

The *semivariation* of μ is the extended nonnegative function $\|\mu\|$ whose value on a set $A \in \mathcal{A}$ is given by

$$\|\mu\|(A) = \sup\{|x^*\mu|(A) : x^* \in X^*, \|x^*\| \leq 1\},$$

where $|x^*\mu|$ is the variation of the real valued measure $x^*\mu$. If $\|\mu\|(\Omega)$ $< \infty$, then μ will be called a *measure of bounded semivariation* or simply a *bounded vector measure* [3]. We define $\|\mu\| = \|\mu\|(\Omega)$.

Let $\mathcal{A}$ and $\mathcal{B}$ be fields of subsets of a set Ω and let μ and ν be vector measures on $\mathcal{A}$ and $\mathcal{B}$, respectively. Say that μ and ν are *consistent* if $\mu(C) = \nu(C)$ for all $C \in \mathcal{A} \cap \mathcal{B}$. Let $\mathcal{A} \vee \mathcal{B}$ be the field generated by $\mathcal{A} \cup \mathcal{B}$.

A Banach space $\mathbf{X}$ is said to have the *Hahn-Banach extension property* if each bounded linear operator T on a subspace of any Banach space $\mathbf{Y}$ with values in $\mathbf{X}$ has a linear extension $\tilde{T}$ carrying all of $\mathbf{Y}$ into $\mathbf{X}$ such that $\|\tilde{T}\| = \|T\|$ [5].

A normed space has the Hahn-Banach extension property if and only if the collection of its spheres has the binary intersection property [7].

Call $\mathbb{R}^n$ the n-dimensional euclidean space considered as a vector space in the usual way, ordered component by component and normed by

$$\|x\| = |x_1| \vee \ldots \vee |x_n| \text{ if } x = (x_1, \ldots, x_n).$$

The collection of spheres of $\mathbb{R}^n$ with the above norm has the binary intersection property [7]. For these and other facts about the Hahn-Banach extension property, we refer the reader to [4], [5] and [7].

Lemma 1.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be fields of subsets of Ω and suppose that μ and ν are given vector measures on $\mathcal{A}$ and $\mathcal{B}$, respectively, with values in a Banach space $\mathbf{X}$. If μ and ν are consistent, then they have a common extension, i.e. there is a vector measure ρ on $\mathcal{A} \vee \mathcal{B}$ such that $\rho(C) = \mu(C)$ for $C \in \mathcal{A}$ and $\rho(C) = \nu(C)$ for $C \in B$.*

PROOF. *Indication:* This is a well-known result. See e.g., theorem 3.6.2 of [2]. □

When do two bounded consistent vector measures have a bounded common extension? By the lemma, some common extension exists, but might not be bounded. The principal result of this paper [theorem 2.5] solves this problem in the case when $\mathbf{X}$ is a normed space with the Hahn-Banach extension property. Earlier, Lipecki [6] gave some examples to show that the answer to the question is "not always".

2 Chain conditions and bounded extensions

We begin with the following definition and a few abbreviations.
Call $\emptyset = C_0 \subseteq C_1 \subseteq C_2 \subseteq \ldots \subseteq C_{N+1} = \Omega$ a *chain* in $\mathcal{A} \cup \mathcal{B}$ if all C_i's are in $\mathcal{A} \cup \mathcal{B}$. The following elementary fact lays the groundwork of our main result.

Lemma 2.1. *Let μ on $\mathcal{A}$ and ν on $\mathcal{B}$ be consistent vector measures with values in a Banach space $\mathbf{X}$. If ρ on $\mathcal{A} \vee \mathcal{B}$ is a common extension of μ and ν, then for any finite chain $\emptyset = C_0 \subseteq C_1 \subseteq C_2 \subseteq \ldots \subseteq C_{N+1} = \Omega$ in $\mathcal{A} \cup \mathcal{B}$*

$$(*) \qquad \sup_{\epsilon_i} \| \sum \epsilon_i (\eta(C_{i+1}) - \eta(C_i)) \| \leq \|\rho\|(\Omega),$$

where the supremum is taken over all finite collections $\{\epsilon_i\}$ satisfying $\epsilon_i = \pm 1$ and $\eta(C) = \mu(C)$ or $\nu(C)$ according as C is in $\mathcal{A}$ or in $\mathcal{B}$.

PROOF. By proposition 11 of [3],

$$\|\rho\|(\Omega) = \sup\{\| \sum_{E_i \in \pi} \epsilon_i \rho(E_i) \|\},$$

where the supremum is taken over all partitions π of Ω into finitely many disjoint members of $\mathcal{A} \vee \mathcal{B}$ and all finite collections $\{\epsilon_i\}$ satisfying $|\epsilon_i| \leq 1$, but an accurate look at the proof shows that

$$\|\rho\|(\Omega) = \sup\{\| \sum_{E_i \in \pi} \epsilon_i \rho(E_i) \|\},$$

where the supremum is taken over all partitions π of Ω into finitely many disjoint members of $\mathcal{A} \vee \mathcal{B}$ and all finite collections $\{\epsilon_i\}$ satisfying $\epsilon_i = \pm 1$. Therefore, it follows that if μ on $\mathcal{A}$ and ν on $\mathcal{B}$ have a bounded common extension, then the supremum of the left-hand side of (*), taken over all possible finite chains and over all finite collections $\{\epsilon_i\}$ satisfying $\epsilon_i = \pm 1$ must be finite. □

Our main result [theorem 2.5] establishes the converse statement in the case when $\mathbf{X}$ is a normed space with the Hahn-Banach extension property.

Let $\mathcal{A}$ and $\mathcal{B}$ be fields of subsets of Ω and let $\mu : \mathcal{A} \to \mathbf{X}$ and $\nu : \mathcal{B} \to \mathbf{X}$ be consistent bounded vector measures. We define

$$I^0 = I^0(\mu, \nu) = \inf\{\|\rho\| : \rho \text{ a common extension of } \mu \text{ and } \nu \text{ to } \mathcal{A} \vee \mathcal{B}\},$$

$$S^0 = S^0(\mu,\nu) = \sup\{\|\int f\,d\mu + \int g\,d\nu\| : f \in F(\mathcal{A}), g \in F(\mathcal{B}), \|f+g\| \leq 1\},$$

$$\begin{aligned} SC^0 = SC^0(\mu,\nu) = \sup\{\|\sum_{i=0}^{N} \|\epsilon_i(\eta(C_{i+1}) - \eta(C_i))\| &: \emptyset = C_0 \subseteq C_1 \subseteq C_2 \\ &\subseteq \ldots \subseteq C_{N+1} = \Omega \text{ a chain in } \mathcal{A} \cup \mathcal{B}, \\ &\{\epsilon_i\} \text{ a finite collection satisfying } \epsilon_i = \pm 1, N \geq 0\}. \end{aligned}$$

In the remainder of this paper $\mathbf{X}$ will be a normed space with the Hahn-Banach extension property.

Theorem 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be fields of subsets of Ω and suppose that μ and ν are consistent vector measures on $\mathcal{A}$ and $\mathcal{B}$, respectively. Then $S^0(\mu,\nu) = I^0(\mu,\nu)$.*
The infimum defining $I^0 = I^0(\mu,\nu)$ is attained at some choice of ρ. If $\mathcal{A}$ and $\mathcal{B}$ are finite, then the supremum defining $S^0 = S^0(\mu,\nu)$ is attained for some f and g.

PROOF. Given $f \in F(\mathcal{A})$, $g \in F(\mathcal{B})$, $\|f+g\| \leq 1$ and some common extension ρ of μ and ν, we have

$$\|\int f\,d\mu + \int g\,d\nu\| = \|\int (f+g)\,d\rho\|.$$

By the Hahn-Banach theorem, there exists $x^* \in X^*$ with $\|x^*\| = 1$ and $x^*(\int (f+g)\,d\rho) = \|\int (f+g)\,d\rho\|$.

Therefore,

$$\begin{aligned} \|\int f\,d\mu + \int g\,d\nu\| &= x^*(\int (f+g)\,d\rho) = \int (f+g) d(x^*\rho) \\ &\leq \int \|f+g\| d|x^*\rho| \leq |x^*\rho|(\Omega) \leq \|\rho\|(\Omega), \end{aligned}$$

so that $S^0 \leq I^0$.

If $S^0 = \infty$, there is nothing to prove. If $S^0 < \infty$, consider the linear subspace M of $F(\mathcal{A} \vee \mathcal{B})$ defined by

$$M = \{f + g : f \in F(\mathcal{A}), g \in F(\mathcal{B})\}.$$

Let $L : M \to \mathbf{X}$ be the linear operator defined by

$$L(f + g) = \int f \, d\mu + \int g \, d\nu.$$

The consistency of μ and ν ensures that L is well-defined. In fact, L is a bounded linear operator on M with norm $\|L\| = S^0$. Since $\mathbf{X}$ has the Hahn-Banach extension property, L may be extended to a linear operator $L_0 : F(\mathcal{A} \vee \mathcal{B}) \to \mathbf{X}$ with $\|L_0\| = \|L\|$.

Then $\rho(C) = L_0(I_C)$ defines a charge ρ on $\mathcal{A} \vee \mathcal{B}$ with $\|\rho\| = \|L_0\| = S^0$, so that $S^0 = I^0$ and the infimum is attained at ρ.

If $\mathcal{A}$ and $\mathcal{B}$ are finite, then $F(\mathcal{A} \vee \mathcal{B})$ is a finite-dimensional space and the last statement of the theorem becomes elementary. □

Corollary 2.3. *In order that consistent $\mathbf{X}$-valued vector measures μ and ν have a bounded common extension, it is necessary and sufficient that $S^0(\mu, \nu) < \infty$.*

The following technical lemma will be used in the proof of our main theorem.

Lemma 2.4. *Let $\mathcal{A}$ be a field of subsets of Ω and let μ be a bounded $\mathbf{X}$-valued vector measures on $\mathcal{A}$. If $\| \int f \, d\mu\| = \|\mu\|$ for $f \in F(\Omega, \mathcal{A})$, with $\|f\| \leq 1$, then there exists $x^* \in X^*$ such that*

(i) $x^*\mu \geq 0$ *on subsets of* $\{x : f(x) = 1\}$;

(ii) $x^*\mu \leq 0$ *on subsets of* $\{x : f(x) = -1\}$;

(iii) $|x^*\mu|(\{x : -1 < f(x) < 1\}) = 0$.

PROOF. By the Hahn-Banach theorem, there exists $x^* \in X^*$ such that $\| \int f \, d\mu\| = x^*(\int f \, d\mu)$. Hence

$$\begin{aligned} \| \int f \, d\mu\| &= x^*(\int f \, d\mu) = \int f d(x^*\mu) \leq \int \|f\| d|x^*\mu| \\ &\leq |x^*\mu|(\Omega) \leq \|x^*\mu\| \leq \|\mu\| \end{aligned}$$

Therefore, $\int f d(x^*\mu) = \|x^*\mu\|$ and the assertions follow from lemma 1.4 of [1]. □

Theorem 2.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be fields of subsets of Ω and suppose that μ and ν are consistent $\mathbf{X}$-valued vector measures on $\mathcal{A}$ and on $\mathcal{B}$, respectively. Then $SC^0(\mu,\nu) = I^0(\mu,\nu)$.*

Proof. That $SC^0 \leq I^0$ follows from lemma 2.1.

In order to prove the reverse inequality, we use theorem 2.2. Suppose that $f_0 \in F(\mathcal{A})$ and $g_0 \in F(\mathcal{B})$ such that $\|f_0 + g_0\| \leq 1$ are given. We shall demonstrate that

$$(**) \qquad SC^0 \geq \| \textstyle\int f_0\, d\mu + \int g_0\, d\nu \|,$$

from which fact follows $SC^0 \geq S^0 = I^0$ as desired.
Let $\mathcal{A}_0$ (respectively $\mathcal{B}_0$) be the smallest field for which f_0 is measurable. Then $\mathcal{A}_0 \subseteq \mathcal{A}$ and $\mathcal{B}_0 \subseteq \mathcal{B}$ are finite. In order to prove $(**)$, we may assume that f_0 and g_0 have been chosen so that

$$\| \int f_0\, d\mu + \int g_0\, d\nu \|$$

is the supremum of $\| \int f\, d\mu + \int g\, d\nu \|$ over all choices of $f \in F(\mathcal{A}_0)$ and $g \in F(\mathcal{B}_0)$; we use the final sentence of theorem 2.2. Applying theorem 2.2 to μ_0 and ν_0, the restrictions of μ and ν to $\mathcal{A}_0$ and $\mathcal{B}_0$, respectively, we find some common extension ρ of μ_0 and ν_0 to $\mathcal{A}_0 \vee \mathcal{B}_0$ such that

$$\| \int f_0\, d\mu + \int g_0\, d\nu \| = \|\rho\|.$$

Hence

$$\| \int (f_0 + g_0)\, d\rho \| = \|\rho\|,$$

and as in lemma 2.4, there exists $x^* \in X^*$ with $f_0 + g_0 = \pm 1 (|x^*\rho| a.e.)$. By the same lemma, $x^*\rho \geq 0$ for subsets of $\{x : f_0(x) + g_0(x) = 1\}$ and $x^*\rho \leq 0$ for subsets of $\{x : f_0(x) + g_0(x) = -1\}$.

Now f_0 and g_0 may be replaced with $f_0 + c$ and $g_0 - c$ for any constant c, with no effect on the norm or integral of their sum. Thus, without loss of generality, we may assume that $f_0 \geq 0$ and $g_0 \leq 0$. Let, as in theorem 1.5 of [1], N be an even integer such that $N \geq \max\{\|f_0\|, \|g_0\|\}$ and define

$$C_i = \begin{cases} \{x \in \Omega : g_0(x) \leq -N + i - 1\} & \text{if } i \text{ is odd,} \\ \{x \in \Omega : f_0(x) \geq N - i + 1\} & \text{if } i \text{ is even.} \end{cases}$$

Then $\emptyset = C_0 \subseteq C_1 \subseteq \ldots \subseteq C_{N+1} = \Omega$ is a chain of sets in $\mathcal{A} \cup \mathcal{B}$. If i is odd, then $f_0 + g_0 > -1$ on $C_{i+1} - C_i$, so that $f_0 + g_0 = 1 (|x^*\rho| a.e.)$ on $C_{i+1} - C_i$.

Likewise, if i is even, then $f_0 + g_0 = -1(|x^*\rho|a.e.)$ on $C_{i+1} - C_i$.

Define functions $f_1 \in F(\mathcal{A})$ and $g_1 \in F(\mathcal{B})$ by putting

$$f_1 = N - 2n - 1 \quad \text{for } x \in C_{2n+2} - C_{2n},$$

$$g_1 = -N + 2n \quad \text{for } x \in C_{2n+1} - C_{2n-1}$$

for $n = 0, 1, \ldots, N/2$, noting that $C_{-1} = \emptyset$ and $C_{N+2} = \Omega$. For i odd $f_1 + g_1 = 1$ on $C_{i+1} - C_i$ and, for i even, $f_1 + g_1 = -1$ on $C_{i+1} - C_i$. Thus

$$\begin{aligned} SC^0 &\geq \sum_{i=0}^{N} |x^*\eta(C_{i+1}) - x^*\eta(C_i)| \\ &= \int (f_1 + g_1)\, d(x^*\rho) = \int (f_0 + g_0)\, d(x^*\rho) \\ &= x^*(\int (f_0 + g_0)\ d\rho) = \| \int (f_0 + g_0)\ d\rho\| \\ &= \| \int f_0\, d\mu + \int g_0\, d\nu\|. \quad \square \end{aligned}$$

Corollary 2.6. *In order that consistent* $\mathbf{X}$*-valued bounded vector measures* μ *and* ν *have a bounded common extension, it is necessary and sufficient that* $SC^0(\mu, \nu) < \infty$.

Inspection of the proof of theorem 2.5 yields a useful sharpening of this result:

Corollary 2.7. *In the supremum used to define* $SC^0(\mu, \nu)$*, it suffices to restrict attention to the chains* $\emptyset = C_0 \subseteq C_1 \subseteq \ldots \subseteq C_{N+1} = \Omega$ *, where* $C_i \in \mathcal{A}$ *if* i *is even, and* $C_i \in \mathcal{B}$ *if* i *is odd.*

3 Global conditions on fields

Let $\mathcal{A}$ and $\mathcal{B}$ be fields of subsets of Ω. Then $\mathcal{A}$ and $\mathcal{B}$ are *independent* if $\mathcal{A} \cap \mathcal{B} = \{\emptyset, \Omega\}$. As in [1] the following result follows from the theory of the previous section.

Theorem 3.1. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be independent fields of subsets of* Ω *and suppose that* μ *and* ν *are consistent vector measures on* $\mathcal{A}$ *and* $\mathcal{B}$*, respectively (consistency means only that* $\mu(\Omega) = \nu(\Omega)$*). Then* μ *and* ν *have a bounded common extension* ρ *on* $\mathcal{A} \vee \mathcal{B}$ *such that* $\|\rho\| = max\{\|\mu\|, \|\nu\|\}$.

PROOF. We apply Theorem 2.5 and Corollary 2.7. Independence essentially limits the length of chains as in Corollary 2.7. It is sufficient to consider chains of the form

$$\emptyset \subseteq A \subseteq \Omega \text{ for } A \in \mathcal{A} \quad \text{or} \quad \emptyset \subseteq B \subseteq \Omega \text{ for } B \in \mathcal{B}.$$

The supremum $SC^0(\mu, \nu)$ is thus taken over quantities of the form

$$\|\epsilon_0 \mu(A) + \epsilon_1 \mu(\Omega - A)\|$$

or

$$\|\epsilon_0 \nu(B) + \epsilon_1 \nu(\Omega - B)\|,$$

where $\epsilon_i = \pm 1$, for $i = 0, 1$. The result follows. □

Fields $\mathcal{A}$ and $\mathcal{B}$ over Ω are weakly independent, a notion due to Lipecki, if whenever $\Omega = A_1 \cup \ldots \cup A_n$ and $\Omega = B_1 \cup \ldots \cup B_m$ are partitions of Ω into nonempty sets $A_i \in \mathcal{A}$ and $B_i \in \mathcal{B}$, then there is some k and some l such that $A_k \cap B_i \neq \emptyset$ (each i) and $A_i \cap B_l \neq \emptyset$ (each i). The following is an improvement on a result of Lipecki [6] and a generalization of a result in [1].

Theorem 3.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be weakly independent fields of subsets of a set Ω and suppose that μ and ν are consistent vector measures on $\mathcal{A}$ and $\mathcal{B}$ (this means only that $\mu(\Omega) = \nu(\Omega)$). Then there is a common extension ρ of μ and ν such that $\|\rho\| \leq \|\mu\| + \|\mu(\Omega)\| + \|\nu\|$.*

PROOF. Apply Theorem 2.5 and Corollary 2.7. Weak independence limits the length the chains as in Corollary 2.7. They are either of the form $\emptyset \subseteq A \subseteq B \subseteq \Omega$ or $\emptyset \subseteq B \subseteq A \subseteq \Omega$ for $A \in \mathcal{A}$ and $B \in \mathcal{B}$. The supremum $SC^0(\mu, \nu)$ is thus taken over quantities

$$\|\epsilon_0 \mu(A) + \epsilon_1(\nu(B) - \mu(A)) + \epsilon_2(\nu(\Omega) - \nu(B))\|$$

or

$$\|\epsilon_0 \nu(B) + \epsilon_1(\mu(A) - \nu(B)) + \epsilon_2(\mu(\Omega) - \mu(A))\|.$$

Both of these are bounded by

$$\|\mu\| + \|\mu(\Omega)\| + \|\nu\|. \quad \square$$

Question: Does theorem 2.5 continue to hold in the case when $\mathbf{X}$ is any Banach space?

References

[1] A. Basile, K. P. S. Bhaskara Rao and R. M. Shortt, *Bounded common extensions of charges*, Proc. Amer. Math. Soc. **121** (1994), 137-143.

[2] K. P. S. Bhaskara Rao and M Bhaskara Rao, *Theory of charges*, Academic Press, London and New York, 1983.

[3] J. Diestel and J. J. Uhl, *Vector measures*, American Mathematical Society, Providence, Rhode Island, 1977.

[4] D. B. Goodner, *Projections in normed linear spaces*, Trans. Amer. Math. Soc. **69** (1950), 89–108.

[5] J. L. Kelley, *Banach spaces with the extension property*, Trans. Amer. Math. Soc. **72** (1952), 323–326.

[6] Z. Lipecki, *On common extensions of two quasi-measures*, Czechoslovak Math. J. **36** (1986), 489–494.

[7] L. Nachbin, *A theorem of the Hahn-Banach type for linear transformations*, Trans. Amer. Math. Soc.**68** (1950), 28–46

[8] K. D. Schmidt and G. Waldschaks, *Common extensions of order bounded vector measures*, Rend. Circ. Mat. Palermo Ser.**II 28**, Suppl. 117–124.