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Marek Balcerzak*, Institute of Mathematics, Łódź Technical University, al. Politechniki 11, 90-924 Łódź, Poland e-mail: `mbalce@krysia.uni.lodz.pl`

Joanna Peredko*, Institute of Mathematics, Łódź Technical University, al. Politechniki 11, 90-924 Łódź, Poland e-mail: `joannape@ck-sg.p.lodz.pl`

DESCRIPTIVE CHARACTER OF SETS OF DENSITY AND $\mathcal{I}$-DENSITY POINTS

Abstract

Let $X = [a, b]$ and $A \subset X^2$. We extend the theorem of Mauldin stating the set of $\langle x, y\rangle \in X^2$ such that y is a density point of A_x, provided that A is Borel is itself a Borel set. We prove the corresponding result if A is analytic or coanalytic and show the analogous statements in the category case.

1 Introduction

Let $X = [a, b]$. If $E \subset X$ is a Lebesgue measurable set, $\varphi(E)$ denotes the set of all density points of E. If $E \subset X$ possesses the Baire property, $\varphi_{\mathcal{I}}(E)$ denotes the set of all $\mathcal{I}$-density points, i.e., the density points in the sense of category, introduced by Wilczyński in [W]. For $A \subset X^2$ and $x \in X$, we put

$$A_x = \{y \in X : \langle x, y\rangle \in A\};$$

the so-called x-section of A. By LM_k (respectively, BP_k) we denote the class of Lebesgue measurable sets (sets with the Baire property) in $\mathbb{R}^k$ for $k = 1, 2$. For $A \subset X^2$ we put

$$D(A) = \{\langle x, y\rangle \in X^2 : A_x \in \mathrm{LM}_1 \ \&\ y \in \varphi(A_x)\};$$

$$D_{\mathcal{I}}(A) = \{\langle x, y\rangle \in X^2 : A_x \in \mathrm{BP}_1 \ \&\ y \in \varphi_{\mathcal{I}}(A_x)\}.$$

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From [S, Chap.IX, Th.11.1] it follows that the symmetric difference $A\triangle D(A)$ is of plane measure zero for each $A \subset X^2$, $A \in \mathrm{LM}_2$. The analogous statement for category is contained in [CW, Th.4]. Thus $D(A)$ (respectively, $D_{\mathcal{I}}(A)$) forms a special kind of a kernel for $A \in \mathrm{LM}_2$ ($A \in \mathrm{BP}_2$).

We set $\omega = \{0, 1, 2, \dots\}$. Let Λ be a pointclass in the sense of Moschovakis [Mo, p.19]. If Y is a given Polish space, then $\Lambda(Y)$ denotes the collection of all sets of Λ contained in Y.

We are interested in the following problem. If $A \in \Lambda(X^2)$, what is a possibly simple pointclass where $D(A)$ or $D_{\mathcal{I}}(A)$ hits? In some cases we can expect that $D(A)$ (or $D_{\mathcal{I}}(A)$) also is in $\Lambda(X^2)$. For instance, Mauldin [Ma, Th.1] proved that $D(A)$ is Borel, provided that $A \subset X^2$ is Borel. We consider the cases where Λ is the pointclass of all Borel sets, or Λ is some of the pointclasses Σ^0_α ($0 < \alpha < \omega_1$), or Λ is the pointclass of analytic sets, or Λ consists of coanalytic sets.

If Y is a metric space, $\mathcal{K}(Y)$ denotes the hyperspace of all compact subsets of Y equipped with the Vietoris topology (or, equivalently with the Hausdorff distance). For details concerning $\mathcal{K}(Y)$ we refer the reader to [Ke, pp.24–28].

2 Measure Case

In this section $X = [0, 1]$. Lebesgue measure on $\mathbb{R}$ will be denoted by λ. As it has been mentioned above, Mauldin in [Ma] proved the following theorem.

Theorem 2.1. *If $A \subset X^2$ is a Borel set, so is $D(A)$.*

Note that if $A = X \times B$, where B is Borel in X, then $D(A) = X \times \varphi(B)$, which (by Theorem 2.1) easily implies that $\varphi(B)$ is Borel. Hence one can derive the well-known fact that $\varphi(E)$ is Borel, provided that $E \subset X$ is Lebesgue measurable. Indeed, it suffices to consider a G_δ set B such that $E \subset B$, $\lambda(B \setminus E) = 0$, and keep in mind that $\varphi(E) = \varphi(B)$.

Now, we will recall the proof of Theorem 2.1 and, additionally, estimate the Borel class of $D(A)$ if the Borel class of $A \subset X^2$ is assumed.

Let $\mathbb{Q}$ denote the set of all rationals.

Lemma 2.1. *If $A \subset X^2$ and all x-sections A_x are measurable, then*

$$D(A) = \bigcap_{n\in\omega} \bigcup_{m\in\omega} \bigcap_{q\in(0,\frac{1}{m+1})\cap\mathbb{Q}} T(n, q) \tag{1}$$

where

$$T(n, q) = \{\langle x, y\rangle \in X^2 : \lambda(A_x \cap (y - q, y + q)) \geq 2q(1 - \frac{1}{n+1})\}. \tag{2}$$

Furthermore, in the definition of $T(n,q)$, the interval $(y-q, y+q)$ can be replaced by $[y-q, y+q]$, and/or $\geq$ can be replaced by $>$. Statement (1) remains true if $T(n,q)$ is replaced by $(X \times [q, 1-q]) \cap T(n,q)$.

The proof is straightforward. The last remark follows from the fact that while considering y as a density point we may assume $[y-q, y+q] \subset X$.

Theorem 2.2. *If $A \subset X^2$ is in Σ^0_α $(0 < \alpha < \omega_1)$, then $D(A)$ is in $\Pi^0_{\alpha+3}$.*

PROOF. Observe that $T(n,q)$ given by (2) is equal to

$$\bigcap_{p\in\omega} \bigcup_{s\in\mathbb{Q}} \left(\left\{ x \in X : \lambda(A_x \cap (s-q, s+q)) > 2q(1 - \frac{1}{n+1}) - \frac{1}{p+1} \right\} \right. \\ \left. \times \left\{ y \in X : |y-s| < \frac{1}{p+1} \right\} \right) \tag{3}$$

which follows from the continuity of the function $y \mapsto \lambda(A_x \cap (y-q, y+q))$. But

$$A_x \cap (s-q, s+q) = (A \cap (X \times (s-q, s+q)))_x$$

and it is known that

$$\{x \in X : \lambda((A \cap (X \times (s-q, s+q)))_x) > c\}$$

is in Σ^0_α if $c \in \mathbb{R}$ and A is in Σ^0_α [Ke, Exercise 22.25]. Now from (1) and (3) we infer that $D(A)$ is in $\Pi^0_{\alpha+3}$. $\square$

Next we observe that the analogue of Theorem 2.1 holds for analytic and coanalytic sets.

Theorem 2.3. *If $A \subset X^2$ is analytic (coanalytic), so is $D(A)$.*

We will start with a lemma and a proposition. If $E \subset Z \times W$, then $\mathrm{pr}_Z(E) = \{z \in Z : (\exists w \in W)\langle z, w\rangle \in E\}$.

Lemma 2.2. *[Ke, Th.29.27] Let Z and W be Polish spaces and $H \subset Z \times W$ be closed. If μ is a Borel probability measure on Z and for some $a \in \mathbb{R}$, $\mu(\mathrm{pr}_Z(H)) > a$, then there is a compact set $K \subset H$ such that $\mu(\mathrm{pr}_Z(K)) > a$.*

Proposition 2.1. *If $A \subset X^2$ is analytic and $h > 0, a \in \mathbb{R}$, then*

$$T = \{\langle x, y\rangle \in X^2 : \lambda(A_x \cap [y-h, y+h]) > a\}$$

is analytic.

PROOF. Observe that

$$T = \bigcup_{p\in\omega} \bigcup_{s\in\mathbb{Q}} \left(T(p,s) \times \{y \in X : |y-s| < \frac{1}{p+1} \right)$$

where

$$T(p,s) = \left\{ x \in X : \lambda(A_x \cap [s-h, s+h]) > a + \frac{1}{p+1} \right\}.$$

It suffices to show that $T(p,s)$ is analytic. So, fix $p \in \omega$ and $s \in \mathbb{Q}$. Since A is analytic, there exists a closed set $E \subset X^2 \times \omega^\omega$ such that $A = \mathrm{pr}_{X^2}(E)$. It is easy to check that for a fixed $x \in X$ we have

$$A_x \cap [s-h, s+h] = \mathrm{pr}_X \left(E_x \cap ([y-h, y+h] \times \omega^\omega) \right).$$

Obviously $E_x \cap ([s-h, s+h] \times \omega^\omega)$ is closed. Then by Lemma 2.2 we infer that

$$\begin{aligned} \lambda(A_x \cap [s-h, s+h]) > a + \frac{1}{p+1} \Leftrightarrow \\ \lambda(\mathrm{pr}_X(E_x \cap ([s-h, s+h] \times \omega^\omega))) > a + \frac{1}{p+1} \Leftrightarrow \\ \big(\exists K \in \mathcal{K}(X \times \omega^\omega)\big)\big(K \subset E_x \cap ([s-h, s+h] \times \omega^\omega) \\ \& \ \lambda(\mathrm{pr}_X(K)) > a + \frac{1}{p+1}\big). \end{aligned} \tag{4}$$

Consider the sets

$$\begin{aligned} M_1 &= \{\langle x, K\rangle \in X \times \mathcal{K}(X \times \omega^\omega) : K \subset E_x \times ([s-h, s+h] \times \omega^\omega)\}, \\ M_2 &= X \times \{K \in \mathcal{K}(X \times \omega^\omega) : \lambda(\mathrm{pr}_X(K)) > a + \frac{1}{p+1}\}. \end{aligned}$$

The set M_1 is closed since from $K \subset E_x \Leftrightarrow \{x\} \times K \subset E \cap (X \times [s-h, s+h] \times \omega^\omega)$ it follows that $M_1 = f^{-1}[W]$ where:

- the mapping $f : X \times \mathcal{K}(X \times \omega^\omega) \to \mathcal{K}(X^2 \times \omega^\omega)$ given by $f(x, K) = \{x\} \times K$ is continuous [Ke, p.27];
- the set $W = \{F \in \mathcal{K}(X^2 \times \omega^\omega) : F \subset E \cap (X \times [s-h, s+h] \times \omega^\omega)\}$ is closed.

The set M_2 is of type F_σ. Indeed, for each $c \in \mathbb{R}$, the set $S(c)$, given by $S(c) = \{F \in \mathcal{K}(X) : \lambda(F) < c\}$, can be expressed as

$$\bigcup \{V(G) : G \text{ open } \ \& \ \lambda(G) < c\}$$

where $V(G) = \{F \in \mathcal{K}(X) : F \subset G\}$ is a set from the subbasis of the Vietoris topology. Hence $S(c)$ is open, and therefore

$$\{F \in \mathcal{K}(X) : \lambda(F) > a + \frac{1}{p+1}\} = \bigcup_{n\in\omega} (\mathcal{K}(X) \setminus S(a + \frac{1}{p+1} + \frac{1}{n+1}))$$

is of type F_σ. Consequently, M_2 is of type F_σ since $\mathrm{pr}_X : \mathcal{K}(X \times \omega^\omega) \to \mathcal{K}(X)$ is continuous.

Now, from (4) it follows that the set $T(p,s)$ is the projection of a Borel set $M = M_1 \cap M_2$ on X. Thus $T(p,s)$ is analytic. $\square$

PROOF OF THEOREM 2.3. Let A be analytic. Using Lemma 2.1 we can express $D(A)$ by (1) where

$$T(n,q) = \{\langle x,y\rangle \in X^2 : \lambda(A_x \cap [y-q, y+q]) > 2q(1 - \frac{1}{n+1})\}.$$

Then the assertion follows from (1) and Proposition 2.1.

Let A be coanalytic. Using Lemma 2.1 we can express $D(A)$ by (1) where $T(n,q)$ is the set

$$(X \times [q, 1-q]) \cap \{\langle x,y\rangle \in X^2 : \lambda(A_x \cap [y-q, y+q]) \geq 2q(1 - \frac{1}{n+1})\}$$

and $[y-q, y+q] \subset X$. Thus

$$\lambda((X^2 \setminus A)_x \cap [y-q, y+q]) = 2q - \lambda(A_x \cap [y-q, y+q])$$

and $T(n,q)$ is equal to

$$(X \times [q, 1-q]) \setminus \{\langle x,y\rangle \in X^2 : \lambda((X^2 \setminus A)_x \cap [y-q, y+q]) > \frac{2q}{n+1}\}.$$

Now we apply Proposition 2.1 to the analytic set $X^2 \setminus A$ and infer that $T(n,q)$ is coanalytic. Then the assertion follows from (1). $\square$

3 Category Case

In this section, for technical reasons, we assume that $X = [-1, 1]$. Let int and cl denote the operators of interior and closure in X. Recall that a set $G \subset X$ is *regular open* if $G = \mathrm{int}(\mathrm{cl}\, F)$, and a set $F \subset X$ is *regular closed* if $F = \mathrm{cl}(\mathrm{int}\, F)$. It is well known that for each set $A \subset X$ with the Baire property there is a unique regular open G such that the symmetric difference $A \triangle G$ is meager [O, Th.4.6]. This regular open set associated with A will be denoted by A°. It is not hard to check that $(X \setminus A)^\circ = \operatorname{int}(X \setminus A^\circ)$. Let $A^\star = \operatorname{cl}(A^\circ)$. Then $A^\star$ is regular closed and $A \triangle A^\star$ is meager. From $(X \setminus A)^\circ = \operatorname{int}(X \setminus A^\circ)$ we also have $A^\star = X \setminus (X \setminus A)^\circ$. Thus $A^\star$ is a (unique) regular closed set F such that $A \triangle F$ is meager.

The σ-ideal of meager subsets of X will be denoted by $\mathcal{I}$. Let us recall the original definition of an $\mathcal{I}$-density point introduced by Wilczyński in [W]. A number $y \in X$ is called an *$\mathcal{I}$-density point* of a set $A \subset X$ with the Baire property iff for each increasing sequence $\{n_m\}_{m\in\omega}$ of positive integers there exists a subsequence $\{n_{m_p}\}_{p\in\omega}$ with the property that the equality

$$\lim_{p\to\infty} \chi_{(n_{m_p}(A-y))\cap X}(x) = 1 \tag{5}$$

holds $\mathcal{I}$-almost everywhere on X. This last part means that the set of points $x \in X$ for which (5) does not hold is meager. Set

$$c(A-y) = \{t \in \mathbb{R} :\, (\exists a \in A)\; t = c(a-y)\}$$

and $\chi_E : X \to \{0,1\}$ stands for the characteristic function of a set $E \subset X$. We say that $y \in X$ is an *$\mathcal{I}$-dispersion point* of A if it is an $\mathcal{I}$-density point of $X \setminus A$.

For our purpose we will use a more convenient version of the definition where the quantifiers $(\forall\{n_m\})$ $(\exists\{n_{m_p}\})$ do not appear and where we have even a greater number of quantifiers but they can deal with countable sets. That version derived from [CLO, Th.2.2.2(vii)] was inspired by a theorem of Łazarow [L, Th.1]. (We give it with small nonessential changes which are caused by the fact that the authors in [CLO] consider subsets of $\mathbb{R}$ rather than of X, and Th.2.2.2(vii) in [CLO] is formulated for an $\mathcal{I}$-dispersion point.) Namely, $y \in X$ is an $\mathcal{I}$-density point of $A \subset X$ with the Baire property iff for every nonempty interval $(a,b) \subset X$ there exist $\varepsilon > 0$ and $m \in \omega$ such that for every $n \geq m$ there is an interval $(c,d) \subset (a,b)$ with the property that

$$|d-c| > \varepsilon \text{ and } (c,d) \cap n\Big((X \setminus A) - y\Big)^\circ = \emptyset. \tag{6}$$

By the relationships between $(\ )^\circ$ and $(\ )^\star$, we easily deduce that $(c,d) \cap n((X \setminus A) - y)^\circ = \emptyset$ can be equivalently written as $(c,d) \subset n(A^\star - y)$. Also, the above statement will not be destroyed if we consider $[c,d] \subset (a,b)$ and $[c,d] \subset n(A^\star - y)$. (Note here that $n(A^\star - y)$ is closed.) Denote by $\mathcal{M}$ the family of all nonempty open intervals with rational endpoints contained in X. Observe that in the above statement we may assume $(a,b),(c,d) \in \mathcal{M}$ and we may replace ε by $\frac{1}{k+1}$ where $k \in \omega$. After these modifications we get the following assertion.

Lemma 3.1. *A number $y \in X$ is an $\mathcal{I}$-density point of a set $A \subset X$ with the Baire property iff for every $(a,b) \in \mathcal{M}$ there exist numbers $k, m \in \omega$ such that for every $n \geq m$ there is an interval $(c,d) \in \mathcal{M}$ with the properties that*

$$[c,d] \subset (a,b) \ \ \& \ \ d-c > \frac{1}{k+1} \ \ \& \ \ [c,d] \subset n(A^\star - y).$$

If $A \subset X$, then let $\Delta(A)$ denote the set of all points $x \in X$ such that $U \cap A$ is nonmeager for each open neighborhood U of x. Following [Ku, p.83], $\Delta(A)$ is called the set of points where A is of the second category.

Lemma 3.2. *If $A \subset X^2$ is Borel of class Σ^0_α, where $0 < \alpha < \omega_1$, (is analytic, coanalytic), then the set*

$$\{\langle x,y\rangle \in X^2 : \, y \in \Delta(A_x)\}$$

is Borel of the class $\Pi^0_{\alpha+1}$ (is analytic, coanalytic).

Proof. Let $\{U_n\}_{n\in\omega}$ be a fixed base of open sets in X. For $\langle x,y\rangle \in X^2$ we have, $y \in \Delta(A_x)$ iff

$$\big(\forall n \in \omega\big) \ \big(y \notin U_n \vee \big((X \times U_n) \cap A\big)_x \notin \mathcal{I}\big).$$

Since (see [Ke, Exercises 22.22 and 32.4, Th. 29.22]) the set

$$\{x \in X : \, \big((X \times U_n) \cap A\big)_x \notin \mathcal{I}\}$$

is of class Σ^0_α (is analytic, coanalytic), provided that A is of class Σ^0_α (is analytic, coanalytic). Therefore we get the assertion. □

We are now in a position to prove the following category analogue of Theorems 2.2 and 2.3.

Theorem 3.1. *If $A \subset X^2$ is in Σ^0_α, $0 < \alpha < \omega_1$, (is analytic, coanalytic), then $D_{\mathcal{I}}(A)$ is in $\Pi^0_{\alpha+5}$ (is analytic, coanalytic).*

Proof. From the definition of $B^\star = \mathrm{cl}(B^\circ)$, for a set $B \subset X$ with the Baire property, it easily follows that $B^\star = \Delta(B)$. Thus we may write $\Delta(A)$ instead of $A^\star$ in Lemma 3.1. Consequently, for our set A, the condition $\langle x,y\rangle \in D_{\mathcal{I}}(A)$ is equivalent to

$$\begin{gathered}\big(\forall (a,b) \in \mathcal{M}\big)\big(\exists k, m \in \omega\big)\big(\forall n \geq m\big)\big(\exists (c,d) \in \mathcal{M}\big) \\ a<c<d<b \ \ \& \ \ d-c>\frac{1}{k+1} \ \ \& \ \ [c,d] \subset n(\Delta(A_x) - y).\end{gathered} \tag{7}$$

It suffices to study the nature of the set

$$F = \{\langle x, y\rangle \in X^2 : [c, d] \subset n(\Delta(A_x) - y)\}$$

if c, d and n are fixed. Note that

$$[c, d] \subset n(\Delta(A_x) - y)$$

is equivalent to

$$\big(\forall t \in \mathbb{Q}\big)\big(t \notin [y + \frac{c}{n}, y + \frac{d}{n}] \vee t \in \Delta(A_x)\big).$$

From Lemma 3.2 it follows that the set

$$H = \big\{\langle x, y, t\rangle \in X^3 : \, t \notin [y + \frac{c}{n}, y + \frac{d}{n}] \vee t \in \Delta(A_x)\big\}$$

is Borel of class $\Pi^0_{\alpha+1}$ (respectively, is analytic, coanalytic). Since

$$F = \bigcap_{t \in \mathbb{Q} \cap X} H^t, \text{ where } H^t = \{\langle x, y\rangle \in X^2 : \, \langle x, y, t\rangle \in H\},$$

F is also of class $\Pi^0_{\alpha+1}$ (respectively, is analytic, coanalytic). Finally we consider the quantifiers in (7) to get the assertion. $\square$

As in the remark following Theorem 2.1 we can deduce from the "Borel part" of Theorem 3.1 that $\varphi_{\mathcal{I}}(E)$ is Borel, provided that $E \subset X$ possesses the Baire property. This was first proved in [JLW].

Remarks. 1. We leave open the question whether our evaluation of a Borel class for $D(A)$ and $D_{\mathcal{I}}(A)$ is sharp, i.e. we do not know for which $\alpha < \omega_1$ there exists $A \in \Sigma^0_\alpha$ such that $D(A)$ is not in $\Sigma^0_{\alpha+3}$ ($D_{\mathcal{I}}(A)$ is not in $\Sigma^0_{\alpha+5}$). Observe that if A is open, then $D(A)$ and $D_{\mathcal{I}}(A)$ are open.
2. Note that there exists an analytic set $A \subset X^2$ for which $D(A)$ and $D_{\mathcal{I}}(A)$ are not coanalytic (thus not Borel). It is enough to put $A = B \times X$ where B is analytic and not coanalytic in X.
3. Two facts concerning section properties of Σ^0_α-sets for measure and category that we use in the proofs of Theorem 2.2 and Lemma 3.2 [Ke, Exercises 22.25, 22.22] are attributed to Montgomery (Amer. J. Math., **56** (1934)) and Novikov (J. Math. Tokyo, (1), (1951)). We were so informed by one of the referees. A nice proof of the both facts can easily be reconstructed from an abstract idea given in [G, Th.2.2].

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