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CHARACTERIZATIONS OF $\mathbf{VB^*G \cap (N)}$

Abstract

We introduce the condition (PAC^*) that is a slight modification of the condition (PAC) of Sarkhel and Kar [10]. The main result is Theorem 4: *A function* $f : [a,b] \to \mathbb{R}$ *is* $VB^*G \cap (N)$ *on a subset* E *of* $[a,b]$ *if and only if* $f \in (PAC^*)$ *on* E. Consequently, *the set* $\{f : [a,b] \to \mathbb{R} : f \in VB^*G \cap (N)$ *on* $E\}$ *is an algebra, whenever* E *is a subset of* $[a,b]$. Using Theorem 1, we find seven characterizations of $VB^*G \cap (N)$ on a Lebesgue measurable set (Theorem 5). We also give fifteen characterizations of the class of AC^*G functions on a closed set E, that are continuous at each point of E (Theorem 6). In the last two sections, using Thomson's outer measure $\mathcal{S}_o$-μ_f, we characterize a $VB^*G \cap (N)$ function f on a Lebesgue measurable set (Theorem 9). As a consequence we obtain that: *A function* $f : [a,b] \to \mathbb{R}$ *is* AC^*G *on a closed subset* E *of* $[a,b]$ *and continuous at each point of* E *if and only if* $\mathcal{S}_o$-$\mu_f(Z) = 0$ *whenever* Z *is a null subset of* E (Theorem 10).

1 Introduction

The purpose of this paper is to give some characterizations of $VB^*G \cap (N)$ on an arbitrary real set.

In [10], Sarkhel and Kar introduced the class (PAC) (Definition 4), showing that it is equivalent to the class $[VBG] \cap (N)$ on a closed set. In [5] we show that the class $(PAC)G$ (generalized (PAC)) is equivalent to $VBG \cap (N)$ on an arbitrary set. In this paper, we introduce the condition (PAC^*), that is a slight modification of (PAC). (We replace expressions like $|f(a) - f(b)|$ by the oscillation of the function f on the interval $[a,b]$.) Clearly the class (PAC^*) is contained in (PAC). Thus we obtain the main result: *A function*

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$f : [a, b] \to \mathbb{R}$ *is* $VB^*G \cap (N)$ *on a subset* E *of* $[a, b]$ *if and only if* $f \in (PAC^*)$ *on* E (see Theorem 4). Consequently, *the set* $\{f : [a, b] \to \mathbb{R} : f \in VB^*G \cap (N)$ *on* $E\}$ *is an algebra, whenever* E *is a subset of* $[a, b]$ (Corollary 3).

In Theorem 1 we obtain the following result. *A function* $f : [a, b] \to \mathbb{R}$ *is* VB^*G *on a Lebesgue measurable subset* E *of* $[a, b]$ *if and only if it is so on any null subset of* E.

As a consequence of Theorems 1 and 4, we find seven characterizations of $VB^*G \cap (N)$ on a Lebesgue measurable set (Theorem 5).

In Theorem 2 we obtain the following result. *A function* $f : [a, b] \to \mathbb{R}$ *is* AC^*G *on a Lebesgue measurable subset* E *of* $[a, b]$ *if and only if it is so on any null subset of* E.

Using Theorems 1 and 2, we find fifteen characterizations of the class of AC^*G functions on a closed set E, that are continuous at each point of E (Theorem 6).

In the last two sections we study the relationship between Thomson's outer measure $\mathcal{S}_o$-μ_f and $VB^*G \cap (N)$ on a Lebesgue measurable set. In Theorem 8 we obtain that: *If* $f : [a, b] \to \mathbb{R}$ *is* VB^*G *and continuous at each point of a set* $A \subseteq [a, b]$*, then* $m^*(f(A)) = 0$ *if and only if* $\mathcal{S}_o$-$\mu_f(A) = 0$. Using this theorem we obtain again that *the set* $\{f : [a, b] \to \mathbb{R} : f \in VB^*G \cap (N)$ *on* $E\}$ *is an algebra, whenever* E *is a subset of* $[a, b]$ (Corollary 7), as well as the following characterization. *A function* $f : [a, b] \to \mathbb{R}$ *is* $VB^*G \cap (N)$ *on a Lebesgue measurable subset* E *of* $[a, b]$ *if and only if there is a countable subset* E_1 *of* E *such that* $\mathcal{S}_o$-$\mu_f(Z) = 0$ *whenever* Z *is a null subset of* $E \setminus E_1$ (Theorem 9).

As a consequence of Theorem 9, it follows that: *A function* $f : [a, b] \to \mathbb{R}$ *is* AC^*G *on a closed subset* E *of* $[a, b]$ *and continuous at each point of* E *if and only if* $\mathcal{S}_o$-$\mu_f(Z) = 0$ *whenever* Z *is a null subset of* E (Theorem 10). Using different techniques, this result was obtained before in [3], [4], and rediscovered by Bongiorno, Di Piazza and Skvortsov in [1].

2 Preliminaries

We denote by $m^*(X)$ the outer measure of the set X and by $m(A)$ the Lebesgue measure of A, whenever $A \subseteq \mathbb{R}$ is Lebesgue measurable. For the definitions of VB, AC, AC^*, VB^* and Lusin's condition (N), see [8].

Definition 1. Let E be a real compact set, $c = \inf(E)$, $d = \sup(E)$ and $f : E \to \mathbb{R}$. Let $\{(c_k, d_k)\}_k$ be the intervals contiguous to E and let $f_E : [c, d] \to \mathbb{R}$, $f_E(x) = f(x)$ if $x \in E$, f_E is linear on each $[c_k, d_k]$.

Definition 2. ([9]). A sequence $\{E_n\}$ of sets whose union is E is called an E- form with parts E_n; if, in addition, each part E_n is closed in E (i.e., $E_n = P_n \cap E$, where P_n is a closed set; so $P_n = \overline{E}_n$), then the E-form is said to be closed. An expanding E-form is called an E-chain.

Lemma 1. ([10]). *For every closed E-form $\{E_n\}$, there is a closed E-chain $\{Q_n\}$ such that $Q_n = \cup_{k\le n} Q_{kn}$, where $Q_{kn} \subseteq Q_{km} \subseteq E_k$ for all k and for $m \ge n \ge k$, and $d(Q_{in}, Q_{jn}) \ge 1/n$ for $i \ne j$. (Here d denotes the usual metric distance.)*

Definition 3. Let $f : [a, b] \to \mathbb{R}$, $E \subseteq [a, b]$, and $c = \inf E$, $d = \sup E$.

- Put $V^*(f; E) = \sup\{\sum_{i=1}^{n} \mathcal{O}(f; [a_i, b_i]) : \{[a_i, b_i]\}_{i=1}^{n}$ is a finite set of nonoverlapping closed intervals with endpoints in $E\}$ ([8], p. 228).
- f is said to be VB^* on E if $V^*(f; E) < +\infty$ ([8], p. 228).
- f is said to be VB^*G (respectively AC^*G, VBG, ACG) on E if there is an E-form $\{E_n\}$ such that f is VB^* (respectively AC^*, VB, AC) on each E_n. f is said to be $[VB^*G]$ (respectively $[AC^*G]$, $[VBG]$, $[ACG]$) on E if the E-form is closed. Note that AC^*G and ACG here differ from the definitions given in [8], because f is not supposed to be continuous.
- **(Krzyzewski)** f is said to be increasing* on E if $f(x) \le f(y)$ whenever $c \le x < y \le d$ and $\{x, y\} \cap E \ne \emptyset$. f is said to be monotone* on E if either f or $-f$ is increasing* on E ([4], p. 47).

Definition 4. Let $Q \subseteq \mathbb{R}$, $f : Q \to \mathbb{R}$, $E \subseteq Q$ and $r > 0$. Then:

- (**Sarkhel, Kar**, [10]) $V(f; E; r) = \sup\{\sum_{i=1}^{n} |f(b_i) - f(a_i)| : \{[a_i, b_i]\}_{i=1}^{m}$ is a finite set of nonoverlapping closed intervals with the endpoints in E and $\sum_{i=1}^{m}(b_i - a_i) < r\}$
- (**Sarkhel, Kar**, [10]) $V(f; E; 0) = \inf_{r>0} V(f; E; r)$.
- (**Sarkhel, Kar**, [10]) $PV(f; E) = \inf\{\sup_n V(f; E_n; 0) : \{E_n\}$ is an E-chain$\}$.
- (**Sarkhel, Kar**, [10]) f is said to be (PAC) on E if $PV(f; E) = 0$.
- $[PV](f; E) = \inf\{\sum_n V(f; E_n; 0) : \{E_n\}$ is a closed E-form$\}$.
- f is said to be $[PAC]$ on E if $[PV](f; E) = 0$.

3 A Characterization of VB*G on a Lebesgue Measurable Set

Lemma 2. *Let $f : [a,b] \to \mathbb{R}$ and let E be a closed subset of $[a,b]$. The following assertions are equivalent.*

(i) f is VB^*G on E.

(ii) f is VB^*G on Z, whenever Z is a null subset of E.

PROOF. See Theorem 1.9.1, (i) of [4] and Theorem 7.1 of [8], p. 229. □

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ and let E be a Lebesgue measurable subset of $[a,b]$. The following assertions are equivalent.*

(i) f is VB^*G on E.

(ii) f is VB^*G on Z, whenever Z is a null subset of E.

PROOF. (i) $\Rightarrow$ (ii) This part is always true, even if E is not assumed to be Lebesgue measurable.

(ii) $\Rightarrow$ (i) Since E is Lebesgue measurable, there exists an increasing sequence of closed sets $\{Q_n\}$ such that $Z = E \setminus (\cup_{n=1}^{\infty} Q_n)$ is of measure zero. Clearly $f \in VB^*G$ on Z. By Lemma 2, f is VB^*G on each Q_n. It follows that $f \in VB^*G$ on E. □

4 A Characterization of AC*G on a Lebesgue Measurable Set

Lemma 3. *Let $f : [a,b] \to \mathbb{R}$ and let E be a closed subset of $[a,b]$. If $f_{|E}$ is continuous, then the following assertions are equivalent.*

(i) f is AC^*G on E.

(ii) f is AC^*G on Z, whenever Z is a null subset of E.

PROOF. See Theorem 1.9.1, (iii) of [4]. □

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ and let E be a Lebesgue measurable subset of $[a,b]$. Then the following assertions are equivalent.*

(i) f is AC^*G on E.

(ii) f is AC^*G on Z, whenever Z is a null subset of E.

PROOF. (i) $\Rightarrow$ (ii) This is always true (without Lebesgue measurability).
(ii) $\Rightarrow$ (i) By Theorem 1, clearly f is VB^*G on E. So f is Lebesgue measurable on E. By Lusin's Theorem ([8], p. 72), it follows that there is an increasing sequence $\{E_n\}$ of closed sets such that $Z = E \setminus (\cup_{n=1}^{\infty} E_n)$ is a null set and $f_{|E_n}$ is continuous. Clearly $f \in AC^*G$ on Z. By Lemma 3, $f \in AC^*G$ on each E_n. Therefore f is AC^*G on E. $\square$

5 The Conditions (PAC*), [PAC*], PAC*

Definition 5. Let $f : [a, b] \to \mathbb{R}$, $E \subseteq [a, b]$ and $r > 0$. Put

- $V^*(f; E; r) = \sup\{\sum_{i=1}^{n} \mathcal{O}(f; [a_i, b_i]) : \{[a_i, b_i]\}_{i=1}^{n}$ is a finite set of nonoverlapping closed intervals with endpoints in E and $\sum_{i=1}^{n}(b_i - a_i) < r\}$;
- $V^*(f; E; 0) = \inf_{r>0} V^*(f; E; r)$;
- $PV^*(f; E) = \inf\{\sup_n V^*(f; E_n; 0) : \{E_n\}$ is an E-chain$\}$;
- $[PV^*](f; E) = \inf\{\sum_n V^*(f; E_n; 0) : \{E_n\}$ is a closed E-form$\}$;
- $\mu_f^*(E) = \inf\{\sum_n V^*(f; E_n; 0) : \{E_n\}$ is an E-form$\}$;
- $V^{**}(f; E; r) = \sup\{\sum_{i=1}^{n} |f(b_i) - f(a_i)| : \{[a_i, b_i]\}_{i=1}^{n}$ is a finite set of nonoverlapping closed intervals with $\sum_{i=1}^{n}(b_i - a_i) < r$ such that each $[a_i, b_i]$ has at least one endpoint in $E\}$;
- $V^{**}(f; E; 0) = \inf_{r>0} V^{**}(f; E; r)$;
- $PV^{**}(f; E) = \inf\{\sup_n V^{**}(f; E_n; 0) : \{E_n\}$ is an E-chain$\}$;
- $[PV^{**}](f; E) = \inf\{\sum_n V^{**}(f; E_n; 0) : \{E_n\}$ is a closed E-form$\}$;
- $\mu_f^{**}(E) = \inf\{\sum_n V^{**}(f; E_n; 0) : \{E_n\}$ is an E-form$\}$;

Definition 6. Let $f : [a, b] \to \mathbb{R}$, $E \subseteq [a, b]$.

- f is said to be (PAC^*) on E if $PV^*(f; E) = 0$;
- f is said to be $[PAC^*]$ on E if $[PV^*](f; E) = 0$;
- f is said to be PAC^* on E if $\mu_f^*(E) = 0$.
- f is said to be (PAC^{**}) on E if $PV^{**}(f; E) = 0$;
- f is said to be $[PAC^{**}]$ on E if $[PV^{**}](f; E) = 0$;

- f is said to be PAC^{**} on E if $\mu_f^{**}(E) = 0$.

Lemma 4. *With the notations of Definition 5, we have each of the following assertions.*

(i) $V^*(f;E;r) \le 2V^{**}(f;E;r)$.

(ii) $V^*(f;E;0) \le 2V^{**}(f;E;0)$.

(iii) $PV^*(f;E) \le 2PV^{**}(f;E)$.

(iv) $[PV^*](f;E) \le 2[PV^{**}](f;E)$.

(v) $\mu_f^*(E) \le 2\mu_f^{**}(E)$.

Moreover, if f is continuous at each point of $\overline{E}$, then

(vi) $V^{**}(f;E;0) \le V^*(f;E;0)$;

(vii) $PV^{**}(f;E) \le PV^*(f;E)$;

(viii) $[PV^{**}](f;E) \le [PV^*](f;E)$;

(ix) $\mu_f^{**}(E) \le \mu_f^*(E)$.

PROOF. (i) For any finite set of non-overlapping closed intervals $\{[a_i,b_i]\}_{i=1}^n$ with the endpoints in E and $\sum_{i=1}^n (b_i - a_i) < r$,

$$\sum_{i=1}^n \mathcal{O}\big(f;[a_i,b_i]\big) \le \sum_{i=1}^n 2 \sup_{x\in[a_i,b_i]} \big|f(x) - f(a_i)\big| \le 2V^{**}(f;E;r).$$

(ii),(iii),(iv),(v) follow by (i).

(vi) Since f is continuous at each point of $\overline{E}$, it is easily seen that $V^*(f;E;r) = V^*(f;\overline{E};r)$ for all $r > 0$. Let $V^*(f;E;r) < \infty$. (Otherwise there is nothing to prove.) Then for $\epsilon > 0$ there is an $r > 0$ such that

$$V^*(f;\overline{E};r) = V^*(f;E;r) < V^*(f;E;0) + \epsilon\,. \tag{1}$$

Let $(c_1,d_1),(c_2,d_2),\dots$ be the intervals contiguous to $\overline{E}$, if any, and let $c_0 = \inf E, d_0 = \sup E$. Choose a positive integer k_0 such that $\sum_{k>k_0}(d_k - c_k) < r$. By (1), $\sum_{k>k_0} \mathcal{O}\big(f;[c_k,d_k]\big) \le V^*\big(f;\overline{E};r\big) < \infty$. Hence there is a positive integer $n_o > k_0$ such that

$$\sum_{k>n_0} \mathcal{O}\big(f;[c_k,d_k]\big) < \epsilon\,. \tag{2}$$

By continuity of f at the points of $\overline{E}$, there is a $\delta \in (0,r)$ such that

$$\sum_{k=0}^{n_0}\big(\mathcal{O}(f;[c_k-\delta,c_k+\delta])+O(f;[d_k-\delta,d_k+\delta])\big)<\epsilon\,. \tag{3}$$

Now, let $\{[a_i,b_i]\}_{i=1}^m$ be a finite set of non-overlapping closed intervals such that each $[a_i,b_i]$ has at least one endpoint in E and $\sum_{i=1}^m(b_i-a_i)<\delta$.

If $a_i,b_i\in\overline{E}$ retain $[a_i,b_i]$. If $a_i\in\overline{E}$ and $b_i>d_0$, split $[a_i,b_i]$ into $[a_i,d_0]$ and $[d_0,b_i]$, and use $|f(b_i)-f(a_i)|\le\mathcal{O}(f;[a_i,d_0])+\mathcal{O}(f;[d_0,b_i])$. If $a_i\in\overline{E}$ and $c_k<b_i<d_k$ for some $k\ge 1$ split $[a_i,b_i]$ into $[a_i,c_k]$ and $[c_k,b_i]$, and use

$$\big|f(a_i)-f(b_i)\big|\le\begin{cases}\mathcal{O}\big(f;[a_i,c_k]\big)+O\big(f;[c_k,c_k+\delta]\big) & \text{if } k\le n_0\,,\\ \mathcal{O}\big(f;[a_i,c_k]\big)+\mathcal{O}\big(f;[c_k,d_k]\big) & \text{if } k>n_0\,.\end{cases}$$

If $b_i\in\overline{E}$ and $a_i<c_0$, split $[a_i,b_i]$ into $[c_0,b_i]$ and $[a_i,c_0]$, and use $|f(b_i)-f(a_i)|\le\mathcal{O}(f;[c_0,b_i])+\mathcal{O}(f;[c_0-\delta,c_0])$. If $b_i\in\overline{E}$ and $c_k<a_i<d_k$ for some $k\ge 1$, split $[a_i,b_i]$ into $[d_k,b_i]$ and $[a_i,d_k]$ and use

$$\big|f(a_i)-f(b_i)\big|\le\begin{cases}\mathcal{O}\big(f;[d_k,b_i]\big)+O\big(f;[d_k-\delta,d_k]\big) & \text{if } k\le n_0\,,\\ \mathcal{O}\big(f;[d-k,b_i]\big)+\mathcal{O}\big(f;[c_k,d_k]\big) & \text{if } k>n_0\,.\end{cases}$$

Since $\sum_{i=1}^m(b_i-a_i)<\delta<r$, by (2) and (3), it follows that

$$\sum_{i=1}^m\big|f(b_i)-f(a_i)\big|<V^*(f;\overline{E};r)+2\epsilon+2\epsilon\,.$$

Hence, by (1), $V^{**}(f;E;\delta)<V^*(f;\overline{E};0)+5\epsilon$. Since $\epsilon>0$ is arbitrary, we obtain that $V^{**}(f;E;0)<V^*(f;\overline{E};0)$.

(vii), (viii), (ix) follow by (vi). □

Corollary 1. *Let $f:[a,b]\to\mathbb{R}$, $E\subseteq[a,b]$.*

(i) If f is (PAC^{**}) (respectively $[PAC^{**}]$; PAC^{**}) on E, then f is (PAC^*) (respectively $[PAC^*]$; PAC^*) on E, and f is continuous at each point of the set E.

(ii) If f is (PAC^*) (respectively $[PAC^*]$; PAC^*) on E and f is continuous at each point of $\overline{E}$, then f is (PAC^{**}) (respectively $[PAC^{**}]$; PAC^{**}) on the set E.

PROOF. (i) For the first part see Lemma 4, (iii), (iv), (v). Let $x_0 \in E$ and suppose for example that $f \in (PAC^{**})$ on E (the other two cases are similar). For $\epsilon > 0$, there exist a sequence of positive numbers $\{r_n\}$ and an E-chain $\{E_n\}$ such that

$$V^{**}(f; E_n; r_n) < \epsilon \quad \text{for all } n\,.$$

Let n_o be a positive integer such that $x_0 \in E_{n_o}$. Clearly for $x \in [a, b]$,

$$\big|f(x) - f(x_0)\big| < V^{**}(f; E_{n_o}; r_{n_o}) < \epsilon \quad \text{whenever } x \in (x_0 - r_{n_o}, x_0 + r_{n_o})\,.$$

Therefore f is continuous at x_0.

(ii) See Lemma 4, (vii), (viii), (ix). □

6 Characterizations of $\mathbf{VB^*G} \cap \mathbf{(N)}$ on a Real Set

Theorem 3. *Let $f, g : [a, b] \to \mathbb{R}$, $E \subseteq [a, b]$, $\alpha, \beta \in \mathbb{R}$. The following hold.*

(i) $PV^*(\alpha f + \beta g; E) \le |\alpha| PV^*(f; E) + |\beta| PV^*(g; E)$ *. Moreover, if* $c = \inf E$*,* $d = \sup E$ *and* $M = \sup_{x \in [c,d]}\{|f(x)|, |g(x)|\} < +\infty$*, then*

$$PV^*(f \cdot g; E) \le M(PV^*(f; E) + PV^*(g; E))$$

and

$$V^*(f \cdot g; E) \le M(V^*(f; E) + V^*(g; E))\,.$$

(ii) If $PV^*(g; E) = 0$*, then* $PV^*(f + g; E) = PV^*(f; E)$ *.*

(iii) $PV(f; E) \le PV^*(f; E)$ *;*

(iv) (**Sarkhel and Kar** [10]) *If* $m^*(E) = 0$*, then* $m^*(f(E)) \le PV(f; E)$ *.*

(v) If $PV^*(f; E) < +\infty$*, then* $f \in [VB^*G]$ *on* E*.*

(vi) $PV^*(f; E) \le \sum_n PV^*(f; E_n)$ *whenever* $\{E_n\}$ *is a closed E-form.*

(vii) $\mu^*_f(E) \le [PV^*](f; E)$ *.*

(viii) $PV^*(f; E) \le [PV^*](f; E)$ *.*

(ix) $[PV^*](f; E) \le \sum_n [PV^*](f; E_n)$ *whenever* $\{E_n\}$ *is a closed E-form.*

(x) $\mu^*_f : \mathcal{P}(E) \to [0, +\infty]$ *is a metric outer measure.*

(xi) $PV^{**}(\alpha f + \beta g; E) \le |\alpha| PV^{**}(f; E) + |\beta| PV^{**}(g; E)$ *. Moreover, if* $c = \inf E$*,* $d = \sup E$ *and* $M = \sup_{x \in [c,d]}\{|f(x)|, |g(x)|\} < +\infty$*, then*

$$PV^{**}(f \cdot g; E) \le M(PV^{**}(f; E) + PV^{**}(g; E))\,.$$

(xii) *If* $PV^{**}(g;E) = 0$, *then* $PV^{**}(f+g;E) = PV^{**}(f;E)$;

(xiii) $\mu_f^{**}(E) \le [PV^{**}](f;E)$;

(xiv) $PV^{**}(f;E) \le [PV^{**}](f;E)$;

(xv) $PV^{**}(f;\cdot) : \mathcal{P}(E) \to [0,+\infty]$ *is a metric outer measure.*

(xvi) $[PV^{**}](f;E) \le \sum_n [PV^{**}](f;E_n)$ *whenever* $\{E_n\}$ *is a closed E-form.*

(xvii) $\mu_f^{**} : \mathcal{P}(E) \to [0,+\infty]$ *is a metric outer measure.*

Proof. (i) We shall use the technique of Theorem 3.1, (i) of [10]. For $\epsilon > 0$ there exist two E-chains $\{A_n\}$, $\{B_n\}$ and two sequences of positive numbers $\{r_n^{'}\}$, $\{r_n^{''}\}$ such that for all n we have

$$V^*(f;A_n;r_n^{'}) \le PV^*(f;E) + \epsilon \quad \text{and} \quad V^*(g;B_n;r_n^{''}) \le PV^*(g;E) + \epsilon\,.$$

Let $E_n = A_n \cap B_n$ and $r_n = \min\{r_n^{'}, r_n^{''}\}$. Then $\{E_n\}$ is an E-chain and

$$\begin{aligned} V^*(\alpha f + \beta g; E_n; 0) &\le V^*(\alpha f + \beta g; E_n; r_n) \\ &\le |\alpha| V^*(f;E_n;r_n) + |\beta| V^*(g;E_n;r_n) \\ &\le |\alpha| V^*(f;A_n;r_n^{'}) + |\beta| V^*(g;B_n;r_n^{''}) \\ &\le |\alpha| PV^*(f;E) + |\beta| PV^*(g;E) + \epsilon(|\alpha| + |\beta|)\,. \end{aligned}$$

Therefore

$$PV^*(\alpha f + \beta g; E) \le |\alpha| PV^*(f;E) + |\beta| PV^*(g;E)\,.$$

We prove the second part. Let $a^{'}, b^{'} \in E$, $a^{'} \le x < y \le b^{'}$. Then

$$\big|f(y)g(y) - f(x)g(x)\big| = \big|g(y)(f(y) - f(x)) + f(x)(g(y) - g(x))\big| \le$$

$$\le M\big(|f(y) - f(x)| + |g(y) - g(x)|\big) \le M \cdot \big(\mathcal{O}(f;[a^{'},b^{'}]) + \mathcal{O}(g;[a^{'},b^{'}])\big)\,.$$

Therefore $\mathcal{O}(f \cdot g;[a^{'},b^{'}]) \le M\big(\mathcal{O}(f;[a^{'},b^{'}]) + \mathcal{O}(g;[a^{'},b^{'}]\big)$. It follows that

$$V^*(f \cdot g; E_n; 0) \le V^*(f \cdot g; E_n; r_n) \le M\big(V^*(f;E_n,r_n) + V^*(g;E_n;r_n)\big) \le$$

$$\le M\big(V^*(f;E_n;r_n^{'}) + V^*(g;E_n;r_n^{''})\big) \le M\big(PV^*(f;E) + PV^*(g;E) + 2\epsilon\big)\,.$$

Therefore

$$PV^*(f \cdot g; E) \le M\big(PV^*(f;E) + PV^*(g;E)\big)\,.$$

Clearly

$$V^*(f\cdot g;E)\le M\bigl(V^*(f;E)+V^*(g;E)\bigr)\,.$$

(ii) We shall use the technique of Theorem 3.1, (ii) of [10]. Since $PV^*(g;E) = 0$ implies that $PV^*(-g;E)=0$, we have

$$\begin{aligned}PV^*(f;E)&=PV^*(f+g-g;E)\le PV^*(f+g;E)+PV^*(-g;E)\\&=PV^*(f+g;E)\le PV^*(f;E)+PV^*(g;E)=PV^*(f;E)\,.\end{aligned}$$

Therefore $PV^*(f;E)=PV^*(f+g;E)$.

(iii) This is obvious.

(iv) See [10].

(v) There exist an E-chain $\{E_n\}$ and a sequence $\{r_n\}$ of positive numbers, such that $V^*(f;E_n;r_n) < PV^*(f;E)+1$, for all n. For every integer k, let $E_{nk}=E_n\cap\left[k\frac{r_n}{2},(k+1)\frac{r_n}{2}\right]$. Then $f\in VB^*$ on each E_{nk}. By Theorem 7.1 of [8] (p. 229), $f\in VB^*$ on $\overline{E_{nk}}$; so $f\in VB^*$ on $E\cap\overline{E_{nk}}$. It follows that $f\in[VB^*G]$ on E.

(vi) We shall use the technique of Theorem 3.4 of [10]. Let $\epsilon>0$. For every k there exist an E_k-chain $\{E_{kn}\}$ and a sequence of positive numbers $\{r_{kn}\}$, such that $V^*(f;E_{kn};r_{kn})\le PV^*(f;E_k)+\frac{\epsilon}{2^k}$ for all n. Now, considering the closed E-chain $\{Q_n\}$ given by Lemma 1 corresponding to the closed E-form $\{E_n\}$, and setting $H_n=\cup_{k\le n}(Q_{kn}\cap E_{kn})$, it is easy to see that $\{H_n\}$ is an E-chain. Let $r_n=\min\left\{\frac{1}{n},r_{1n},r_{2n},\dots,r_{nn}\right\}$. If $\{[a_p,b_p]\}$ is a finite set of nonoverlapping closed intervals with the endpoints in H_m, m fixed, with $\sum(b_p-a_p)<r_m$, then, since $d(Q_{im},Q_{jm})\ge 1/m$ for $i\ne j$, the endpoints of an interval $[a_p,b_p]$ must both belong to precisely one of the sets $Q_{km}\cap E_{km}$, $k=1,2,\dots,m$, and so we clearly have

$$\sum_p\mathcal{O}(f;[a_p,b_p])\le\sum_{k\le m}V^*(f;Q_{km}\cap E_{km};r_m)\le$$

$$\le\sum_{k\le m}V^*(f;E_{km};r_{km})\le\sum_{k\le m}\left(PV^*(f;E_k)+\frac{\epsilon}{2^k}\right).$$

Hence $V^*(f;H_m;r_m)\le\sum_n PV^*(f;E_n)+\epsilon$ for all m. Therefore $PV^*(f;E)\le\sum_n PV^*(f;E_n)+\epsilon$. But ϵ is arbitrary; so

$$PV^*(f;E)\le\sum_n PV^*(f;E_n)\,.$$

(vii) This is obvious.

(viii) Suppose that $[PV^*](f;E) = M < +\infty$. (If $M = +\infty$, there is nothing to prove.) Then for $\epsilon > 0$, it follows that there exist a closed E-form $\{E_n\}$ and a sequence of positive numbers $\{r_n\}$ such that $\sum_n V^*(f;E_n;r_n) < M + \epsilon$. By Lemma 1, there exists a closed E-chain $\{Q_n\}$ such that $Q_n = \cup_{k=1}^n Q_{kn}$, $Q_{kn} \subseteq Q_{km} \subseteq E_k$ for all k and $m \geq n \geq k$, and

$$d(Q_{in}, Q_{jn}) \geq \frac{1}{n} \quad \text{for } i \neq j\,. \tag{4}$$

Let $\rho_n = \min\left\{r_1, r_2, \ldots, r_n, \frac{1}{2n}\right\}$. Let $\{[a_p, b_p]\}_{p=1}^q$ be a finite set of nonoverlapping closed intervals with the endpoints in Q_n and $\sum_{p=1}^q (b_p - a_p) < \rho_n$. By (4), both endpoints of an interval $[a_p, b_p]$ belong to some Q_{in}. It follows that

$$\sum_{p=1}^q \mathcal{O}(f;[a_p,b_p]) \leq \sum_{i=1}^n V^*(f;Q_{in};\rho_n) \leq \sum_{i=1}^n V^*(f;E_i;r_i) < M + \epsilon \;\text{ for all } n\,.$$

Therefore $PV^*(f;E) \leq M$.

(ix) We may suppose that $\sum_n [PV^*](f;E_n) < +\infty$. (Otherwise there is nothing to prove.) Let $\epsilon > 0$. Then for every positive integer k, there exist a closed E_k-form $\{E_{kn}\}$ and a sequence of positive numbers $\{r_{kn}\}$ such that

$$\sum_n V^*(f;E_{kn};r_{kn}) < [PV^*](f;E_k) + \frac{\epsilon}{2^k}\,.$$

But $\{E_{kn}\}_{k,n}$ is a closed E-form, and

$$\sum_k \sum_n V^*(f;E_{kn};r_{kn}) < \epsilon + \sum_k [PV^*](f;E_k)\,.$$

It follows that $[PV^*](f;E) \leq \epsilon + \sum_k [PV^*](f;E_k)$. Since ϵ is arbitrary, we obtain that $[PV^*](f;E) \leq \sum_k [PV^*](f;E_k)$.

(x) Clearly $\mu_f^*(\emptyset) = 0$ and μ_f^* is an increasing set-function, i.e., $\mu_f^*(A) \leq \mu_f(B)$ whenever $A \subseteq B \subseteq E$. As in (ix) we obtain that

$$\mu_f^*(\cup_n E_n) \leq \sum_n \mu_f^*(E_n)\,. \tag{5}$$

Let E_1, E_2 be such that $d(E_1, E_2) = r > 0$. Suppose that $\mu_f^*(E_1 \cup E_2) < +\infty$. (If $\mu_f^*(E_1 \cup E_2) = +\infty$, by (5), it follows that $\mu_f^*(E_1 \cup E_2) = \mu_f^*(E_1) + \mu_f^*(E_2)$.) For $\epsilon > 0$ there exist an $E_1 \cup E_2$-form $\{P_n\}$ and a sequence of positive numbers $\{r_n\}$ such that

$$\sum_n V^*(f;P_n;r_n) < \mu_f^*(E_1 \cup E_2) + \epsilon\,.$$

Let $P_{1n} = E_1 \cap P_n$, $P_{2n} = E_2 \cap P_n$ and $\rho_n = \min\{r_n, r\}$. Fix some n and let $\{[a_i^{'}, b_i^{'}]\}$ be a finite set of nonoverlapping closed intervals with the endpoints in P_{1n} and $\sum(b_i^{'} - a_i^{'}) < \rho_n/2$. Let $\{[a_j^{''}, b_j^{''}]\}$ be a finite set of nonoverlapping closed intervals with the endpoints in P_{2n} and $\sum(b_j^{''} - a_j^{''}) < \rho_n/2$. Suppose that there exists $a_{ij} \in [a_i^{'}, b_i^{'}] \cap [a_j^{''}, b_j^{''}]$. Then

$$d(a_i^{'}, a_{ij}) < \frac{\rho_n}{2} \quad \text{and} \quad d(a_{ij}, b_j^{''}) < \frac{\rho_n}{2}\,;$$

so $d(a_i^{'}, b_j^{''}) < \rho_n \le r$, a contradiction. Therefore $[a_i^{'}, b_i^{'}] \cap [a_j^{''}, b_j^{''}] = \emptyset$. Hence

$$\sum \bigl|f(b_i^{'}) - f(a_i^{'})\bigr| + \sum \bigl|f(b_j^{''}) - f(a_j^{''})\bigr| \le V^*(f; P_n; \rho_n)\,.$$

It follows that $V^*\Bigl(f; P_{1n}; \frac{\rho_n}{2}\Bigr) + V^*\Bigl(f; P_{2n}; \frac{\rho_n}{2}\Bigr) \le V^*(f; P_n; \rho_n)$. Then

$$\begin{aligned}\mu_f^*(E_1) + \mu_f^*(E_2) &\le \sum_n V^*\Bigl(f; P_{1n}; \frac{\rho_n}{2}\Bigr) + \sum_n V^*\Bigl(f; P_{2n}; \frac{\rho_n}{2}\Bigr) \\ &\le \sum_n V^*(f; P_n; \rho_n) \le \sum_n V^*(f; P_n; r_n) \le \mu_f^*(E_1 \cup E_2) + \epsilon\,.\end{aligned}$$

Since ϵ is arbitrary and μ_f^* is an outer measure, we obtain that $\mu_f^*(E_1 \cup E_2) = \mu_f^*(E_1) + \mu_f^*(E_2)$.

(xi) The proof is similar to (i).

(xii) The proof is similar to (ii).

(xiii) This is obvious.

(xiv) Suppose that $[PV^{**}](f; E) = M < +\infty$. (If $M = +\infty$, there is nothing to prove.) For $\epsilon > 0$ there exist a closed E-form $\{E_n\}$ and a sequence of positive numbers $\{r_n\}$ such that $\sum_n V^{**}(f; E_n; r_n) < M + \epsilon$. Let $Q_n = \cup_{i=1}^n E_i$. Then $\{Q_n\}$ is a closed E-chain. Fix some n and let $\rho_n = \min\{r_1, r_2, \ldots, r_n\}$. Let $\{[a_p, b_p]\}_{p=1}^q$ be a finite set of nonoverlapping closed intervals having at least one endpoint in Q_n and $\sum_{q=1}^p (b_p - a_p) < \rho_n$. It follows that for each n,

$$\sum_{p=1}^q |f(b_p) - f(a_p)| \le \sum_{i=1}^n V^{**}(f; E_i; \rho_n) \le \sum_{i=1}^n V^{**}(f; E_i; r_i) < M + \epsilon\,.$$

Therefore $PV^{**}(f; E) \le M$.

(xv) Clearly $PV^{**}(f; \emptyset) = 0$ and $PV^{**}(f; \cdot)$ is an increasing set function. Let $\{E_k\}$ be an E-form and $\epsilon > 0$. For every k there exist an E_k-chain $\{E_{kn}\}$ and a sequence of positive numbers $\{r_{kn}\}$ such that

$$V^{**}(f; E_{kn}; r_{kn}) \le PV^{**}(f; E_k) + \frac{\epsilon}{2^k}\,, \quad \text{for all } n\,.$$

If $H_n = \cup_{k=1}^n E_{kn}$, then $\{H_n\}$ is an E-chain. Let $r_n = \min\{r_{1n}, \dots, r_{nn}\}$. Fix some m and let $\{[a_p, b_p]\}_{p=1}^q$ be a finite set of nonoverlapping closed intervals having at least one endpoint in H_m and $\sum_{q=1}^p (b_p - a_p) < r_m$. Then we have

$$\begin{aligned}\sum_{p=1}^q |f(b_p) - f(a_p)| &\le \sum_{k=1}^m V^{**}(f; E_{km}; r_m) \le \sum_{k=1}^m V^{**}(f; E_{km}; r_{km}) \\ &\le \sum_{k=1}^m \left(PV^{**}(f; E_k) + \frac{\epsilon}{2^k} \right) \le \sum_{k=1}^\infty PV^{**}(f; E_k) + \epsilon\,.\end{aligned}$$

Therefore $PV^{**}(f; E) \le \sum_n PV^{**}(f; E_n) + \epsilon$. Since ϵ is arbitrary, we obtain that $PV^{**}(f; E) \le \sum_n PV^{**}(f; E_n)$. That $PV^{**}(f; E_1 \cup E_2) = PV^{**}(f; E_1) + PV^{**}(f; E_2)$ whenever $d(E_1, E_2) = r > 0$, follows as in the proof of (x).

(xvi) The proof is similar to (ix).

(xvii) The proof is similar to (x). □

Lemma 5. *Let $f : [a, b] \to \mathbb{R}$ and $E \subseteq [a, b]$, $c = \inf E$, $d = \sup E$. If $f \in VB^*$ on E, then there exists a function $F : [a, b] \to \mathbb{R}$ having the following properties.*

(i) $F_{|\overline{E}} = f$ *and* $F \in VB$ *on* $[a, b]$.

(ii) $\mathcal{O}(f; [\alpha, \beta]) = \mathcal{O}(F; [\alpha, \beta])$ *whenever* $\alpha, \beta \in \overline{E}$, $\alpha < \beta$.

PROOF. Let $\{(c_k, d_k)\}_k$ be the set of intervals contiguous to $\overline{E}$. For every positive integer k, let $c_k < \alpha_k < \beta_k < d_k$, and

$$M_k = \sup_{x \in [c_k, d_k]} f(x)\,, \qquad m_k = \inf_{x \in [c_k, d_k]} f(x)\,.$$

Define $F : [a, b] \to \mathbb{R}$ by

$$F(x) = \begin{cases} f(c) & \text{if } x \in [a, c] \\ f(d) & \text{if } x \in [d, b] \\ f(x) & \text{if } x \in \overline{E} \\ M_k & \text{if } x = \alpha_k \\ m_k & \text{if } x = \beta_k \\ \textit{linear} & \text{on each } [c_k, \alpha_k],\ [\alpha_k, \beta_k],\ [\beta_k, d_k] \end{cases}$$

(i) Clearly $F_{|\overline{E}} = f$. Let $\Delta : a = x_0 < x_1 < \ldots < x_n = b$ be a partition of $[a,b]$. If, for example, $(x_{i-1}, x_i) \cap \overline{E} \neq \emptyset$, then let $x^*_{i-1} = \inf(x_{i-1}, x_i) \cap \overline{E}$ and $y^*_{i-1} = \sup(x_{i-1}, x_i) \cap \overline{E}$. This means that there exists a new partition Δ_1 of $[a,b]$, finer than Δ, such that for each component interval I of Δ_1 we have $\mathrm{int}(I) \cap \overline{E} = \emptyset$, or both endpoints of I belong to $\overline{E}$. Therefore

$$V_\Delta(F) := \sum_{i=1}^{n} |F(x_{i-1}) - F(x_i)| \leq V_{\Delta_1}(F) \leq V(F;\overline{E}) + \sum_k V(F;[c_k,d_k])\,.$$

By Theorem 7.1 of [8] (p. 229), f is VB^* on $\overline{E}$; so VB on $\overline{E}$. But

$$V(F;[c_k,d_k]) \leq 3\mathcal{O}(F;[c_k,d_k]) = 3\mathcal{O}(f;[c_k,d_k])$$

and $\sum_k \mathcal{O}(f;[c_k,d_k]) < +\infty$ (see Theorem 8.5 of [8], p. 232). Therefore $V(F;[a,b]) < +\infty$. Hence $F \in VB$ on $[a,b]$.

(ii) Let $\alpha < \beta$, $\alpha, \beta \in \overline{E}$. Then $\sup_{x\in[\alpha,\beta]} f(x) = \sup\{\sup(f([\alpha,\beta] \cap \overline{E}), M_k : k$ is a positive integer such that $(c_k,d_k) \subset (\alpha,\beta)\} = \sup_{x\in[\alpha,\beta]} F(x)$. Analogously, it follows that $\inf_{x\in[\alpha,\beta]} f(x) = \inf_{x\in[\alpha,\beta]} F(x)$. Thus we obtain that $\mathcal{O}(f;[\alpha,\beta]) = \mathcal{O}(F;[\alpha,\beta])$. □

Lemma 6. *Let $f : [a,b] \to \mathbb{R}$, and let E be a closed subset of $[a,b]$, $x_0 \in E$. If $f \in VB^*$ on E, then for every $\epsilon > 0$ there is a $\delta > 0$ such that*

$$V^*(f;E \cap (x_0, x_0+\delta)) < \epsilon \quad \text{and} \quad V^*(f;E \cap (x_0-\delta, x_0)) < \epsilon\,.$$

Moreover, if $\{I_n\}_n$ is a sequence of abutting closed intervals with $\cup I_n = (x_0 - \delta, x_0)$ or $\cup I_n = (x_0, x_0+\delta)$, then $\sum_n V^*(f;E\cap I_n) \leq \epsilon$.

Proof. Let $F : [a,b] \to \mathbb{R}$ be the function given by Lemma 5, and define $V_F : [a,b] \to \mathbb{R}$ by

$$V_F(x) = \begin{cases} 0 & \text{if } x = a \\ V(F;[a,x]) & \text{if } x \in (a,b] \end{cases}$$

Clearly V_F is an increasing function on $[a,b]$. It follows that there exist $V_F(x_0-) = \ell^-$ and $V_F(x_0+) = \ell^+$, and that they are both finite. Then there is a $\delta > 0$ such that

$$V_F\big((x_0-\delta, x_0)\big) \subset (\ell^- - \epsilon, \ell^-) \quad \text{and} \quad V_F\big((x_0, x_0+\delta)\big) \subset (\ell^+, \ell^+ + \epsilon)\,.$$

Let $\alpha, \beta \in (x_0, x_0+\delta) \cap E$. By Lemma 5, (ii). We have

$$\mathcal{O}\big(f;[\alpha,\beta]\big) = \mathcal{O}\big(F;[\alpha,\beta]\big) \leq V\big(F;[\alpha,\beta]\big) = V_F(\beta) - V_F(\alpha)\,.$$

Therefore $V^*(f;E\cap(x_0,x_0+\delta))\le \ell^+ +\epsilon-\ell^+=\epsilon$. Clearly $\sum_n V^*(f;E\cap I_n)\le \sum_n V(F;I_n)=\sum_n(V_F(\beta_n)-V_F(\alpha_n))<\epsilon$, where $\{I_n\}_n=\{[\alpha_n,\beta_n]\}_n$ are as in the hypothesis. □

Lemma 7. *Let $f:[a,b]\to\mathbb{R}$, $E\subseteq[a,b]$. If $f\in AC^*G$ on E, then $\mu_f^*(E)=0$.*

PROOF. Since $f\in AC^*G$ on E, there exists an E-form $\{E_n\}$ such that f is AC^* on each E_n. Let $\epsilon>0$. For $\epsilon/2^n$, let $r_n>0$ be given by the fact that $f\in AC^*$ on E_n. Then $V^*(f;E_n;r_n)<\epsilon/2^n$. Hence

$$\mu_f^*(E)\le\sum_n V^*(f;E_n;0)\le\sum_n V^*(f;E_n;r_n)<\epsilon\,.$$

It follows that $\mu_f^*(E)=0$. □

Lemma 8. *Let $f:[a,b]\to\mathbb{R}$, $E\subseteq[a,b]$, $m^*(f(E))=0$. If there exists an E-form $\{E_n\}$ such that f is monotone* on each E_n, then $\mu_f^*(E)=0$.*

PROOF. Clearly $m^*(f(E_n))=0$ for each n. We may suppose without loss of generality that f is increasing* on each E_n. Let $\epsilon>0$. Then there exists an open set $G_n=\cup_{i=1}^{\infty}(\alpha_{ni},\beta_{ni})$ such that $f(E_n)\subset G_n$ and $m(G_n)<\epsilon/2^n$. Let $E_{ni}=\{x\in E_n: f(x)\in(\alpha_{ni},\beta_{ni})\}$. For $\alpha,\beta\in E_{ni}$, $\alpha<\beta$, we have $\mathcal{O}(f;[\alpha,\beta])=f(\beta)-f(\alpha)$. It follows that $V^*(f;E_{ni})\le\beta_{ni}-\alpha_{ni}$. Hence

$$\mu_f^*(E)\le\sum_n V^*(f;E_n;0)\le\sum_n\sum_i(\beta_{ni}-\alpha_{ni})<\epsilon\,.$$

Therefore $\mu_f^*(E)=0$. □

Lemma 9. *Let $f:[a,b]\to\mathbb{R}$, $f\in VB$ on $[a,b]$. Consider the curve*

$$C: X(t)=t;\quad Y(t)=f(t),\quad t\in[a,b]$$

and let $Z=\{x\in[a,b]: f^{'}(x)$ does not exist (finite or infinite)$\}$. Let $S:[a,b]\to\mathbb{R}$, where $S(x)$ is the length of the curve C on the interval $[a,x]$. Then $m^*(S(Z))=0$.

PROOF. Let $C_f=\{x\in[a,b]: f$ is continuous at $x\}$. Then $[a,b]\setminus C_f$ is countable (see [7], p. 219). Let $N=Z\cap C_f$. Then $m^*(S(N))=0$ (see [8], pp. 125–126). It follows that $m^*(S(Z))=0$. □

Lemma 10. *Let $f:[a,b]\to\mathbb{R}$, $f\in VB^*$ on $[a,b]$. Let $Z=\{x\in[a,b]: f^{'}$ does not exist, finite or infinite$\}$. Then $\mu_f^*(Z)=0$.*

PROOF. Let S be the function from Lemma 9. Then $m^*(S(Z)) = 0$. Let $\epsilon > 0$ and $G = \cup_{i=1}^{\infty}(\alpha_i, \beta_i)$, with $\{(\alpha_i, \beta_i)\}_i$ a sequence of nonoverlapping open intervals, such that $S(Z) \subset G$, $m(G) < \epsilon$ and $S(Z) \cap (\alpha_i, \beta_i) \neq \emptyset$. Let $Z_i = \{x \in Z : S(x) \in (\alpha_i, \beta_i)\}$. For $a \le \alpha < \beta \le b$ we have that $\mathcal{O}(f; [\alpha, \beta]) \le S(\beta) - S(\alpha)$ (because S is increasing). It follows that $V^*(f; Z_i) \le \beta_i - \alpha_i$. Therefore $\mu_f^*(Z) \le \sum_i V^*(f; Z_i; 0) \le \sum_i V^*(f; Z_i) \le \sum_i(\beta_i - \alpha_i) < \epsilon$. Since ϵ is arbitrary we obtain that $\mu_f^*(Z) = 0$. □

Lemma 11 (Bruckner). ([2], pp. 196–197). *Let $f : [a, b] \to \mathbb{R}$, $E \subseteq [a, b]$. If $f \in VB^*G$ on E, then there exists a countable set $E_1 \subseteq E$ such that f is continuous at each point of $E \setminus E_1$.*

Lemma 12. *Let $f : [a, b] \to \mathbb{R}$, $E \subseteq [a, b]$. If $f \in VB^* \cap (N)$ on E, then $\mu_f^*(E) = 0$.*

PROOF. Let $F : [a, b] \to \mathbb{R}$ be the function from Lemma 5. By Lemma 5, (ii) we have that $\mu_f^*(E) = \mu_F^*(E)$. Let $A = \{x \in [a, b] : F'(x)$ exists and is finite$\}$. By Lemma 7, $\mu_F^*(A) = 0$. Hence $\mu_F^*(A \cap E) = 0$. Let $B = \{x \in E : F'(x) = \pm\infty\}$. Clearly $m^*(F(B)) = 0$ and there exists a B-form $\{B_n\}$ such that F is monotone* on each B_n (see the technique of [8], p. 235). By Lemma 8, $\mu_F^*(B) = 0$. Let $C = \{x \in [a, b] : F'(x)$ does not exist, finite or infinite$\}$. It follows that $\mu_F^*(C) = 0$ (see Lemma 10). Hence $\mu_F^*(C \cap E) = 0$. It follows that $\mu_F^*(E) = 0$ (see Theorem 3, (x)). □

Lemma 13. *Let $f : [a, b] \to \mathbb{R}$, $E \subseteq [a, b]$ and $D = \{x \in \overline{E} : f$ is not continuous at $x\}$. If $f \in VB^*$ on E, then:*

(i) D is a countable set;

(ii) $V^*(f; Q; r) \le V^*(f; E; r)$ whenever Q is a closed subset of $\overline{E} \setminus D$ and $r > 0$.

PROOF. By Theorem 7.1 of [8] (p. 229), $f \in VB^*$ on $\overline{E}$.

(i) This follows by Lemma 11.

(ii) Let $\{[a_i, b_i]\}_{i=1}^m$ be a finite set of nonoverlapping closed intervals with the endpoints in Q and $\sum_{i=1}^m(b_i - a_i) < r$. Since f is continuous at each point of Q, for $\epsilon > 0$, there exists $\{[\alpha_i, \beta_i]\}_{i=1}^m$ a finite set of nonoverlapping closed intervals, with the endpoints in E and $\sum_{i=1}^m(\beta_i - \alpha_i) < r$, such that

$$\mathcal{O}(f; I_i') < \frac{\epsilon}{4m} \quad \text{and} \quad \mathcal{O}(f; I_i'') < \frac{\epsilon}{4m},$$

where I_i' is the closed interval with the endpoints a_i, α_i, and I_i'' is the closed interval with the endpoints b_i, β_i. We have four situations.

If $[a_i, b_i] \subseteq [\alpha_i, \beta_i]$, then $\mathcal{O}(f; [a_i, b_i]) \le \mathcal{O}(f; [\alpha_i, \beta_i])$.

If $[\alpha_i, \beta_i] \subset [a_i, b_i]$, then $\mathcal{O}(f; [a_i, b_i]) \le \mathcal{O}(f; [a_i, \alpha_i]) + \mathcal{O}(f; [\alpha_i, \beta_i]) + \mathcal{O}(f; [\beta_i, b_i]) < \mathcal{O}(f; [\alpha_i, \beta_i]) + \frac{\epsilon}{2m}$.

If $a_i < \alpha_i < b_i < \beta_i$, then $\mathcal{O}(f; [a_i, b_i]) \le \mathcal{O}(f; [a_i, \beta_i]) \le \mathcal{O}(f; [a_i, \alpha_i]) + \mathcal{O}(f; [\alpha_i, \beta_i]) < \mathcal{O}(f; [\alpha_i, \beta_i]) + \frac{\epsilon}{4m}$.

If $\alpha_i < a_i < \beta_i < b_i$, then $\mathcal{O}(f; [a_i, b_i]) \le \mathcal{O}(f; [\alpha_i, b_i]) \le \mathcal{O}(f; [\alpha_i, \beta_i]) + \mathcal{O}(f; [\beta_i, b_i]) < \mathcal{O}(f; [\alpha_i, \beta_i]) + \frac{\epsilon}{4m}$.

It follows that $\sum_{i=1}^m \mathcal{O}(f; [a_i, b_i]) < \frac{\epsilon}{2} + \sum_{i=1}^m \mathcal{O}(f; [\alpha_i, \beta_i]) < \frac{\epsilon}{2} + V^*(f; E; r)$. Therefore $V^*(f; Q; r) \le \frac{\epsilon}{2} + V^*(f; E; r)$. Since ϵ is arbitrary, we obtain that $V^*(f; Q; r) \le V^*(f; E; r)$. ☐

Lemma 14. *Let $f : [a, b] \to \mathbb{R}$, $E \subseteq [a, b]$. If $f \in VB^*$ on E and $\mu_f^*(E) = 0$, then $[PV^*](f; E) = 0$. Hence $PV^*(f; E) = 0$.*

PROOF. Let $\epsilon > 0$. Then there exist an E-form $\{E_n\}$ and a sequence of positive numbers $\{r_n\}$ such that $\sum_n V^*(f; E_n; r_n) < \frac{\epsilon}{2}$ (because $\mu_f^*(E) = 0$). Since f is VB^* on E, it follows that f is VB^* on $\overline{E}$. Let $D = \{d_1, d_2, \ldots\}$ be the set of all discontinuity points of f in $\overline{E}$. (That D is a countable set follows by Lemma 13.) By Lemma 6, there exist $I_n = (p_n, d_n)$ and $J_n = (d_n, q_n)$ such that if $I_n = \cup_k I_{nk}$, $J_n = \cup_k J_{nk}$ and $\{I_{nk}\}_k$, $\{J_{nk}\}_k$ are nonoverlapping closed intervals, then

$$\sum_k V^*(f; \overline{E} \cap I_{nk}) + \sum_k V^*(f; \overline{E} \cap J_{nk}) < \frac{\epsilon}{2^{n+1}}\,.$$

Let $Q = \overline{E} \setminus (\cup_n (p_n, q_n))$. Then Q is a compact set and f is continuous at each point of Q. Let $Q_n = Q \cap \overline{E}_n$. By Lemma 13, (ii), it follows that $V^*(f; Q_n; r) \le V^*(f; E_n; r)$. Then

$$\{E \cap Q_n\}_n \cup \{E \cap I_{nk}\}_{n,k} \cup \{E \cap J_{nk}\}_{n,k} \cup \{d_n\}_n$$

is a closed E-form. It follows that

$$\sum_n V^*(f; Q_n; r_n) + \sum_n \sum_k V^*(f; E \cap I_{nk}) + \sum_n \sum_k V^*(f; E \cap J_{nk}) < \epsilon\,.$$

Since $V^*(f; \{d_n\}) = 0$ for each n and ϵ is arbitrary, we obtain that $[PV^*](f; E) = 0$. That $PV^*(f; E) = 0$ follows by Theorem 3, (viii). ☐

Corollary 2. *Let $f : [a, b] \to \mathbb{R}$, $E \subseteq [a, b]$. If $f \in VB^* \cap (N)$ on E, then $[PV^*](f; E) = 0$.*

PROOF. By Lemma 12, $f \in VB^*$ on E and $\mu_f^*(E) = 0$. Now by Lemma 14 it follows that $[PV^*](f; E) = 0$. □

Lemma 15. *Let $f : [a, b] \to \mathbb{R}$, $E \subseteq [a, b]$. If $\mu_f^*(E) < +\infty$, then $f \in VB^*G$ on E.*

PROOF. Since $\mu_f^*(E) < +\infty$, there exist an E-form $\{E_n\}$ and a sequence $\{r_n\}$ of positive numbers such that $\sum_n V^*(f; E_n; r_n) < \mu_f^*(E) + 1$. It follows that $V^*(f; E_n; r_n) < \mu_f^*(E) + 1$. Consequently, $f \in VB^*$ on E_{nk}, where

$$E_{nk} = E_n \cap \left[k\frac{r_n}{2}, (k+1)\frac{r_n}{2}\right], \quad k = 0, \pm 1, \pm 2, \pm 3, \dots .$$

It follows that $f \in VB^*G$ on E. □

Theorem 4 (Main Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$, $E \subseteq [a, b]$. The following assertions are equivalent.*

(i) $f \in VB^*G \cap (N)$ on E.

(ii) $f \in [PAC^*]$ on E.

(iii) $f \in PAC^*$ on E.

(iv) $f \in (PAC^*)$ on E.

PROOF. (i) ⇒ (ii) By Theorem 7.1 of [8] (p. 229), $f \in [VB^*G] \cap (N)$ on E. Then there exists a closed E-form $\{E_n\}$ such that $f \in VB^* \cap (N)$ on each E_n. By Corollary 2, $f \in [PAC^*]$ on each E_n. By Theorem 3, (ix), it follows that $f \in [PAC^*]$ on E.

(ii) ⇒ (iii) See Theorem 3, (vii).

(iii) ⇒ (ii) By Lemma 15, $f \in VB^*G = [VB^*G]$ on E. Then there is a closed E-form $\{E_n\}$ such that $f \in VB^*$ on each E_n. But $\mu_f^*(E_n) = 0$. By Lemma 14 we obtain that $[PV^*](f; E_n) = 0$. Now by Theorem 3, (ix) we have that $[PV^*](f; E) = 0$. Hence $f \in [PAC^*]$ on E.

(ii) ⇒ (iv) See Theorem 3, (viii).

(iv) ⇒ (i) By Theorem 3, (v), $f \in VB^*G$ on E, and by Theorem 3, (iii) and (iv), we obtain that $f \in (N)$ on E. □

Corollary 3. *Let $E \subseteq [a, b]$ and $\mathcal{A} = \{f : [a, b] \to \mathbb{R} : f \in VB^*G \cap (N)$ on $E\}$. Then $\mathcal{A}$ is an algebra.*

PROOF. Let $f, g \in \mathcal{A}$, $\alpha, \beta \in \mathbb{R}$. By Theorem 4, (i), (iv) we obtain that $f, g \in (PAC^*)$ on E. Hence $PV^*(f; E) = PV^*(g; E) = 0$. By Theorem 3, (i), $PV^*(\alpha f + \beta g; E) = 0$; so $\alpha f + \beta g \in (PAC^*) = VB^*G \cap (N)$ (see Theorem 4, (i), (iv)). It follows that $\mathcal{A}$ is a real linear space. Let $\{E_n\}_n$ be an E-form such that $f, g \in VB^* \cap (N)$ on each E_n. But $f, g \in VB^*$ on $\overline{E}_n$; so f and g are bounded on each $[c_n, d_n]$, where $c_n = \inf E_n$, $d_n = \sup E_n$. By Theorem 4, (i), (iv), we have that $f, g \in (PAC^*)$ on E_n. By Theorem 3, (i), $PV^*(f{\cdot}g; E_n) = 0$. Hence $f \cdot g \in (PAC^*)$ on each E_n and $f \cdot g \in VB^*$ on E_n. Again by Theorem 4, (i), (iv), it follows that $f \cdot g \in (N)$ on each E_n; so $f \cdot g \in VB^*G \cap (N)$ on E. □

7 Characterizations of $\mathbf{VB^*G \cap (N)}$ on a Lebesgue Measurable Set

Theorem 5. *Let $f : [a, b] \to \mathbb{R}$ and let E be a Lebesgue measurable subset of $[a, b]$. The following assertions are equivalent.*

(i) $f \in VB^*G \cap (N)$ *on* E.

(ii) $f \in [PAC^*]$ *on* E.

(iii) $f \in PAC^*$ *on* E.

(iv) $f \in (PAC^*)$ *on* E.

(v) $f \in VB^*G \cap (N)$ *on* Z*, whenever* Z *is a null subset of* E.

(vi) $f \in [PAC^*]$ *on* Z*, whenever* Z *is a null subset of* E.

(vii) $f \in PAC^*$ *on* Z*, whenever* Z *is a null subset of* E.

(viii) $f \in (PAC^*)$ *on* Z*, whenever* Z *is a null subset of* E.

PROOF. By Theorem 4, we obtain that (i) ⇔ (ii) ⇔ (iii) ⇔ (iv) and (v) ⇔ (vi) ⇔ (vii) ⇔ (viii). For (i) ⇔ (v) see Theorem 1. □

Theorem 6. *Let $f : [a, b] \to \mathbb{R}$ and let E be a closed subset of $[a, b]$. The following assertions are equivalent.*

(i) $f \in AC^*G$ *on* E *and* f *is continuous at each point of* E.

(ii) $f \in VB^*G \cap (N)$ *on* E *and* f *is continuous at each point of* E.

(iii) $f \in (PAC^*)$ *on* E *and* f *is continuous at each point of* E.

(iv) $f \in [PAC^*]$ *on* E *and* f *is continuous at each point of* E*.*

(v) $f \in PAC^*$ *on* E *and* f *is continuous at each point of* E*.*

(vi) $f \in (PAC^{**})$ *on* E*.*

(vii) $f \in [PAC^{**}]$ *on* E*.*

(viii) $f \in PAC^{**}$ *on* E*.*

(ix) $f \in AC^*G$ *on* Z *whenever* Z *is a null subset of* E *and* f *is continuous at each point of* E*.*

(x) $f \in VB^*G \cap (N)$ *on* Z*, whenever* Z *is a null subset of* E *and* f *is continuous at each point of* E*.*

(xi) $f \in [PAC^*]$ *on* Z*, whenever* Z *is a null subset of* E *and* f *is continuous at each point of* E*.*

(xii) $f \in PAC^*$ *on* Z*, whenever* Z *is a null subset of* E *and* f *is continuous at each point of* E*.*

(xiii) $f \in (PAC^*)$ *on* Z*, whenever* Z *is a null subset of* E *and* f *is continuous at each point of* E*.*

(xiv) f *is* $[PAC^{**}]$ *on* Z*, whenever* Z *is a null subset of* E*.*

(xv) f *is* PAC^{**} *on* Z*, whenever* Z *is a null subset of* E*.*

(xvi) f *is* (PAC^{**}) *on* Z*, whenever* Z *is a null subset of* E*.*

PROOF. (i) $\Leftrightarrow$ (ii) follows by Theorem 8.8 of [8] (p. 233). (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) follow from Theorem 4. (iii) $\Leftrightarrow$ (vi), (iv) $\Leftrightarrow$ (vii) and (v) $\Leftrightarrow$ (viii) follow from Corollary 1. (i) $\Leftrightarrow$ (ix) follows by Lemma 3. (ii) $\Leftrightarrow$ (x) $\Leftrightarrow$ (xi) $\Leftrightarrow$ (xii) $\Leftrightarrow$ (xiii) follow by Theorem 5, (i), (v), (vi), (vii), (viii). (xiv) $\Leftrightarrow$ (xi), (xv) $\Leftrightarrow$ (xii) and (xvi) $\Leftrightarrow$ (xiii) follow from Corollary 1. $\square$

8 Thomson's Outer Measure $\mathcal{S}_{\mathbf{o}}$-$\mu_{\mathbf{f}}$

Definition 7. ([11], pp. 99–101). Let $E \subseteq [a,b]$ and let $\delta : E \to (0,+\infty)$.

- $\beta_\delta^o[E] = \{([y,z];x) : x \in [y,z] \subset (x-\delta(x), x+\delta(x)) \text{ and } x \in E\}$ and $\mathcal{A}_\delta^o = \{[y,z] : ([y,z];x) \in \beta_\delta^o[E]\}$.
- $\beta_\delta[E] = \{([y,z];x) : x \in E \cap \{y,z\} \text{ and } [y,z] \subset (x-\delta(x), x+\delta(x))\}$ and $\mathcal{A}_\delta = \{[y,z] : ([y,z];x) \in \beta_\delta[E]\}$.

- A family $\mathcal{A}$ of intervals is said to be a $\mathcal{S}_o$-cover of E if there exists a $\delta : E \to (0, +\infty)$ such that $\mathcal{A} \supseteq \mathcal{A}_\delta$. Clearly $\mathcal{A}_\delta$ is a $\mathcal{S}_o$-cover of E [12].

Definition 8 (Thomson)**.** [12]. Let $f : [a,b] \to \mathbb{R}$, $E \subseteq [a,b]$. Let $\mathcal{A}$ be a $\mathcal{S}_o$-cover of E and $\delta : E \to (0, +\infty)$. Put

- $V^*(f;\mathcal{A}) = \sup\{\sum_{i=1}^{n} |f(b_i) - f(a_i)| : \{[a_i, b_i]\}_{i=1}^{n}$ is a finite set of nonoverlapping closed intervals belonging to $\mathcal{A}\}$;
- $\mathcal{S}_o$-$\mu_f(E) = \inf\{V^*(f;\mathcal{A}) : \mathcal{A}$ is a $\mathcal{S}_o$-cover$\}$;
- $V^*_\delta(f;E) = V^*(f;\mathcal{A}_\delta)$ and $V^{*,o}_\delta(f;E) = V^*(f;\mathcal{A}^o_\delta)$;

Proposition 1. *Let $f : [a,b] \to \mathbb{R}$, $E \subseteq [a,b]$ and $\delta : E \to (0, +\infty)$. Then $V^*_\delta(f;E) = V^*(f;\mathcal{A}_\delta) = V^*(f;\mathcal{A}^0_\delta) = V^{*,0}_\delta(f;E)$ and $\mathcal{S}_0$-$\mu_f(E) = \inf_\delta V^*(f;\mathcal{A}_\delta)$.*

PROOF. By definitions, we clearly have

$$V^*_\delta(f;E) = V^*(f;\mathcal{A}_\delta) \le V^*(f;\mathcal{A}^0_\delta) = V^{*,0}_\delta(f;E)\,.$$

Let $\{[a_i, b_i]\}_{i=1}^{m}$ be any finite set of non-overlapping closed intervals with $[a_i, b_i] \in \mathcal{A}^0_\delta$. Then there exists $x_i \in E$ such that $x_i \in [a_i, b_i] \subset \big(x_i - \delta(x_i), x_i + \delta(x_i)\big)$. Hence $[a_i, x_i], [x_i, b_i] \in \mathcal{A}_\delta$. Then

$$\sum_{i=1}^{m} \big|f(b_i) - f(a_i)\big| \le \sum_{i=1}^{m} \big|f(x_i) - f(a_i)\big| + \sum_{i=1}^{m} \big|f(b_i) - f(x_i)\big| \le V^*(f;\mathcal{A}_\delta)\,.$$

Hence $V^*(f;\mathcal{A}^0_\delta) \le V^*(f;\mathcal{A}_\delta)$, as remained to be shown.

The second part is obvious from definitions. □

Definition 9. ([4], p. 89). Let $f : [a,b] \to \mathbb{R}$ and $E \subseteq [a,b]$. f is said to be Y_{D^o} (respectively Y_D) on E if for every null subset Z of E and for every $\epsilon > 0$, there is a $\delta : Z \to (0, +\infty)$ such that $\sum_{i=1}^{n} |f(d_i) - f(c_i)| < \epsilon$, whenever $\{[c_i, d_i]\}_{i=1}^{n}$ is a finite set of nonoverlapping closed intervals, with $([c_i, d_i], t_i) \in \beta^o_\delta[Z]$ (respectively $([c_i, d_i], t_i) \in \beta_\delta[Z]$).

The condition Y_{D^o} was introduced by P. Y. Lee in [6]. He called it "the strong Lusin condition" (abbreviated SLC).

Corollary 4. *Let $f : [a,b] \to \mathbb{R}$ and $E \subseteq [a,b]$. The following assertions are equivalent.*

(i) $f \in Y_D$ *on* E.

(ii) $f \in Y_{D^o}$ *on* E.

(iii) $\mathcal{S}_o$-$\mu_f(Z) = 0$ *whenever* Z *is a null subset of* E *(i.e.* $\mathcal{S}_o$-μ_f *is absolutely continuous on* E*).*

PROOF. See Proposition 1. □

Theorem 7. *Let* $f, g : [a,b] \to \mathbb{R}$, $E \subseteq [a,b]$, $c = \inf E$, $d = \sup E$, $\alpha, \beta \in \mathbb{R}$.

(i) $\mathcal{S}_o$-$\mu_{\alpha f+\beta g}(E) \le |\alpha| \cdot \mathcal{S}_o$-$\mu_f(E) + |\beta| \cdot \mathcal{S}_o$-$\mu_g(E)$.

(ii) If $\mathcal{S}_o$-$\mu_g(E) = 0$, *then* $\mathcal{S}_o$-$\mu_{f+g}(E) = \mathcal{S}_o$-$\mu_f(E)$.

(iii) If $\sup_{x\in[c,d]}\{|f(x)|, |g(x)|\} = M < +\infty$, *then*

$$\mathcal{S}_o\text{-}\mu_{f\cdot g}(E) \le M \cdot \left(\mathcal{S}_o\text{-}\mu_f(E) + \mathcal{S}_o\text{-}\mu_g(E)\right).$$

(iv) $PV^{**}(f;E) \le \mathcal{S}_o$-$\mu_f(E)$.

PROOF. Recall Proposition 1. Let $\delta : E \to (0,+\infty)$.

(i) We have

$$\mathcal{S}_o\text{-}\mu_{\alpha f+\beta g}(E) \le V_\delta^*(\alpha f + \beta g; E) \le |\alpha| \cdot V_\delta^*(f;E) + |\beta| \cdot V_\delta^*(g;E).$$

Hence $\mathcal{S}_o$-$\mu_{\alpha f+\beta g}(E) \le |\alpha| \cdot \mathcal{S}_o$-$\mu_f(E) + |\beta| \cdot \mathcal{S}_o$-$\mu_g(E)$.

(ii) Clearly $\mathcal{S}_o$-$\mu_g(E) = 0$ implies that $\mathcal{S}_o$-$\mu_{-g}(E) = 0$. By (i), we have

$$\begin{aligned}\mathcal{S}_o\text{-}\mu_f(E) &= \mathcal{S}_o\text{-}\mu_{f+g-g}(E) \le \mathcal{S}_o\text{-}\mu_{f+g}(E) + \mathcal{S}_o\text{-}\mu_{-g}(E)\\ &= \mathcal{S}_o\text{-}\mu_{f+g}(E) \le \mathcal{S}_o\text{-}\mu_f(E) + \mathcal{S}_o\text{-}\mu_g(E) = \mathcal{S}_o\text{-}\mu_f(E).\end{aligned}$$

Therefore $\mathcal{S}_o$-$\mu_f(E) = \mathcal{S}_o$-$\mu_{f+g}(E)$.

(iii) Let $x, y \in [c,d]$, $c \le x < y \le d$. Then

$$\begin{aligned}\left|f(y)\cdot g(y) - f(x)\cdot g(x)\right| &= \left|g(y)\cdot(f(y)-f(x)) + f(x)(g(y)-g(x))\right|\\ &\le M \cdot \left(|f(y)-f(x)| + |g(y)-g(x)|\right).\end{aligned}$$

We have $\mathcal{S}_o$-$\mu_{f\cdot g}(E) \le V_\delta^*(f\cdot g;E) \le M \cdot \left(V_\delta^*(f;E) + V_\delta^*(g;E)\right)$. Therefore $\mathcal{S}_o$-$\mu_{f\cdot g}(E) \le M \cdot \left(\mathcal{S}_o\text{-}\mu_f(E) + \mathcal{S}_o\text{-}\mu_g(E)\right)$.

(iv) We may suppose that $\mathcal{S}_o$-$\mu_f(E) = M < +\infty$. For $\epsilon > 0$ there is a $\delta : E \to (0,+\infty)$ such that $V_\delta^*(f;E) < M + \epsilon$. Let

$$E_k = \left\{x \in E : \delta(x) > \frac{1}{k}\right\}, \quad k = 1, 2, 3, \dots .$$

Then $\{E_k\}$ is an E-chain. Fix some k and let $\{[a_i,b_i]\}_{i=1}^m$ be a finite set of nonoverlapping closed intervals having at least one endpoint in E_k, such that

$\sum_{i=1}^{m}(b_i - a_i) < 1/k$. We may suppose without loss of generality that each $a_i \in E_k$. Then $b_i \in \left(a_i, a_i + \frac{1}{k}\right) \subset \big(a_i, a_i + \delta(a_i)\big)$; so

$$\sum_{i=1}^{m}\big|f(b_i) - f(a_i)\big| < V_\delta^*(f;E) < M + \epsilon\,.$$

Then $V^{**}(f;E_k;1/k) < M+\epsilon$. Hence $V^{**}(f;E_k;0) \le M+\epsilon$ for each k. Since ϵ is arbitrary, we obtain that $PV^{**}(f;E) \le M$. ☐

Lemma 16 (Thomson)**.** (Theorem 43.1 of [12], p. 101). *Let $f : [a,b] \to \mathbb{R}$, $E \subseteq [a,b]$. Then $m^*(f(E)) \le \mathcal{S}_o\text{-}\mu_f(E)$.*

Lemma 17. *Let $f : [a,b] \to \mathbb{R}$, $E \subseteq [a,b]$.*

(i) If f is increasing* on E, then $\mathcal{S}_o\text{-}\mu_f(A) \le 2m^*(f(A))$, whenever $A \subseteq \{x \in E : f \text{ is continuous at } x\}$.

(ii) If f is increasing on $[a,b]$, then $\mathcal{S}_o\text{-}\mu_f(A) \le m^*(f(A))$, whenever $A \subseteq \{x \in E : f \text{ is continuous at } x\}$.

Proof. Suppose that $m^*(f(A)) < +\infty$. (If $m^*(f(A)) = +\infty$, there is nothing to prove.) For $\epsilon > 0$, let G be an open set such that $f(A) \subset G$ and $m(G) < m^*(f(A)) + \epsilon$. Let $\{(\alpha_i, \beta_i)\}_i$ be the components of G. Since f is continuous at each point of A, there exists a $\delta : A \to (0,+\infty)$ such that

$$f\big((x-\delta(x), x+\delta(x))\big) \subset (\alpha_i, \beta_i), \quad \text{whenever} \quad f(x) \in (\alpha_i, \beta_i)\,.$$

Let $\{[a_i, b_i]\}_{i=1}^{m}$ be a finite set of nonoverlapping closed intervals such that each $[a_i, b_i]$ contains a point $x_i \in A$ with $[a_i, b_i] \subset (x_i - \delta(x_i), x_i + \delta(x_i))$. Suppose that $a_1 < b_1 \le a_2 < b_2 \le \ldots \le a_m < b_m$. Then each $[f(a_i), f(b_i)] \subset G$.

(i) Clearly, $\big\{[f(a_i), f(b_i)]\big\}_{i=1, i=\text{even}}^{m}$ and $\big\{[f(a_i), f(b_i)]\big\}_{i=1, i=\text{odd}}^{m}$, consist both of nonoverlapping closed intervals. It follows that

$$\begin{aligned}\sum_{i=1}^{m}\big(f(b_i) - f(a_i)\big) = &\sum_{i=1, i=\text{even}}^{m}\big(f(b_i) - f(a_i)\big) + \sum_{i=1, i=\text{odd}}^{m}\big(f(b_i) - f(a_i)\big)\\ &< 2\cdot m(G) < 2m^*(f(A)) + 2\epsilon\,.\end{aligned}$$

Hence $V_\delta^{*,o}(f;A) \le 2m^*(f(A)) + 2\epsilon$. Now by Proposition 1, we obtain that $\mathcal{S}_o\text{-}\mu_f(A) \le 2m^*(f(A))$.

(ii) Clearly $\{[f(a_i, f(b_i)]\}_{i=1}^{m}$ are nonoverlapping closed intervals. It follows that

$$\sum_{i=1}^{m}\big(f(b_i) - f(a_i)\big) < m(G) < m^*(f(A)) + \epsilon\,.$$

Hence $V_\delta^{*,o}(f;A) \le m^*(f(A)) + \epsilon$. Now by Proposition 1, we obtain that $\mathcal{S}_o$-$\mu_f(A) \le m^*(f(A))$. □

Corollary 5. *Let $f : [a,b] \to \mathbb{R}$ and $E \subseteq \{x \in [a,b] : f$ is continuous at $x\}$.*

(i) If f is increasing* on E and $m^*(f(E)) = 0$, then $\mathcal{S}_o$-$\mu_f(E) = 0$.

(ii) If f is increasing on $[a,b]$, then $m^*(f(E)) = \mathcal{S}_o$-$\mu_f(E)$. (This is the second part of Theorem 13.3 of [8], p. 100.)

Corollary 6. *Let $f : [a,b] \to \mathbb{R}$ and $E = \{x \in [a,b] : \underline{D}f(x) > 0$ and f is continuous at $x\}$. If $m^*(f(E)) = 0$, then $\mathcal{S}_o$-$\mu_f(E) = 0$.*

PROOF. Let

$$E_n = \left\{ x \in E : \frac{f(t) - f(x)}{t - x} \ge \frac{1}{n}, \quad 0 < |t - x| < \frac{1}{n} \right\}, \ n = 1, 2, \dots .$$

Let $E_n^i = \left[\frac{i}{2n}, \frac{i+1}{2n} \right] \cap E_n$, $i = 0, \pm 1, \pm 2, \dots$. Then $E = \cup_{n,i} E_n^i$. Let J_n^i be an open interval such that $E_n^i \subset J_n^i$ and $m(J_n^i) < 3/(4n)$. Let $x, y \in J_n^i$, $x < y$. At least one of them belonging to E_n^i. Then $f(y) - f(x) > \frac{1}{n}(y - x)$. Hence f is increasing* on each E_n^i. Clearly $m^*(f(E_n^i)) = 0$. It follows that $\mathcal{S}_o$-$\mu_f(E_n^i) = 0$ (see Corollary 5, (i)). Since $\mathcal{S}_o$-μ_f is an outer measure, we obtain that $\mathcal{S}_o$-$\mu_f(E) = 0$. □

Lemma 18. *Let $f : [a,b] \to \mathbb{R}$ and let $X = \{x \in [a,b] : f'(x) = 0\}$. Then $\mathcal{S}_o$-$\mu_f(X) = 0$.*

PROOF. See Lemma 42.1 of [12], p. 99. □

Lemma 19. (Theorem 9.1 of [8], p. 125). *Let $f : [a,b] \to \mathbb{R}$, and let $N = \{x \in [a,b] : f$ is continuous at x; $f'(x)$ does not exist (finite or infinite)$\}$. If $f \in VB$ on $[a,b]$, then $m^*(f(N)) = \mathcal{S}_o$-$\mu_f(N) = m^*(N) = 0$.*

PROOF. That $m^*(f(N)) = m^*(N) = 0$ follows immediately from Theorem 9.1 of [8] (see (9.2) and (9.3), p. 125). Consider the curve:

$$C : X(t) = t\,, \quad Y(t) = f(t)\,, \quad t \in [a,b]\,,$$

and let $S(t)$ be its length on the interval $[a,t]$. In the proof of Theorem 9.1 of [8] (p. 126), it is shown that $m^*(S(N)) = 0$. By Corollary 5, $\mathcal{S}_o$-$\mu_S(N) = m^*(S(N)) = 0$ (because S is a strictly increasing function on $[a,b]$). But $|f(t_2) - f(t_1)| \le S(t_2) - S(t_1)$, whenever $a \le t_1 < t_2 \le b$; so

$$0 \le \mathcal{S}_o\text{-}\mu_f(N) \le \mathcal{S}_o\text{-}\mu_S(N) = 0\,.$$

Therefore $\mathcal{S}_o$-$\mu_f(N) = 0$. □

Lemma 20. *Let $f : [a,b] \to \mathbb{R}$, $E \subseteq [a,b]$, $A \subseteq \{x \in E : f$ is continuous at $x\}$, and let $\tilde{f} : [a,b] \to \mathbb{R}$, $\tilde{f} = f_{\overline{E}\cup\{a,b\}}$ (see Definition 1). If $f \in VB^*$ on E, then $\mathcal{S}_o$-$\mu_f(A) = \mathcal{S}_o$-$\mu_{\tilde{f}}(A)$.*

PROOF. Let $g = \tilde{f} - f$. Since $\tilde{f}$ is continuous at each point of A, the function g has this property as well. Suppose that there are infinitely many intervals contiguous to $\overline{E} \cup \{a,b\}$, and let's denote them by $\{(a_i, b_i)\}_{i=1}^{\infty}$. Let

$$A_1 = A \cap \{a, b, a_1, b_1, a_2, b_2, \ldots\} \quad \text{and} \quad A_2 = A \setminus A_1 \,.$$

Since g is continuous at each point of A, we have that $\mathcal{S}_o$-$\mu_g(\{x\}) = 0$ for every $x \in A$. It follows that $\mathcal{S}_o$-$\mu_g(A_1) = 0$ (because A_1 is at most countable and $\mathcal{S}_o$-μ_g is an outer measure). For $\epsilon > 0$ let n_o be a positive integer such that $\sum_{i=n_o+1}^{\infty} \mathcal{O}(f;[a_i,b_i]) < \epsilon$. Then

$$\sum_{i=n_o+1}^{\infty} \mathcal{O}(g;[a_i,b_i]) < 2\epsilon \,. \tag{6}$$

Let $G = (a,b) \setminus \left(\cup_{i=1}^{n_o}[a_i,b_i]\right)$ and let $\delta : A_2 \to (0,+\infty)$ be a positive function such that $(x-\delta(x), x+\delta(x)) \subset G$. Let $\{[c_j,d_j]\}_{j=1}^{n}$ be a finite set of nonoverlapping closed intervals such that each $[c_j,d_j]$ contains a point $x_j \in A_2$ with $[c_j,d_j] \subset (x_j - \delta(x_j), x_j + \delta(x_j))$. Since any interval (a_i,b_i) with $i \geq n_o + 1$ contains at most two points of the set $\{c_1,d_1,c_2,d_2,\ldots,c_n,d_n\}$, and $g = 0$ on $\overline{E}$, by (6), $\sum_{j=1}^{n} |g(d_j) - g(c_j)| < 2\epsilon$; so $V_{\delta}^{*,o}(f;A_2) < 4\epsilon$. By Proposition 1, it follows that $\mathcal{S}_o$-$\mu_g(A_2) = 0$. Clearly $\mathcal{S}_o$-$\mu_g(A) = 0$. Now, by Theorem 7, (ii), we obtain that $\mathcal{S}_o$-$\mu_f(A) = \mathcal{S}_o$-$\mu_{\tilde{f}}(A)$. □

Lemma 21. *Let $f : [a,b] \to \mathbb{R}$, $E \subseteq [a,b]$, $N = \{x \in E : f'(x)$ does not exist (finite or infinite)$\}$ and $N_o = N \cap \{x \in [a,b] : f$ is continuous at $x\}$. If $f \in VB^*G$ on E, then*

(i) f is derivable almost everywhere on E and $m^*(f(N)) = 0$;

(ii) $\mathcal{S}_o$-$\mu_f(N_o) = 0$.

PROOF. (i) See Theorem 7.2 of [8], p. 230.
(ii) Since $\mathcal{S}_o$-μ_f is an outer measure, it is sufficient to suppose that $f \in VB^*$ on E. Let $\tilde{f}$ be the function defined in Lemma 20. Then $\mathcal{S}_o$-$\mu_f(N_o) = \mathcal{S}_o$-$\mu_{\tilde{f}}(N_o)$. Let

$N_1 = \{x \in N_o : \tilde{f}'(x) = 0\}$;

$N_2 = \{x \in N_o : \tilde{f}'(x) > 0\}$;

$N_3 = \{x \in N_o : \tilde{f}^{'}(x) < 0\}$;

$N_4 = \{x \in N_o : \tilde{f}^{'}(x)$ does not exist (finite or infinite)$\}$;

$\tilde{N} = \{x \in [a,b] : \tilde{f}$ is continuous at x; $\tilde{f}^{'}(x)$ does not exist (finite or infinite)$\}$.

Then $N_4 \subset \tilde{N}$ and $\tilde{f}$ is VB on $[a,b]$. By Lemma 19, $\mathcal{S}_o$-$\mu_{\tilde{f}}(\tilde{N}) = 0$. Therefore $\mathcal{S}_o$-$\mu_{\tilde{f}}(N_4) = 0$. By Lemma 18, $\mathcal{S}_o$-$\mu_{\tilde{f}}(N_1) = 0$, and by (i), $m^*(\tilde{f}(N_2)) = 0$. Hence $\mathcal{S}_o$-$\mu_{\tilde{f}}(N_2) = 0$ (see Corollary 6). Analogously, it follows that $\mathcal{S}_o$-$\mu_{\tilde{f}}(N_3) = 0$. Therefore $\mathcal{S}_o$-$\mu_{\tilde{f}}(N_o) = 0$. □

Remark 1. Lemma 21 is an extension of Theorem 7.2 of [8] (p. 230), Theorem 44.2 and Theorem 44.1 of [12] (pp. 103–104).

Theorem 8. (An Extension of Corollary 43.4 of [12], p. 103).
Let $f : [a,b] \to \mathbb{R}$, $E \subseteq [a,b]$ and $A \subseteq \{x \in E : f$ is continuous at $x\}$. If $f \in VB^*G$ on E, then the following assertions are equivalent.

(i) $m^*(f(A)) = 0$.

(ii) $\mathcal{S}_o$-$\mu_f(A) = 0$.

PROOF. (i) ⇒ (ii) Let $N = \{x \in A : f^{'}(x)$ does not exist (finite or infinite)$\}$. By Lemma 21, (ii), we have that $\mathcal{S}_o$-$\mu_f(N) = 0$.

Let $B = A \setminus N$.

Let $B_1 = \{x \in B : f^{'}(x) = 0\}$. Then $\mathcal{S}_o$-$\mu_f(B_1) = 0$ (see Lemma 18).

Let $B_2 = \{x \in B : f^{'}(x) > 0\}$. Then $\mathcal{S}_o$-$\mu_f(B_2) = 0$ (see Corollary 6).

Let $B_3 = \{x \in B : f^{'}(x) < 0\}$. Then $\mathcal{S}_o$-$\mu_f(B_3) = 0$ (see Corollary 6).

Therefore $\mathcal{S}_o$-$\mu_f(A) = 0$.
(ii) ⇒ (i) See Lemma 16. □

Corollary 7. (Identical with Corollary 3). *Let $E \subseteq [a,b]$. Then*

$$\mathcal{A} = \{f : [a,b] \to \mathbb{R} : f \in VB^*G \cap (N) \text{ on } E\}$$

is an algebra.

Proof. Let $f, g \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{R}$. By Lemma 11, there exists a countable set $E_1 \subseteq E$ such that both functions f and g are continuous at each point of $E \setminus E_1$. Clearly $\alpha f + \beta g \in VB^*G$ on E. We have to show that $\alpha f + \beta g \in (N)$ on $E \setminus E_1$. Let Z be a null subset of $E \setminus E_1$. Then $m^*(f(Z)) = m^*(g(Z)) = 0$. By Theorem 8, $\mathcal{S}_o$-$\mu_f(Z) = \mathcal{S}_o$-$\mu_g(Z) = 0$. It follows that $\mathcal{S}_o$-$\mu_{\alpha f+\beta g}(Z) = 0$ (see Theorem 7, (i)). Hence by Lemma 16, we obtain that $m^*((\alpha f + \beta g)(Z)) = 0$. Therefore $\alpha f + \beta g \in (N)$ on $E \setminus E_1$.

It is well known that $f \cdot g \in VB^*G$ on E. We show that $f \cdot g \in (N)$ on E. Since $f, g \in VB^*G$ on E, there exists a sequence $\{E_n\}_n$ of sets such that $E = \cup_n E_n$ and $f, g \in VB^*$ on each E_n. Then $f, g \in VB^*$ on $\overline{E}_n$ (see Theorem 7.1 of [8], p. 229). Let $c_n = \inf E_n$ and $d_n = \sup E_n$. Then f and g are bounded by some number M_n on $[c_n, d_n]$. By Lemma 11, there exists a countable subset $E_n^{'} \subseteq E_n$ such that f and g are both continuous at each point of $E_n \setminus E_n^{'}$. Let Z be a null subset of $E_n \setminus E_n^{'}$. Then $m^*(f(Z)) = m^*(g(Z)) = 0$, and by Theorem 8, $\mathcal{S}_o$-$\mu_f(Z) = \mathcal{S}_o$-$\mu_g(Z) = 0$. It follows that $\mathcal{S}_o$-$\mu_{f\cdot g}(Z) = 0$ (see Theorem 7, (iii)). Now, by Lemma 16, we obtain that $m^*((f \cdot g)(Z)) = 0$. Hence $f \cdot g \in (N)$ on each E_n. Therefore $f \cdot g \in (N)$ on E. □

9 Characterizations of a $\mathbf{VB^*G \cap (N)}$ Function f on a Lebesgue Measurable Set, Using $\mathcal{S}_\mathbf{o}$-$\mu_\mathbf{f}$

Theorem 9. *Let $f : [a, b] \to \mathbb{R}$ and let E be a Lebesgue measurable subset of $[a, b]$. The following assertions are equivalent.*

(i) $f \in VB^*G \cap (N)$ on E.

(ii) $f \in VB^*G \cap (N)$ on Z, whenever Z is a null subset of E.

(iii) there exists a countable subset E_1 of E such that $\mathcal{S}_o$-$\mu_f(Z) = 0$, whenever Z is a null subset of $E \setminus E_1$.

Proof. Let $E_1 = \{x \in E : f \text{ is not continuous at } x\}$.

(i) ⇒ (ii) This is obvious.

(ii) ⇒ (i) Clearly $f \in (N)$ on E, and by Theorem 1, $f \in VB^*G$ on E. Therefore $f \in VB^*G \cap (N)$ on E.

(i) ⇒ (iii) By Lemma 11, E_1 is at most countable. Let Z be a null subset of $E \setminus E_1$. Then $m^*(f(Z)) = 0$. By Theorem 8, we obtain that $\mathcal{S}_o$-$\mu_f(Z) = 0$.

(iii) ⇒ (ii) Let Z be a null subset of E. Then $Z = Z_1 \cup Z_2$, where $Z_1 = Z \cap E_1$ and $Z_2 = Z \cap (E \setminus E_1)$. By Lemma 16, we obtain that $m^*(f(Z_2)) = \mathcal{S}_o$-$\mu_f(Z_2) = 0$. By Theorem 40.1 of [12] (p. 94), it follows that $f \in VB^*G$ on Z_2. Hence $f \in VB^*G$ on Z. Since the set $f(Z_1)$ is at most countable, it follows that $m^*(f(Z)) = 0$. □

Lemma 22. *Let $f : [a, b] \to \mathbb{R}$, and let E be a null subset of $[a, b]$. If $f \in AC^*G$ on E, then $\mathcal{S}_o$-$\mu_f(E) = 0$.*

PROOF. Suppose that $f \in AC^*$ on E, and for $\epsilon > 0$ let $\delta > 0$ be given by this fact. Let G be an open set such that $E \subset G$ and $m(G) < \delta$. Let $\eta : E \to (0, +\infty)$, with $(x - \eta(x), x + \eta(x)) \subset G$. Then $V^*_\eta(f; E) < \epsilon$; so $\mathcal{S}_o$-$\mu_f(E) = 0$. Now, if $f \in AC^*G$ on E, since $\mathcal{S}_o$-μ_f is an outer measure, it follows that $\mathcal{S}_o$-$\mu_f(E) = 0$. □

Theorem 10. (An extension of Theorem 45.3, (i), (ii) of [12], p. 106) *Let $f : [a, b] \to \mathbb{R}$ and let E be a closed subset of $[a, b]$. The following assertions are equivalent*

(i) $f \in AC^*G$ *on E and f is continuous at every point of E.*

(ii) $f \in VB^*G \cap (N)$ *on E and f is continuous at every point of E.*

(iii) $f \in VB^*G \cap (N)$ *on any null subset of E and f is continuous at every point of E.*

(iv) $\mathcal{S}_o$-$\mu_f(Z) = 0$, *whenever Z is a null subset of E.*

(v) $f \in Y_{D^o}$ *on E (i.e., $f \in SLC$ on E).*

PROOF. By Theorem 6 ((i),(ii),(x)), (i) ⇔ (ii) ⇔(iii). By Theorem 9 ((ii),(iii)), (iii) ⇔ (iv). By Corollary 4 ((ii),(iii)), (iv) ⇔ (v). □

Remark 2. Theorem 10, (i), (v) was obtained before in [3] (see Corollary 1, (i), (vii)) and [4] (see Corollary 2.27.1, (i), (vii)). The same result is also shown by Bongiorno, Di Piazza and Skvortsov in [1], using a different technique.

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