RESEARCH

Real Analysis Exchange
Vol. 36(2), 2010/2011, pp. 297–306

Jack Grahl, Department of Mathematics, University College London WC1E63T, London, U.K. email: jgrahl@math.ucl.ac.uk

Togo Nishiura, Department of Mathematics and Computer Science, Dickinson College, Carlisle, PA 17013, USA; and Department of Mathematics, Wayne State University, Detroit, MI 48203, U.S.A. email: nishiura@dickinson.edu

A FACTORIZATION PROBLEM

Abstract

A solution is presented of a problem proposed at the Summer Symposium in Real Analysis XXXIII.

The factorization problem was proposed by the first author during the Summer Symposium in Real Analysis XXXIII, which was held at Southeastern Oklahoma State University. The statement of the problem (given in Section 1) requires some definitions.

A *partition* Π of $[0,1)$ is a finite disjointed collection $\{\, I_i \colon i = 1, 2, \dots, n \,\}$ whose union is $[0,1)$, where I_i is a half-open interval of the form $[a,b)$. The *size* of a partition, $\delta(\Pi)$, is the maximum length of the intervals of Π.

A function $\varphi\colon [0,1] \to [0,1]$ is called a *permutation* of a partition Π of $[0,1)$ if $\Pi' = \{\, \varphi[I_i] \colon I_i \in \Pi \,\}$ is a partition of $[0,1)$, φ restricted to I_i is a translation for each i, and $\varphi(1) = 1$. Clearly a permutation is bijective and has the property that the Lebesgue measures of $\varphi[E]$ and $\varphi^{-1}[E]$ are equal to the Lebesgue measure of E for every Lebesgue measurable set $E \subset [0,1]$. (Lebesgue measure will be denoted by λ, and the modifying "Lebesgue" will be dropped.) The set of all permutations will be denoted by $\mathcal{P}$.

The set of all functions $h\colon [0,1] \to [0,1]$ that are almost everywhere limits of sequences in $\mathcal{P}$ will be denoted by $\mathcal{H}$. This is the same as the set of functions that are limit-in-measure of sequences in $\mathcal{P}$. (See [2, Chapter 5].)

The collection of partitions of $[0,1)$ can be used to define another set $\mathcal{H}'$. A function $h\colon [0,1] \to [0,1]$ is in $\mathcal{H}'$ if, for each sequence Π_n of partitions of $[0,1)$

Mathematical Reviews subject classification: Primary: 26, 28
Key words: measure preserving function, rearrangement of functions
Received by the editors April 8, 2010
Communicated by: Paul D. Humke

such that $\delta(\Pi_n) \to 0$ as $n \to \infty$, there exists a permutation φ_n of Π_n such that φ_n converges almost everywhere to h. Obviously, $\mathcal{H}' \subset \mathcal{H}$. It will be shown in Section 2 that $\mathcal{H} = \mathcal{H}'$ and that $h \in \mathcal{H}$ if and only if h is λ-measurable and $\lambda = h_{\#}\lambda$.[1]

1 Factorization problems

The statement of the proposed factorization problem is the following: Find a minimal[2] set $\mathcal{G}$ of functions $g\colon [0,1] \to \mathbb{R}$ such that for each λ-measurable function $f\colon [0,1] \to \mathbb{R}$ there is an h in $\mathcal{H}$ and there is a g in $\mathcal{G}$ such that the composition[3] $gh\colon [0,1] \to \mathbb{R}$ is λ-equivalent to f. Observe that the set $\mathcal{B}_2$ of Baire class 2 functions has the property that each λ-measurable function f has a factorization $f = gh$ almost everywhere with $g \in \mathcal{B}_2$ and h being the identity function; but $\mathcal{B}_2$ is not minimal. (The first author conjectured that a possible $\mathcal{G}$ is the set of nondecreasing functions.) See Theorems 7 and 5 for our solution.

Factorization Problem. *Give a minimal set $\mathcal{G}$ of functions $g\colon [0,1] \to [0,1]$ having the property that for each Borel measurable function $f\colon [0,1] \to (0,1)$ there is an h in $\mathcal{H}$ and there is a g in $\mathcal{G}$ such that gh is λ-equivalent to f.*

Note that the λ-measurability of $f\colon [0,1] \to \mathbb{R}$ has been replaced with Borel measurability of $f\colon [0,1] \to (0,1)$ and that $g\colon [0,1] \to \mathbb{R}$ has been replaced by $g\colon [0,1] \to [0,1]$. Clearly there is no loss in generality in making these replacements. We shall show that the set $\mathcal{G}$ of continuous-from-above, nondecreasing functions will solve the above factorization problem. Also, we strengthen the solution to include connections between the Baire classes of the functions f and h.

Our solution of the factorization problem is achieved by employing measures on certain linearly ordered spaces $(S, \prec)$ which are defined in Section 4. The order topology of $(S, \prec)$ will not be used, only its measure theoretic properties induced by the order $\prec$ are exploited.

Before embarking on the construction of the function h let us characterize the sets $\mathcal{H}$ and $\mathcal{H}'$.

---

[1] For any λ-measurable function $f\colon E \to \mathbb{R}$, where E is a Borel set of $\mathbb{R}$, the measure $f_{\#}\lambda$ is defined on the Borel sets B of $\mathbb{R}$ by $f_{\#}\lambda(B) = \lambda(f^{-1}[B])$.

[2] As usual, we mean minimal in the sense of the partial order $\subset$. Of course, minimal sets need not be unique.

[3] As real-valued functions will not be multiplied, compositions will be indicated in the multiplicative notation.

2 Characterization of $\mathcal{H}$ and $\mathcal{H}'$

The 2 sets $\mathcal{H}$ and $\mathcal{H}'$ were introduced to define the factorization problem. $\mathcal{H}' \subset \mathcal{H}$ has been observed already. We have the following characterisation.

Theorem 1. *$\mathcal{H}' = \mathcal{H}$, and $h \in \mathcal{H}$ if and only if h is a λ-measurable function such that $\lambda = h_{\#}\lambda$.*

A λ-measurable function $h\colon [0,1] \to [0,1]$ is said to be *measure preserving*[4] if $\lambda = h_{\#}\lambda$. Note that $\lambda(h^{-1}(1)) = 0$.

The next 2 propositions provide the proof of the theorem.

Proposition 2. *If $h \in \mathcal{H}$, then h is measure preserving.*

Proof. Let φ_n be a sequence of permutations converging in measure to h, and let ε_n be such that $\lambda(E_n) < \varepsilon_n$, where $E_n = \{\, x\colon |\varphi_n(x) - h(x)| \geq \varepsilon_n \,\}$, with $\varepsilon_n \to 0$ as $n \to \infty$. For closed sets K in $[0,1]$ let K_n be the ε_n-neighborhood of K. As ${\varphi_n}^{-1}[K] \subset h^{-1}[K_n] \cup E_n$, it follows that $\lambda(K) \leq h_{\#}\lambda(K_n) + \varepsilon_n$, whence $\lambda(K) \leq h_{\#}\lambda(K)$. Consequently, $\lambda(U) \leq h_{\#}\lambda(U)$ for every open set U in $[0,1]$. It now follows that $\lambda(K) = h_{\#}\lambda(K)$ since $h_{\#}\lambda([0,1]) = 1$. $\square$

It remains to prove that if h is measure preserving then $h \in \mathcal{H}'$. The proof is a "pigeonhole" process. That is, a partition Π of pigeons are to be assigned to a partition Π' of pigeonholes under certain rules φ. Unfortunately the pigeons may be too fat to achieve this assignment. The final step of the proof of the characterization will depend on the following modified pigeonhole lemma.

Lemma 3. *Let $\Pi' = \{\, I'_i\colon i = 1, 2, \ldots, n \,\}$ be a partition of $[0,1)$ and let $\{\, K_i\colon i = 1, 2, \ldots, n \,\}$ be a disjointed collection of compact sets of $[0,1)$ such that $\lambda(I'_i) > \lambda(K_i)$ for each i. If $\varepsilon > 0$, then there is a $\delta > 0$ such that for each partition $\Pi = \{\, I_j\colon j = 1, 2, \ldots, m \,\}$ with $\delta(\Pi) < \delta$ there is a permutation $\varphi\colon [0,1] \to [0,1]$ of Π such that, for each i,*

$$\lambda(I'_i) - 2\delta(\Pi) > \lambda(\varphi[H_i]) > \lambda(\varphi[K_i]) - \varepsilon/n,$$

where $H_i = \bigcup_j \{\, I_j\colon \varphi[I_j] \subset I'_i \text{ and } I_j \cap K_i \neq \emptyset \,\}$.

Proof. Let γ be such that $0 < 3\gamma < \lambda(I'_i) - \lambda(K_i)$ and $\lambda(U_i \setminus K_i) < \varepsilon/n$ for each i, where U_i is the γ-neighborhood of K_i, and such that 3γ is less than the distances between distinct K_i's.

[4] Here, h need not be bijective; in ergodic theory, "measure preserving" requires that h be bijective and both h and h^{-1} be measurable and measure preserving in our sense (see, for example, [1, page 7]). Such h is often called an automorphism.

With $W(i,\Pi) = \bigcup\{\, I_j \in \Pi\colon I_j \cap K_i \neq \emptyset \,\}$, observe that $\lambda(I'_i) - \lambda(W(i,\Pi)) + \lambda\big(W(i,\Pi) \setminus K_i\big) = \lambda(I'_i) - \lambda(K_i) > 3\gamma$. As $\lambda\big(W(i,\Pi) \setminus K_i\big) \to 0$ as $\delta(\Pi) \to 0$, there is a δ such that $0 < \delta < \gamma$ and such that $\lambda\big(W(i,\Pi) \setminus K_i\big) < \gamma$ whenever $\delta(\Pi) < \delta$. So, if $\delta(\Pi) < \delta$, then

$$\lambda(I'_i) - 2\delta(\Pi) > \textstyle\sum\{\, \lambda(I_j)\colon I_j \in \Pi,\ I_j \cap K_i \neq \emptyset \,\}.$$

Let us construct the required φ. Let $\delta(\Pi) < \delta$. Note that no interval I_j intersects more than one of the K_i. We separate the intervals I_j into classes B_i, which consists of those intervals which intersect K_i, $1 \leq i \leq n$, and C, those intervals which do not intersect any K_i. Denote $\sum\{\, \lambda(I_j)\colon I_j \in B_i \,\}$ by l_i and index the collection C as $J_1, J_2, \dots, J_m$. Then $\sum_{i=1}^{n} l_i + \sum_{k=1}^{m} \lambda(J_k) = 1$. Call a_i the right endpoint of I'_i.

Let us describe the first step of the construction of φ. As $l_1 < a_1 - \delta(\Pi)$, there is an m_1 such that $1 < m_1$ and $\sum_{k=1}^{m_1-1} \lambda(J_k) \leq a_1 - l_1 < \sum_{k=1}^{m_1} \lambda(J_k)$. Define $a'_1 = l_1 + \sum_{k=1}^{m_1} \lambda(J_k)$. Let φ be a permutation of B_1 onto $[0, l_1)$ and $\{\, J_k\colon k \leq m_1 \,\}$ onto $[l_1, a'_1)$ with $a_1 \in \varphi(J_{m_1})$.

We now repeat this procedure for B_2. We have $a_1 < a'_1 < a'_1 + l_2 < a_2 - \delta(\Pi)$, whence $[a'_1, a'_1 + l_2) \subset [a_1, a_2)$, and there is an m_2 such that $m_1 < m_2$ and $\sum_{k=m_1+1}^{m_2-1} \lambda(J_k) \leq a_2 - (a'_1 + l_2) < \sum_{k=m_1+1}^{m_2} \lambda(J_k)$. Define $a'_2 = \sum_{i=1}^{2} l_i + \sum_{k=1}^{m_2} \lambda(J_k)$. Let φ be a permutation of B_2 onto $[a'_1, a'_1 + l_2)$ and $\{\, J_k\colon m_1 < k \leq m_2 \,\}$ onto $[a'_1 + l_2, a'_2)$ with $a_2 \in \varphi(J_{m_2})$.

This process continues up to the n^{th} stage, where $a_{n-1} \in \varphi(J_{m_{n-1}})$ and $a'_{n-1} = \sum_{i=1}^{n-1} l_i + \sum_{k=1}^{m_{n-1}} \lambda(J_k)$ satisfies $a_{n-1} < a'_{n-1} < a'_{n-1} + l_n < 1 - \delta(\Pi)$. The remainder of the construction of φ is left to the reader.

As $W(i,\Pi) \setminus K_i \subset U_i \setminus K_i$, the construction is completed. $\square$

Proposition 4. *If h is measure preserving, then $h \in \mathcal{H}'$.*

PROOF. Let $h\colon [0,1] \to [0,1]$ be measure preserving and, for each m, let Π'_m be a partition of $[0,1)$ such that $\delta(\Pi'_m) < 2^{-m}$. Denote by n_m the number of intervals in Π'_m. Then $\{\, h^{-1}[I'(m,i)]\colon I'(m,i) \in \Pi_m \,\}$ and $h^{-1}(1)$ form a decomposition of $[0,1]$. For each $I'(m,i)$, let $K(m,i)$ be a compact subset of $[0,1) \cap h^{-1}[I'(m,i)]$ such that $\lambda\big(h^{-1}[I'(m,i)] \setminus K(m,i)\big) < (n_m 2^m)^{-1}$.

For each Π'_m and $\varepsilon = 2^{-m}$, let δ_m be as provided by the modified pigeonhole lemma. We may assume $\delta_m > 2\,\delta_{m+1}$. For each m let

$$D_m = h^{-1}(1) \cup \textstyle\bigcup_{m \leq m'} \bigcup\{\, h^{-1}[I'(m',i)] \setminus K(m',i)\colon I'(m',i) \in \Pi'_{m'} \,\}.$$

Note $D_{m'} \subset D_m$ whenever $m' \geq m$, and $\lambda(D_m) \leq 2^{-(m-1)}$. Also, if $x \notin D_m$ and $m' > m$, then x is in some $K(m',i)$, whence $h(x) \in I'(m',i) \in \Pi'_{m'}$.

Suppose that Π_k is a sequence of partitions such that $\delta(\Pi_k) \to 0$ as $k \to 0$. Let k_m be the least k such that $\delta(\Pi_{k'}) < \delta_m$ whenever $k' < k$. Observe that k_m is nondecreasing and converges to $+\infty$. If $k \leq k_1$, then let φ_k be the identity function. If $k_m < k \leq k_{m+1}$, then let φ_k be as given by the modified pigeonhole lemma for the partitions Π_k and Π'_m. Clearly φ_k, $k = 1, 2, \ldots$, is a well defined sequence.

The constructed sequence φ_k will converge almost everywhere to h. Indeed, let $\varepsilon > 0$ and let m be such that $2^{-(m-1)} < \varepsilon$. Suppose $x \notin D_m$ and $m' \geq m$. Let m'' and k be such that $k_{m'} = k_{m''} < k_{m''+1}$ and $k_{m''+1} \geq k > k_{m''}$. There is an $I(k,j)$ in Π_k such that $x \in I(k,j) \cap K(m'', i)$ for some i. By the modified pigeonhole lemma, $\varphi_k[I(k,j)] \subset I'(m'', i) \in \Pi'_{m''}$. Hence $h(x)$ and $\varphi_k(x)$ are in the same $I'(m'', i)$. As $\delta(\Pi'_{m''}) < 2^{-m''}$, $|\varphi_k(x) - h(x)| < 2^{-m''}$ whenever $k_{m''+1} \geq k > k_{m''}$. We infer from this that φ_k converges to h except on a subset of D_m. Hence the set in which φ_k does not converge to h has measure less than ε. $\square$

3 Measure induced by $f\colon [0,1] \to [0,1]$

For a Borel measurable function $f\colon [0,1] \to [0,1]$ let $g\colon [0,1] \to [0,1]$ be its nondecreasing, continuous-from-above distribution function (see Remark 8 below) that satisfies

$$g_{\#}\lambda([0,y]) = f_{\#}\lambda([0,y]), \quad y \in [0,1].$$

For each y in $[0,1]$, the level sets of g and f are given by $g^{-1}(y)$ and $f^{-1}(y)$, respectively. Moreover, $f_{\#}\lambda(\{y\}) = \lambda(g^{-1}(y))$. As g is nondecreasing and continuous-from-above, each nonempty level set $g^{-1}(y)$ is a connected set such that

$$a(y) = \min g^{-1}(y) \qquad \text{and} \qquad b(y) = \sup g^{-1}(y)$$

satisfy $\lambda\big([0, a(y))\big) = \lambda\big(f^{-1}\big[[0,y)\big]\big)$ and $\lambda(g^{-1}(y)) = b(y) - a(y) = \lambda(f^{-1}(y))$.

It will be shown in the following section that the set $\mathcal{G}$ of all nondecreasing, continuous-from-above functions $g\colon [0,1] \to [0,1]$ fulfills the requirements of the factorization. Indeed, for each f, a measure-preserving h is constructed so that $f = gh$ almost everywhere. Moreover, if $f \in \mathcal{G}$ then the g in $\mathcal{G}$ and the constructed h yielding $f = gh$ almost everywhere are f and the identity function, respectively. To see that $\mathcal{G}$ is a minimal set, it is enough to observe that if two nondecreasing, continuous-from-above functions g_1 and g_2 are different, then they differ on a set of positive measure. Hence there exists some r such that

$$\lambda(\{\, x : g_1(x) > r \,\}) \neq \lambda(\{\, x : g_2(x) > r \,\}).$$

So there can be no measure preserving function h such that $g_1 h = g_2$ almost everywhere. Consequently, upon the successful construction of the measure preserving h in the next section, the following theorem yields a solution of the factorization problem.

Theorem 5. *The set $\mathcal{G}$ of all functions $g\colon [0,1] \to [0,1]$ that are nondecreasing and continuous-from-above is a minimal set having the property that for each Borel measurable function $f\colon [0,1] \to (0,1)$ there is an h in $\mathcal{H}$ and there is a g in $\mathcal{G}$ such that gh is λ-equivalent to f.*

(Observe: If each g in $\mathcal{G}$ of the theorem is reassigned the value y_0 at 0 and at 1, then the resulting set is also minimal. This will be useful for Theorem 7.)

4 Linearly ordered spaces

To prove Theorem 5 let $g \in \mathcal{G}$ correspond to the function f. Our task is to construct a measure preserving function h such that $gh = f$ almost everywhere. It will be convenient to designate by Z the space $[0,1]$ through which the factorization is accomplished. That is, $h\colon [0,1] \to Z$ and $g\colon Z \to [0,1]$.

Turning to the construction, we will need a Baire class 1 function η that retracts $[0,1]$ onto $g[Z]$. To this end, define

$$\eta(y) = \begin{cases} \inf\,[y,1] \cap g[Z], & \text{if } [y,1] \cap g[Z] \neq \emptyset; \\ \sup\, g[Z], & \text{if } [y,1] \cap g[Z] = \emptyset. \end{cases}$$

Clearly η is nondecreasing (hence is Baire class 1); and, as $g_{\#}\lambda([0,y])$ is continuous-from-above, $g[Z] = \eta\big[g[Z]\big]$ and $\eta\eta = \eta$. Moreover, ηf is equal to f almost everywhere and $\eta f\big[[0,1]\big] \subset g[Z]$.

The tool used in the construction of the function $h\colon [0,1] \to Z$ is the following linear order[5] $\prec$ of $[0,1]^2$.

$$(s,t) \prec (s',t') \quad \text{ if } t < t', \text{ or if } t = t' \text{ and } s < s'.$$

Denote the graph of g by $\mathrm{Graph}(g)$. The linear order $\prec$ restricted to $\mathrm{Graph}(g)$, denoted $\prec_g$, yields a linearly ordered space $(\mathrm{Graph}(g), \prec_g)$ which is order

[5]This is a dictionary order on $[0,1]^2$, though not the usual one. If $f\colon [0,1] \to [0,1]$ is nondecreasing, then $\psi\colon x \mapsto (x, f(x))$ is an increasing map of $[0,1]$ into the linearly ordered space $([0,1]^2, \prec)$. As the factorization problem is a measure theoretic one, the emphasis will be on the σ-algebra $\overline{\mathcal{B}}$ of $([0,1]^2, \prec)$ generated by the open half-rays $H(\bar{x},\bar{y}) = \{\,(x,y)\colon (x,y) \prec (\bar{x},\bar{y})\,\}$. $\overline{\mathcal{B}}$ is contained in the collection of the usual Borel sets of $[0,1]^2$. Moreover, the sets of $\overline{\mathcal{B}}$ that are contained in horizontal lines of $[0,1]^2$ are exactly the usual Borel sets of the lines.

isomorphic to $(Z,<)$, where Z is the domain of g. Clearly, $\pi(z,y) \mapsto z$ is the order-preserving isomorphism of $\mathrm{Graph}(g)$ onto Z.[6]

Consider next the linearly ordered space $(\mathrm{Graph}(\eta f), \prec_{\eta f})$. This linearly ordered space need not be order isomorphic to the domain of ηf. Nonetheless the following 2 statements hold for each $\bar{y}$ in $g[Z]$ and $\bar{x} \in (\eta f)^{-1}(\bar{y})$.

1. $\{\, (x, \eta f(x))\colon \eta f(x) = \bar{y} \,\} = (\eta f)^{-1}(\bar{y}) \times \bar{y}$.
2. $\{\, x\colon (x, \eta f(x)) \prec (\bar{x}, \bar{y}) \,\} = (\eta f)^{-1}\big[[0,\bar{y})\big] \cup \{\, x \in (\eta f)^{-1}(\bar{y})\colon x < \bar{x} \,\}$.

The linear order $\prec_{\eta f}$ induces a natural σ-algebra $\mathcal{B}$ generated by the "half-rays" $\{\, (x,y)\colon (x,y) \prec (\bar{x},\bar{y}) \,\} \cap \mathrm{Graph}(\eta f)$. As $\psi\colon x \mapsto (x, \eta f(x))$ is such that $\psi^{-1}[\mathcal{B}]$ is contained in the collection of Borel subsets of $[0,1]$ whenever ηf is Borel measurable[7], the measure $\nu = \psi_{\#}\lambda$ on $\mathrm{Graph}(\eta f)$ is well defined and is non-atomic. Consequently, if ηf is Borel measurable, then

$$\nu(H(x,y)) = (\eta f)_{\#}\lambda([0,y)) + \lambda\big([0,x] \cap (\eta f)^{-1}(y)\big),$$

where $H(x,y)$ is the half-ray for $\eta f(x) = y$.

We are now ready to construct the measure preserving function h. In our construction we shall assume that f is Borel measurable.

Define a ν-measurable map h_0 from $\mathrm{Graph}(\eta f)$ to $\mathrm{Graph}(g)$ as follows. For each y in $g[Z]$, define I_y to be the characteristic function of $(\eta f)^{-1}(y)$. If $x \in (\eta f)^{-1}(y)$ and $g^{-1}(y)$ is a closed subset of Z, define $h_0(x,y)$ to be (z,y), where $z = a(y) + \int_0^x I_y\, dt$. For the contrary case of $g^{-1}(y) = [a(y), b(y))$ and $\lambda((\eta f)^{-1}(y)) = b(y) - a(y) > 0$ the definition of $h_0(x,y)$ is a slight modification of the above. That is, if $x \in (\eta f)^{-1}(y)$, define $h_0(x,y)$ as before whenever $b(y) - a(y) > \int_0^x I_y\, dt$ and define $h_0(x,y)$ to be $(a(y), y)$ whenever $b(y) - a(y) \le \int_0^x I_y\, dt$. As $\lambda(\{\, x \in (\eta f)^{-1}(y)\colon b(y) - a(y) \le \int_0^x I_y\, dt \,\}) = 0$, the above defined h_0 maps into $\mathrm{Graph}(g)$. The ν-measurability of h_0 is easily shown by the statements 1 and 2 above. Moreover, $h_0[(\eta f)^{-1}(y) \times y] \subset g^{-1}(y) \times y$.

Observe that if $f = g$, where $g \in \mathcal{G}$, then the function that corresponds to f in $\mathcal{G}$ is g itself and the constructed h_0 is the identity map.

Let us return to the linearly ordered space $(\mathrm{Graph}(g), \prec_g)$. As this space is linearly isomorphic to $(Z,<)$ there is a natural "Lebesgue" measure λ_0 on it generated by $\lambda_0(\{\, (z', g(z'))\colon (z', g(z')) \prec_g (z, g(z)) \,\}) = \lambda([0,z)) = z$. Let us verify $\lambda_0 = (h_0)_{\#}\nu$. To this end, for $y_0 = g(z_0)$, we exhibit the mutually exclusive and exhaustive cases of $M(z_0) = {h_0}^{-1}\big[\{\, (z, g(z))\colon (z, g(z)) \prec_g (z_0, y_0) \,\}\big]$. Denoting the set $\{\, (x, \eta f(x))\colon \eta f(x) < y_0 \,\}$ by $N(y_0)$, we have

[6] Observe that the 2 topologies on $\mathrm{Graph}(g)$ induced by $\prec_g$ and induced by the product topology of $[0,1]^2$ will be the same if and only if g is continuous as well as nondecreasing. We have not assumed that g is continuous.

[7] ψ maps Borel subsets of $[0,1]$ to usual Borel sets of $[0,1]^2$.

1. If $z_0 = a(y_0)$, then $M(z_0) = N(y_0)$.
2. If $a(y_0) < z_0 \le b(y_0)$ and $b(y_0) \in g^{-1}(y_0)$, then

$$M(z_0) = N(y_0) \cup \{\, (x, \eta f(x)) \colon \eta f(x) = y_0,\ a(y_0) + \textstyle\int_0^x I_{y_0}\, dt < z_0 \,\}.$$

3. If $a(y_0) < z_0 \le b(y_0)$ and $b(y_0) \notin g^{-1}(y_0)$, then $z_0 \neq b(y_0)$ and

$$\begin{aligned} M(z_0) = N(y_0) &\cup \{\, (x, \eta f(x)) \colon \eta f(x) = y_0,\ a(y_0) + \textstyle\int_0^x I_{y_0}\, dt < z_0 \,\} \\ &\cup \{\, (x, \eta f(x)) \colon \eta f(x) = y_0,\ a(y_0) + \textstyle\int_0^x I_{y_0}\, dt \ge b(y_0) \,\}. \end{aligned}$$

As $\nu(N(y_0)) = a(y_0)$, we have for each case that $\nu(M(z_0)) = z_0$, whence $\lambda_0 = h_{0\#}\nu$. It is easily seen that $h = \pi h_0 \psi$ is a measurable function such that $\lambda = h_\# \lambda$.

Note that $h\big[(\eta f)^{-1}(y)\big] \subset g^{-1}(y)$ whenever $y \in g[Z]$, whence $\eta f = gh$. As $f = \eta f$ almost everywhere, the promised h has been constructed.

Proposition 6. *If f is a Baire class α function, then the above constructed h is a Baire class $\alpha + 3$ function with $\eta f = gh$ everywhere.*

PROOF. If $f\colon [0,1] \to [0,1]$ is a function of Baire class α, then ηf is a function of Baire class $\alpha + 1$. Employing the bijection $\psi\colon [0,1] \to \mathrm{Graph}(\eta f)$, we infer from the three cases above that $h^{-1}\big[[0,z)\big]$ is the finite union of Borel sets of additive class $\alpha + 2$. Hence $h^{-1}\big[[0,z]\big]$ is a Borel set of multiplicative class $\alpha + 3$. From this we infer that h is a function of Baire class $\alpha + 3$. □

Define $\mathcal{G}^*$ to be the set of all functions $g\colon [0,1] \to \mathbb{R}$ such that $g(0) = g(1) = 0$ and such that g restricted to $(0,1)$ is nondecreasing and continuous-from-above. As every Lebesgue measurable function $f\colon [0,1] \to \mathbb{R}$ is equal almost everywhere to a Baire class 2 function, Theorem 5 and the observation that follows it yield the next theorem.

Theorem 7. *$\mathcal{G}^*$ is a minimal set having the property that for each Lebesgue measurable function $f\colon [0,1] \to \mathbb{R}$ there is an h in $\mathcal{H}$ and there is a g in $\mathcal{G}^*$ such that $f = gh$ almost everywhere.*

The factorization problem is posed in the context of almost everywhere convergence of a sequence of permutations to the function h. The following question remains.

Question. If h is measure preserving then does there exist a λ-equivalent H such that H is the everywhere convergent limit of a sequence of permutations? Clearly, such an H is a Baire class 2 function.

Remark 8. The construction of a nondecreasing function, called a *distribution function*, was known to Hardy and Littlewood for measurable functions defined on the open interval $(0, 1)$. That is, for each real-valued measurable function f on $(0, 1)$, there corresponds a nondecreasing real-valued function g on $(0, 1)$ that is continuous-from-above such that $f_{\#}\lambda((0, y]) = g_{\#}\lambda((0, y])$ for every y. As $\mathbb{R}$ and $(0, 1)$ are order isomorphic, there is no loss in assuming f and g map into $(0, 1)$. Our function in Section 3 is related to the function constructed by the Hardy-Littlewood rearrangement method (see [3, pages 91–92], [5, pages 29–30], and [4, page 272]). Indeed, we infer from their result that each Lebesgue measurable function $f\colon [0, 1] \to (0, 1)$ corresponds to a nondecreasing, continuous-from-above distribution function $g\colon [0, 1] \to [0, 1]$ with $g^{-1}[\{0, 1\}] \subset \{0, 1\}$. Simply restrict f to the open interval $(0, 1)$ and adjust the resulting Hardy-Littlewood distribution function that is defined on $(0, 1)$ to the closed interval $[0, 1]$ in the obvious way.

References

[1] S. Alpern and V. S. Prasad, *Typical Dynamics of Volume Preserving Homeomorphims*, Cambridge Tracts in Mathematics No. 139, Cambridge University Press, Cambridge, 2000.

[2] R. F. Gariepy and W. P. Ziemer, *Modern Real Analysis*, PWS Publishing Company, Boston, 1995.

[3] G. H. Hardy and J. E. Littlewood, *A maximal theorem with function-theoretic applications*, Acta Math. **54(1)** (1930), 81–116.

[4] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Mathematical Series No. 30, Princeton University Press, Princeton, 1970.

[5] A. Zygmund, *Trigonometric Series*, second ed., Cambridge University Press, Cambridge, 1977, (Combined volumes I and II).