

Richárd Balka,* Institute of Mathematics and Informatics, Eszterházy
Károly College, Leányka str. 4., H-3300 Eger, Hungary. email:
balkar@cs.elte.hu

DUALITY BETWEEN MEASURE AND CATEGORY IN UNCOUNTABLE LOCALLY COMPACT ABELIAN POLISH GROUPS

Abstract

We show that there is no addition preserving Erdős-Sierpiński mapping on any uncountable locally compact abelian Polish group. This generalizes results of Bartoszyński and Kysiak.

1 Introduction

Let G be a locally compact abelian (LCA) Polish group. Let \mathcal{M} and \mathcal{N} be the ideals of meager and null (with respect to Haar measure) subsets of G .

Definition 1.1. A bijection $F: G \rightarrow G$ is called an *Erdős-Sierpiński mapping* if

$$X \in \mathcal{N} \Leftrightarrow F[X] \in \mathcal{M} \quad \text{and} \quad X \in \mathcal{M} \Leftrightarrow F[X] \in \mathcal{N}.$$

Theorem 1.2. (*Erdős-Sierpiński*) *Assume the Continuum Hypothesis. Then there exists an Erdős-Sierpiński mapping on \mathbb{R} .*

The existence of such a function is independent from ZFC. Our main question is the following:

Is it consistent that there is an Erdős-Sierpiński mapping F that preserves addition, namely

$$\forall x, y \in G \quad F(x + y) = F(x) + F(y)?$$

Mathematical Reviews subject classification: Primary: 03E15, 22B05; Secondary: 28A99
Key words: measure, category, duality, Erdős-Sierpiński mapping, LCA groups

Received by the editors January 25, 2009

Communicated by: Krzysztof Chris Ciesielski

*Partially supported by Hungarian Scientific Foundation grant no. 72655.

This question is attributed to Ryll-Nardzewski in the case $G = \mathbb{R}$. Besides intrinsic interest, another motivation was the following: If this statement were consistent then the so called strong measure zero and strongly meager sets would consistently form isomorphic ideals. (For the definitions see [1].)

First Bartoszyński gave a negative answer to the question in the case $G = 2^\omega$, see [2], then Kysiak proved this for $G = \mathbb{R}$ and answered the question of Ryll-Nardzewski, see [3], where he used and improved Bartoszyński's idea. We answer the general case, the goal of this article is to prove the following theorem:

Theorem 1.3. (*Main Theorem*) *There is no addition preserving Erdős-Sierpiński mapping on any uncountable locally compact abelian Polish group.*

Let $(\varphi_{\mathcal{M}})$ denote the following statement (considered by Carlson in [1]): For every $F \in \mathcal{M}$ there exists a set $F' \in \mathcal{M}$ such that

$$\forall x_1, x_2 \in G \exists x \in G \quad (F + x_1) \cup (F + x_2) \subseteq F' + x.$$

Let $(\varphi_{\mathcal{N}})$ be the dual statement obtained by replacing \mathcal{M} by \mathcal{N} . If there exists an Erdős-Sierpiński mapping preserving addition then $(\varphi_{\mathcal{M}})$ and $(\varphi_{\mathcal{N}})$ are equivalent. First we show that $(\varphi_{\mathcal{M}})$ holds in LCA Polish groups. In the second part we begin to show that $(\varphi_{\mathcal{N}})$ fails for all uncountable LCA Polish groups by reducing the general case to three special cases. Finally, in part three we settle these three special cases.

2 $(\varphi_{\mathcal{M}})$ holds for all LCA Polish groups

As the known proofs only work for the reals, we had to come up with a new, topological proof.

Notation 2.1. Let X be a metric space, $x \in X$ and $r > 0$. Let $B(x, r)$ denote the closed ball of radius r centered at the point x .

Lemma 2.2. *Let X be a metric space and $K \subseteq X$ a nowhere dense compact set. Then there exists a function $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that for all $x \in X$ and $r > 0$ there exists $y \in X$ such that $B(y, f(r)) \subseteq B(x, r) \setminus K$.*

PROOF. Suppose towards a contradiction that $r > 0$ is such a number that there exists a sequence $r_n \rightarrow 0$ ($n \in \mathbb{N}$) and a set $\{x_n \in X : n \in \mathbb{N}\}$ such that for all $y \in X$ and $n \in \mathbb{N}$

$$B(y, r_n) \not\subseteq B(x_n, r) \setminus K. \quad (2.1)$$

$r_n < r$ holds for large enough n , so in the case $y = x_n$ we get that for large enough n there exist $z_n \in B(x_n, r_n) \cap K$. By the compactness of K there exists a convergent subsequence $\lim_{k \rightarrow \infty} z_{n_k} = z \in K$, and so $\lim_{k \rightarrow \infty} x_{n_k} = z \in K$. There is an $N \in \mathbb{N}$ such that $x_{n_k} \in B(z, \frac{r}{2})$ holds for all $k > N$. Then $B(z, \frac{r}{2}) \subseteq B(x_{n_k}, r)$, so by (2.1) for all $k > N$ and $y \in B(z, \frac{r}{2})$ we get $B(y, r_{n_k}) \not\subseteq B(z, \frac{r}{2}) \setminus K$, which contradicts that K is nowhere dense, and we are done. \square

Every abelian Polish group admits a compatible invariant complete metric, because it admits a compatible invariant metric by [[8], Thm. 7.3.], and a compatible invariant metric is automatically complete by [[8], Lem. 7.4.]. So we may assume that the metric on our group is invariant.

Lemma 2.3. *Let G be an abelian Polish group and $K \subseteq G$ a nowhere dense compact set. Then there exists a function $l: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that for all $x, x_1, x_2 \in G$ and $r > 0$ there is a $y \in G$ such that*

$$B(y, l(r)) \subseteq B(x, r) \setminus ((K + x_1) \cup (K + x_2)).$$

PROOF. It is easy to see, using the invariance of our metric, that $l = f \circ f$ works, where f is defined in the previous lemma. \square

We may assume that $l(r) < r$ for all $r > 0$. Now we prove (φ_M) .

Theorem 2.4. *Let S be a meagre set in an LCA Polish group G . Then there is a meagre set $T \subseteq G$ such that for all $s_1, s_2 \in G$ there is a $t \in G$ such that $(S + s_1) \cup (S + s_2) \subseteq T + t$.*

PROOF. We may assume by local compactness (by taking closures of the nowhere dense subsets and decomposing each of them to countably many compact sets) that $S = \cup_{n \in \mathbb{N}} S_n$, where the S_n 's ($n \in \mathbb{N}$) are nowhere dense compact sets. The idea is that we construct nowhere dense T_n 's for the S_n 's simultaneously, and T will be the union of the T_n 's. Later we will set $T_n = \cap_{k \in \mathbb{N}} T_n^k$ for some open sets T_n^k , and we will simultaneously construct a decreasing sequence of closed balls $B(x_k, r_k)$ such that $t = t_{s_1, s_2}$ will be found as $\cap_{k \in \mathbb{N}} B(x_k, r_k)$.

Let $h: \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$ be an enumeration of $\mathbb{N} \times \mathbb{N}$, let $h_1, h_2: \mathbb{N} \rightarrow \mathbb{N}$ be the first and second coordinate functions of h and let $\{g_n: n \in \mathbb{N}\}$ be a dense set in G . Let l_n be the function in the previous lemma for the compact set S_n . We define T by induction, independently from s_1, s_2 . Let $x_0 \in G$ be arbitrary and $r_0 > 0$ so small that $B(x_0, r_0)$ is compact (we can do this by local compactness of G), and $r_k = \frac{l_{h_1(k)}(r_{k-1})}{2}$ if $k > 0$ and $n \in \mathbb{N}$. Let

$$T_n^k = \begin{cases} G & \text{if } n \neq h_1(k), \\ G \setminus B(g_{h_2(k)}, r_k) & \text{if } n = h_1(k). \end{cases} \quad (2.2)$$

The key step of the proof is the following lemma.

Lemma 2.5. *Assume that x_0, x_1, \dots, x_{k-1} are already defined. Then there exists $x_k \in G$ (which depends on s_1 and s_2) such that $B(x_k, r_k) \subseteq B(x_{k-1}, r_{k-1})$ and for all $n \in \mathbb{N}$ and $t \in B(x_k, r_k)$*

$$(S_n + s_1) \cup (S_n + s_2) \subseteq T_n^k + t. \quad (2.3)$$

PROOF. (of the Lemma) If $n = h_1(k)$, then $T_n^k = G \setminus B(g_{h_2(k)}, r_k)$. Using $2r_k = l_{h_1(k)}(r_{k-1}) = l_n(r_{k-1})$ and Lemma 2.3 there exists $y_k \in B(x_{k-1} + g_{h_2(k)}, r_{k-1})$ such that

$$B(y_k, 2r_k) \subseteq B(x_{k-1} + g_{h_2(k)}, r_{k-1}) \setminus ((S_n + s_1) \cup (S_n + s_2)). \quad (2.4)$$

Let us define $x_k := y_k - g_{h_2(k)}$. Using the definition of x_k and (2.4) we get that

$$B(x_k, r_k) = B(y_k, r_k) - g_{h_2(k)} \subseteq B(y_k, 2r_k) - g_{h_2(k)} \subseteq B(x_{k-1}, r_{k-1}).$$

We will use the following easy equation, where the first $+$ is the Minkowski sum.

$$B(g_{h_2(k)}, r_k) + B(x_k, r_k) \subseteq B(g_{h_2(k)} + x_k, 2r_k). \quad (2.5)$$

Using (2.4) again, the definition of x_k and (2.5) in this order we get that for all $t \in B(x_k, r_k)$

$$\begin{aligned} (S_n + s_1) \cup (S_n + s_2) &\subseteq G \setminus B(y_k, 2r_k) = G \setminus B(g_{h_2(k)} + x_k, 2r_k) \\ &\subseteq G \setminus B(g_{h_2(k)}, r_k) + t. \end{aligned}$$

Hence

$$(S_n + s_1) \cup (S_n + s_2) \subseteq T_n^k + t,$$

so (2.3) holds for $n = h_1(k)$. If $n \neq h_1(k)$ then $T_n^k = G$ and (2.3) is obvious, so we are done. \square

Now we return to the proof of Theorem 2.4. By the compactness of $B(x_0, r_0)$ the closed sets $B(x_k, r_k)$ are compact, so the intersection of decreasing sequence of compact sets $\cap_{k \in \mathbb{N}} B(x_k, r_k) \neq \emptyset$. Let

$$t_{s_1, s_2} \in \cap_{k \in \mathbb{N}} B(x_k, r_k)$$

be the common shift. By (2.3) and the definition of t_{s_1, s_2} ,

$$(S_n + s_1) \cup (S_n + s_2) \subseteq T_n^k + t_{s_1, s_2} \quad (2.6)$$

holds for all $k, n \in \mathbb{N}$. For every $n \in \mathbb{N}$ the set $T_n = \bigcap_{k \in \mathbb{N}} T_n^k$ is nowhere dense by (2.2). By (2.6) we easily get for every $n \in \mathbb{N}$

$$(S_n + s_1) \cup (S_n + s_2) \subseteq T_n + t_{s_1, s_2}.$$

The set $T = \bigcup_{n \in \mathbb{N}} T_n$ is meagre, and clearly

$$(S + s_1) \cup (S + s_2) \subseteq T + t_{s_1, s_2},$$

and the proof is complete. \square

3 Reduction to \mathbb{T} , \mathbb{Z}_p and $\prod_{n \in \mathbb{N}} G_n$

In this section we reduce the general uncountable LCA Polish groups to some special groups. We follow the strategy developed in [5].

Definition 3.1. Let us say that an LCA Polish group G is *nice* if (φ_N) fails in G , that is, there is a nullset N , such that for every nullset N' there are $x_1, x_2 \in G$ such that $(N + x_1) \cup (N + x_2) \not\subseteq (N' + x)$ for all $x \in G$.

Lemma 3.2. *If an LCA Polish group G has a nice open subgroup U then G is nice.*

PROOF. Let μ be the Haar measure of the LCA Polish subgroup U and $N \subseteq U$ be the set that witnesses that U is nice. Because of U is open and G is separable, we can write a disjoint countable decomposition $G = \bigcup_{n=0}^{\infty} (U + g_n)$. It is easy to see that $\nu(B) = \sum_{n=0}^{\infty} \mu((B - g_n) \cap U)$ is a Haar measure on G . We show that the ν nullset $N_G = \bigcup_{n=0}^{\infty} (N + g_n)$ is a witness that G is nice. Let $\nu(N'_G) = 0$. Since N witnesses that U is nice, there are $u_1, u_2 \in U$ such that

$$((N + u_1) \cup (N + u_2)) \cap ((N'_G \cap U) + u)^c \neq \emptyset \text{ for all } u \in U. \quad (3.1)$$

Suppose towards a contradiction that there exists a $g \in G$ such that $(N_G + u_1) \cup (N_G + u_2) \subseteq (N'_G + g)$. There exists $i \in \mathbb{N}$ such that $g - g_i \in U$. Then

$$(N + u_1) \cup (N + u_2) \subseteq (N'_G + (g - g_i)) \cap U = (N'_G \cap U) + (g - g_i)$$

where $g - g_i \in U$, a contradiction. \square

Lemma 3.3. *Assume that G is an LCA Polish group, $H \subseteq G$ is a compact subgroup and G/H is nice. Then G is nice, too.*

PROOF. If μ is a Haar measure on G , $\pi: G \rightarrow G/H$ is the canonical homomorphism, then $\mu_G \circ \pi^{-1}$ is a Haar measure on G/H by [[4], 63. Thm. C.]. So the inverse image of a nullset in G/H under π is a nullset in G . Hence if $N \subseteq G/H$ is a nullset witnessing that G/H is nice then $\pi^{-1}(N) \subseteq G$ is a nullset witnessing that G is also nice since the translated copies of $\pi^{-1}(N)$ are composed of cosets of H , so if we would like to cover it with a translation of a set N' , then we may assume without loss of generality, that N' consists of cosets, too. It is easy to see that $\pi(N')$ shows that N is not a witness, a contradiction. \square

Now we start reducing the problem to simpler groups.

Definition 3.4. Let G be a group and p be a prime number, $G_{p^n} = \{g \in G : p^n g = 0\}$ for every $n \in \mathbb{N}$, and also let $G_{p^\infty} = \bigcup_{n \in \mathbb{N}} G_{p^n}$. We say that G is a p -group, if $G = G_{p^\infty}$.

Definition 3.5. Let p be a prime number. An abelian group G is called *quasicyclic* if it is generated by a sequence $\langle g_n \rangle_{n \in \mathbb{N}}$ with the property that $g_0 \neq 0$ and $pg_{n+1} = g_n$ for every $n \in \mathbb{N}$. For a fixed prime p the unique (up to isomorphism) quasicyclic group is denoted by C_{p^∞} .

Notation 3.6. We denote by \mathbb{T} the circle group, by \mathbb{Z}_p the group of p -adic integers for every prime p and by $\prod_{n \in \mathbb{N}} G_n$ the product of the finite abelian groups G_n 's.

Remark 3.7. \mathbb{Z}_p is the topological space $\{0, 1, \dots, p-1\}^\omega$ equipped with the product topology. Addition is coordinate-wise with carried digits from the n^{th} coordinate to the $n+1^{st}$. Both \mathbb{Z}_p and $\prod_{n \in \mathbb{N}} G_n$ are Polish with the product topology.

We will use the following theorem from [5].

Theorem 3.8. *Every infinite abelian group contains a subgroup isomorphic to one of the following:*

- \mathbb{Z} ,
- C_{p^∞} for some prime p ,
- $\bigoplus_{n \in \mathbb{N}} G_n$, where each G_n is a finite abelian group of at least two elements.

Theorem 3.9. \mathbb{T} , \mathbb{Z}_p , and $\prod_{n \in \mathbb{N}} G_n$ are nice.

PROOF. We postpone the proof to the next section. \square

Theorem 3.10. Every uncountable LCA Polish group is nice, that is, (φ_N) fails for every uncountable LCA Polish group.

PROOF. Let G be an LCA Polish group. From the Principal Structure Theorem of LCA groups [[6], 2.4.1] follows that G has an open subgroup H which is of the form $H = K \otimes \mathbb{R}^n$, where K is a compact subgroup and $n \in \mathbb{N}$. G is uncountable, so non-discrete and H is open, therefore H is a non-discrete, so uncountable group. By Lemma 3.2 it is enough to prove that H is nice, so we can assume $G = H$.

Assume that $n \geq 1$. \mathbb{R} is nice by [3], let N be a nullset witnessing this fact. It is obvious that $K \times N \times \mathbb{R}^{n-1}$ witnesses that $G = K \otimes \mathbb{R}^n$ is nice. So we may assume $n = 0$, and we get that G is compact. It is enough to find a closed subgroup $H \subseteq G$ such that G/H is nice by the previous lemma. Using [[6], 2.1.2] and the Pontryagin Duality Theorem, see [[6], 1.7.2]), we get that factors of G are the same as (isomorphically homeomorphic to) dual groups of closed subgroups of \widehat{G} . If G is compact, then \widehat{G} is discrete, see [[6], 1.2.5]. So it is enough to find a subgroup $M \subseteq \widehat{G}$ such that \widehat{M} is nice.

From the previous theorem follows (using that $G = K$ is an infinite group) that \widehat{G} has a subgroup isomorphic either to \mathbb{Z} , or to C_{p^∞} for some prime p , or to $\bigoplus_{n \in \mathbb{N}} G_n$ (where each G_n is a finite abelian group of at least two elements). It is sufficient to show that the duals of these groups are nice. It is well-known that $\widehat{\mathbb{Z}} = \mathbb{T}$, and \mathbb{T} is nice, see [3].

By [[6], 2.2.3] the dual of a direct sum (equipped with the discrete topology) is the direct product of the dual groups (equipped with the product topology), so $\widehat{(\bigoplus_{n \in \mathbb{N}} G_n)} = \prod_{n \in \mathbb{N}} \widehat{G_n}$. If G_n is finite so is $\widehat{G_n}$, so we are done with the second case.

Finally, $\widehat{C_{p^\infty}} = \mathbb{Z}_p$, see [[7], 25.2]. Hence we are done by the previous theorem. \square

4 (φ_N) Fails For \mathbb{T} , \mathbb{Z}_p and $\prod_{n \in \mathbb{N}} G_n$

We will prove that \mathbb{T} , \mathbb{Z}_p and $\prod_{n \in \mathbb{N}} G_n$ are nice. \mathbb{T} is nice, see [3], so we need to handle the last two cases. Our proofs are very similar to the Main Theorem of Kysiak's paper, see [[3], Main Thm. 4.3.], so we only describe the necessary modifications. Hence in order to follow our proof one has to read Kysiak's paper parallelly. Unless stated otherwise, the numbered references in

the following proofs refer to the present paper, and the page numbers refer to [3].

Theorem 4.1. \mathbb{Z}_p is nice.

PROOF. We use the description of \mathbb{Z}_p that can be found in Remark 3.7. We always write \mathbb{Z}_p instead of $(0, 1]$. Let μ be the Haar measure on \mathbb{Z}_p , and let $\mu_2 = \mu \times \mu$ be the Haar measure on \mathbb{Z}_p^2 . The proof of Lemma 4.7. in [3] is the same with these modifications.

For the construction of I_n 's we fix a partition of the set ω instead of $\omega \setminus \{0\}$. For the inequalities we write p^{I_n} and $p^{|I_n|}$ instead of 2^{I_n} and $2^{|I_n|}$. With these changes the following inequality holds.

$$\frac{1}{p^{|I_n|}} \leq \frac{1}{12} \left(\frac{1}{n^2} - \frac{1}{n^5} \right) \text{ for } n > 1. \quad (4.1)$$

Let $J_n \subseteq p^{I_n}$ such that

$$1 - \frac{1}{n^2} + \frac{1}{n^5} > \frac{|J_n|}{p^{|I_n|}} > 1 - \frac{1}{n^2}, \quad (4.2)$$

and J_n consist of the first $|J_n|$ consecutive elements of p^{I_n} with respect to the antilexicographical ordering (sequences are ordered according to the largeness of the rightmost coordinate where they differ).

Let \mathcal{N}^* be the filter of full measure sets of \mathbb{Z}_p . If $I \subseteq \omega$ and $J \subseteq p^I$ then $[J]$ denote the set $\{x \in \mathbb{Z}_p : x \upharpoonright I \in J\}$. We write $p^{<\omega}$ instead of $2^{<\omega}$ and for $s \in p^{<\omega}$ let $[s]$ be the set $\{x \in \mathbb{Z}_p : x \text{ extends } s\}$.

For the definition of C (at the end of the page 274) we omit the $C \subseteq \mathbb{R} \setminus \mathbb{Q}$ condition. We may assume about C the following: for every $s \in p^{<\omega}$ we have $[s] \cap C = \emptyset$ or $\mu([s] \cap C) > 0$ (if not, consider $C' = C \setminus \cup\{[s] : s \in p^{<\omega} \text{ and } \mu([s] \cap C) = 0\}$ instead).

The inductive definition and its proof (beginning on page 275) is almost the same, too. Write $p^{I_1 \cup \dots \cup I_l}$ and $p^{(\max I_l, \infty)}$ instead of $2^{I_1 \cup \dots \cup I_l}$ and $2^{(\max I_l, \infty)}$. Let 0^n be the element of $p^{\{0, \dots, n-1\}}$ with zero coordinates. For the proof of

$$\mu([r_s] \cap C) > \frac{t-1}{t} \mu([r_s]) \quad (4.3)$$

(that can be found at the end of the page 275) we use (instead of the Lebesgue density theorem) the following more general density theorem.

Definition 4.2. Let X be a topological group. A *B-sequence* in X is a non-increasing sequence $\langle V_n \rangle_{n \in \mathbb{N}}$ of closed neighborhoods of the identity, constituting a base of neighborhoods of the identity, such that there is some $M \in \mathbb{N}$

such that for every $n \in \mathbb{N}$ the set $V_n - V_n$ can be covered by at most M translates of V_n .

Theorem 4.3. *Let X be a topological group with a left Haar measure μ , and $\langle V_n \rangle_{n \in \mathbb{N}}$ a B-sequence in X . Then for any Haar measurable set $E \subseteq X$,*

$$\lim_{n \rightarrow \infty} \frac{\mu(E \cap (x + V_n))}{\mu(V_n)} = \chi_E(x)$$

for almost every $x \in X$.

For the proof of Theorem 4.3 and for Definition 4.2 see [[9], 447D Thm.] and [[9], 446L Def.]. It is easy to see that $V_n = [0^{n+1}]$ ($n \in \mathbb{N}$) form a B-sequence, and (4.3) follows by applying Theorem 4.3 for $\langle V_n \rangle_{n \in \mathbb{N}}$.

Remark 4.4. For every countable product space of discrete groups $\prod_{n \in \mathbb{N}} X_n$ the sets $V_n = [\{e_0, \dots, e_n\}]$ ($n \in \mathbb{N}$) form a B-sequence (where e_i is the neutral element of X_i), so (4.3) is valid, too.

We follow the proof till the definition of U (at the end of the page 277), then we deviate from it. We jump to the formula of the following lemma that is at the end of the proof in [3] (in the middle of the page 279). For the following lemma we need a new idea, Kysiak's arguments are not applicable here, he uses a specific relation between the metric and the addition in \mathbb{T} .

Lemma 4.5. *For every $k \in U$ there is a $v_k \in [J'_{n_k}]$ such that*

$$z \upharpoonright I_{n_k} = v_k \upharpoonright I_{n_k} \Rightarrow z \notin (x - y) + [J_{n_k}].$$

PROOF. (of the Lemma) Assume $k \in U$. Remember that $\varepsilon_{n_k} = \frac{1}{4}\lambda_{n_k}$, and as a consequence of (4.2) $\lambda_{n_k} = 1 - \mu(J_{n_k}) \geq \frac{1}{n_k^2} - \frac{1}{n_k^5}$. Using the definition of U , the previous facts and (4.1) in this order we get

$$\begin{aligned} \mu([J'_{n_k}] \setminus ((x - y) + [J_{n_k}])) &= \mu((y + [J'_{n_k}]) \setminus (x + [J_{n_k}])) \\ &\geq \frac{\lambda_{n_k} - \varepsilon_{n_k}}{2} > \frac{\lambda_{n_k}}{12} \geq \frac{1}{12} \left(\frac{1}{n_k^2} - \frac{1}{n_k^5} \right) \geq \frac{1}{p^{|I_{n_k}|}}. \end{aligned}$$

Hence

$$\mu([J'_{n_k}] \setminus ((x - y) + [J_{n_k}])) > \frac{1}{p^{|I_{n_k}|}}. \quad (4.4)$$

Let us define

$$L_k := \{((x - y) + [J_{n_k}]) \upharpoonright I_{n_k}\} \subseteq p^{I_{n_k}}. \quad (4.5)$$

Since the elements of J_{n_k} are consecutive (with respect to the antilexicographical ordering), we get $|L_k| \leq |J_{n_k}| + 1$. Using this inequality and the translation invariance of μ

$$\mu((x - y) + [J_{n_k}]) = \mu([J_{n_k}]) \geq \mu([L_k]) - \frac{1}{p^{|I_{n_k}|}}. \quad (4.6)$$

Using $(x - y) + [J_{n_k}] \subseteq [L_k]$, (4.6) and finally (4.4) we get

$$\mu([J'_{n_k}] \setminus [L_k]) \geq \mu([J'_{n_k}] \setminus ((x - y) + [J_{n_k}])) - \frac{1}{p^{|I_{n_k}|}} > 0.$$

Hence there is a $v_k \in [J'_{n_k}] \setminus [L_k] \neq \emptyset$, and it means by the definition of L_k that

$$z \upharpoonright I_{n_k} = v_k \upharpoonright I_{n_k} \Rightarrow z \notin (x - y) + [J_{n_k}],$$

and Lemma 4.5 follows. □

We can simply follow the last few lines of Kysiak's proof after the analogous formula. □

Theorem 4.6. $\prod_{n \in \mathbb{N}} G_n$ is nice.

PROOF. We write \mathbb{N} and $\prod_{n \in \mathbb{N}} G_n$ instead of ω and $(0, 1]$. Let μ be the Haar measure on $\prod_{n \in \mathbb{N}} G_n$, and let $\mu_2 = \mu \times \mu$ be the Haar measure on $(\prod_{n \in \mathbb{N}} G_n)^2$. The proof of Lemma 4.7. in [3] is the same.

For the construction of I_n 's we fix a partition of the set \mathbb{N} instead of $\omega \setminus \{0\}$. Write $\prod_{i \in I_n} G_i$ and $|\prod_{i \in I_n} G_i|$ instead of 2^{I_n} and $2^{|I_n|}$. At the choice of the sets J_n we omit the ordering condition.

Let \mathcal{N}^* be the filter of full measure sets of $\prod_{n \in \mathbb{N}} G_n$. If $I \subseteq \mathbb{N}$ and $J \subseteq \prod_{n \in I} G_n$ then $[J]$ denote the set $\{x \in \prod_{n \in \mathbb{N}} G_n : x \upharpoonright I \in J\}$. We write $\{\prod_{i \in I} G_i : |I| < \infty\}$ instead of $2^{<\omega}$ and for $s \in \{\prod_{i \in I} G_i : |I| < \infty\}$ let $[s]$ be the set $\{x \in \prod_{n \in \mathbb{N}} G_n : x \text{ extends } s\}$.

For the definition of C we omit the $C \subseteq \mathbb{R} \setminus \mathbb{Q}$ condition, and we may assume that for every $s \in \{\prod_{i \in I} G_i : |I| < \infty\}$ we have $[s] \cap C = \emptyset$ or $\mu([s] \cap C) > 0$. (4.3) follows from Remark 4.4 instead of the Lebesgue density theorem.

The inductive definition and its proof need the following modifications. Use $\prod_{i \in I_1 \cup \dots \cup I_l} G_i$ and $\prod_{i=\max I_l}^\infty G_i$ instead of $2^{I_1 \cup \dots \cup I_l}$ and $2^{[\max I_l, \infty)}$.

We follow the proof till the definition of U , then we deviate from it. Note that by the definition of U , for every $k \in U$

$$\mu([J'_{n_k}] \setminus ((x - y) + [J_{n_k}])) = \mu(y + [J'_{n_k}] \setminus ((x + [J_{n_k}])) > 0. \quad (4.7)$$

We jump to the formula of the following lemma that is at the end of the proof in [3]. The following lemma is analogous to Lemma 4.5, but the proof is much easier than in the case \mathbb{Z}_p .

Lemma 4.7. *For every $k \in U$ there is a $v_k \in [J'_{n_k}]$ such that*

$$z \restriction I_{n_k} = v_k \restriction I_{n_k} \Rightarrow z \notin (x - y) + [J_{n_k}].$$

PROOF. (of the Lemma) Let us define

$$L_k := \{((x - y) + [J_{n_k}]) \restriction I_{n_k}\} \subseteq \prod_{i \in I_{n_k}} G_i,$$

then obviously $(x - y) + [J_{n_k}] = [L_k]$. Using (4.7) we get

$$\mu([J'_{n_k}] \setminus [L_k]) = \mu([J'_{n_k}] \setminus ((x - y) + [J_{n_k}])) > 0.$$

So there is a $v_k \in [J'_{n_k}] \setminus [L_k] \neq \emptyset$, and it means by the definition of L_k that

$$z \restriction I_{n_k} = v_k \restriction I_{n_k} \Rightarrow z \notin (x - y) + [J_{n_k}],$$

and Lemma 4.7 follows. □

The last few lines of the proof is the same as in Kysiak's paper after the analogous formula. □

Remark 4.8. Theorem 4.6 could be proved easier based on Bartoszyński's paper [2], but we do not know to generalize Bartoszyński's method to the case \mathbb{Z}_p in Theorem 4.1.

Acknowledgement The author is indebted to Tomek Bartoszyński, Márton Elekes and Sławomir Solecki for some ideas.

References

- [1] T. J. Carlson, *Strong measure zero and strongly meager sets*, Proc. Amer. Math. Soc., **118**(2) (1993), 577–586.
- [2] T. Bartoszyński, *A note on duality between measure and category*, Proc. Amer. Math. Soc., **128**(9) (2000), 2745–2748.
- [3] M. Kysiak, *Another note on duality between measure and category*, Bull. Pol. Acad. Sci. Math., **51**(3) (2003), 269–281.

- [4] P. R. Halmos, *Measure theory*, Springer, 1974.
- [5] M. Elekes and Á. Tóth, *Covering locally compact groups by less than 2^ω many translates of a compact nullset*, Fund. Math., **193(3)** (2007), 243–257.
- [6] W. Rudin, *Fourier analysis on groups*, John Wiley & Sons, 1990.
- [7] E. Hewitt and K. A. Ross, *Abstract harmonic analysis*, Vol. I., Springer, 1963.
- [8] G. Hjorth, *Classification and orbit equivalence relations*, American Mathematical Society, Providence, RI, 2000.
- [9] D. H. Fremlin, *Measure Theory*, Vol. 4, Torres Fremlin, 2003.