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MULTIPLYING BALLS IN $C^{(N)}[0,1]$

Abstract

Let $C^{(n)}[0,1]$ stand for the Banach space of functions $f:[0,1]\to\mathbb{R}$ with continuous n-th derivative. We prove that if B_1, B_2 are open balls in $C^{(n)}[0,1]$ then the set $B_1\cdot B_2=\{f\cdot g: f\in B_1, g\in B_2\}$ has non-empty interior in $C^{(n)}[0,1]$. This extends the result of [1] dealing with the space of continuous functions on $[0,1]$.

For $n\in\mathbb{N}$, let $C^{(n)}=C^{(n)}[0,1]$ denote the Banach space of functions $f:[0,1]\to\mathbb{R}$ with continuous n-th derivative, equipped with the norm

$$||f||=\max_{0\le i\le n}\max_{x\in[0,1]}|f^{(i)}(x)|.$$

Let us recall that for $f,g\in C^{(n)}$ the inequality

$$||f\cdot g||\le 2^n||f||\cdot||g||$$

holds. For $[a,b]\subset[0,1]$, we also consider the space $C^{(n)}[a,b]$ of all functions $f:[a,b]\to\mathbb{R}$ with continuous n-th derivative, equipped with an analogous norm (the interval $[0,1]$ is replaced by $[a,b]$), and we denote the norm in $C^{(n)}[a,b]$ by $||\cdot||_{[a,b]}$. Let $B(f,r)$ stand for an open ball in $C^{(n)}$ (with center f and radius r), then we denote $B(f,r)|_{[a,b]}=\{g\in C^{(n)}[a,b]: ||g-f||_{[a,b]}<r\}$. If $D,E\subset C^{(n)}$ we write $D\cdot E=\{f\cdot g: f\in D,\, g\in E\}$. Finally, let *int* denote the interior in $C^{(n)}$.

Observe that if $f(x)=x-1/2$, $x\in[0,1]$, then $f^2\notin int(B(f,\frac{1}{2})\cdot B(f,\frac{1}{2}))$ (see [1]). So, analogously as in the space $C[0,1]$, the result of multiplication of two open balls in $C^{(n)}$ need not be an open set. Observe also that if B_1, B_2 are open balls in $C^{(n)}$ and $\Phi: C^{(n)}\times C^{(n)}\to C^{(n)}$ is the operation

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of multiplication, and (for example) there is a function $f \in B_1$ such that $f(x) \neq 0$ for any $x \in [0,1]$, then $\Phi(\{f\} \times B_2)$ is an open set in $C^{(n)}$ (a function $\Phi_f : C^{(n)} \to C^{(n)}$, defined by formula $\Phi_f(g) = f \cdot g$, $g \in C^{(n)}$, is a linear continuous bijection), and therefore $\Phi(B_1 \times B_2) = B_1 \cdot B_2$ is a set with non-empty interior. So, interesting considerations appear in the case when both balls B_1 and B_2 consist only of functions having zeros.

Our main goal is to show that if $B(f,r), B(g,r)$ are open balls in $C^{(n)}$ then $B(f,r) \cdot B(g,r)$ has non-empty interior in $C^{(n)}$. In [1] an analogous theorem was proved for open balls in the space $C[0,1]$ of continuous functions. Here we use a similar method, but the details are different and more difficult. Let us start from the following remark.

Remark 1. *Without loss of generality we may assume that functions f, g are polynomials with disjoint non-empty sets of zeros, and that there is a partition $x_0 = 0 < x_1 < \dots < x_m = 1$ of $[0,1]$ such that*

$$(\forall k \in \{1,\dots,m\})\,(k \textit{ is odd } \Rightarrow (\forall x \in [x_{k-1}, x_k]) \quad f(x) \neq 0) \tag{1}$$

and

$$(\forall k \in \{1,\dots,m\})\,(k \textit{ is even } \Rightarrow (\forall x \in [x_{k-1}, x_k]) \quad g(x) \neq 0)\,. \tag{2}$$

In our further considerations we will need the following:

Lemma 1. *Let $\varphi, h \in C^n$, $x_0 \in [0,1]$, $\varepsilon > 0$ and $|\varphi^{(j)}(x_0) - h^{(j)}(x_0)| < \varepsilon$ for $j = 0,1,\dots,n$. Then the function $k : [0,1] \to \mathbb{R}$ defined by the formula*

$$(\forall x \in [0,1]) \quad k(x) = h(x) + \sum_{j=0}^{n} (\varphi^{(j)}(x_0) - h^{(j)}(x_0)) \frac{(x - x_0)^j}{j!}$$

fulfills the following two conditions:

(i) $k^{(j)}(x_0) = \varphi^{(j)}(x_0)$ for $j = 0,1,\dots,n$;
(ii) $k \in B(h, e\varepsilon)$.

PROOF. Condition (i) is easy to check. We will prove only (ii). Fix $x \in [0,1]$. Then $|x - x_0| \leq 1$, hence we have

$$|k(x) - h(x)| \leq \sum_{j=0}^{n} |\varphi^{(j)}(x_0) - h^{(j)}(x_0)| \frac{|x - x_0|^j}{j!} < \varepsilon \sum_{j=0}^{n} \frac{1}{j!} < \varepsilon e.$$

Fix $p \in \{1, \dots, n\}$. Then we have

$$|k^{(p)}(x) - h^{(p)}(x)| = \left|\sum_{j=p}^{n} (\varphi^{(j)}(x_0) - h^{(j)}(x_0)) \frac{(x-x_0)^{j-p}}{(j-p)!}\right| \le \varepsilon \sum_{j=p}^{n} \frac{1}{(j-p)!} < \varepsilon e.$$

Therefore $||k - h|| < e\varepsilon$. □

Remark 2. *In particular, if* $0 < x_0 < y_0 \le 1$ *and* $\varphi \in B(h, \varepsilon)|_{[0,x_0]}$ *then there exists a function* $k \in B(h, e\varepsilon)|_{[x_0,y_0]}$ *which fulfills condition (i) from Lemma 1. Such a function* k *will be called an extension of* φ *to the interval* $[x_0, y_0]$.

Now we prove a basic lemma used in the proof of our main theorem (compare with Lemma 8 from [1]). By $f(x^+)$, $f(x^-)$ we denote the respective one-sided limits of a function f at a point x.

Lemma 2. *For functions* f, g *fulfilling conditions (1) and (2) (respectively) from Remark 1, the following condition holds:*

$$\left(\underset{\mu>0}{\exists}\right)\left(\underset{\beta_1,\dots,\beta_m>0}{\exists}\right)\left(\underset{\varepsilon\in(0,\mu]}{\forall}\right)\left(\underset{\varphi\in B(f\cdot g,\varepsilon)}{\forall}\right)\left(\underset{\xi,\psi\in C^{(n)}}{\exists}\right)$$
$$\left(\varphi = \xi \cdot \psi,\ \underset{i=1,\dots,m}{\forall}(||f-\xi||_i < \beta_i\varepsilon,\ ||g-\psi||_i < \beta_i\varepsilon)\right),$$

where $||h||_i = ||h||_{[x_{i-1},x_i]}$ *for* $h \in C^{(n)}$ *and* $i = 1, \dots, m$.

PROOF. Our reasoning is divided into m steps. In the i-th step ($i = 1, \dots, m$) we define $\beta_i > 0$ and $\mu_i > 0$. The numbers μ_i will fulfill $\mu_1 > \mu_2 > \dots > \mu_m$. Finally, we will set $\mu = \mu_m$.

Step 1. Let $\mu_1 > 0$. Define $\beta_1 = 2^n||1/f||_1$ (by assumption $f \neq 0$ on $[0, x_1]$). If $\varepsilon \in (0, \mu_1]$ and $\varphi \in B(f \cdot g, \varepsilon)$ then for $f_1 = f$, $g_1 = \varphi/f$ on $[0, x_1]$ we have

$$||f - f_1||_1 = 0,\ ||g - g_1||_1 = ||g - \varphi/f||_1 \le 2^n||1/f||_1||f \cdot g - \varphi||_1 < \beta_1\varepsilon. \quad (3)$$

Of course $f_1 \cdot g_1 = \varphi|_{[0,x_1]}$.

Step 2. Observe that for the function φ from step 1, we have

$$g_1 = \frac{\varphi}{f}|_{[0,x_1]} \in B(g, \beta_1\mu_1)_{[0,x_1]},$$

so, to extend our consideration to $[x_1, x_2]$ we have to modify μ_1 as follows:

$$e\beta_1\mu_1 < \min_{x\in[x_1,x_2]} |g(x)|. \tag{4}$$

Let $\mu_2 \in (0, \mu_1)$, where μ_1 fulfills condition (4) (by assumption, $g \neq 0$ on $[x_1, x_2]$). Fix $\varepsilon \in (0, \mu_2]$ and $\varphi \in B(f \cdot g, \varepsilon)$. Analogous to step 1, define functions f_1, g_1. Of course, condition (3) holds. By Lemma 1 and Remark 2 there exists an extension g_2 of the function g_1 to the interval $[x_1, x_2]$ such that $||g - g_2||_2 < e\beta_1\varepsilon$. Moreover, by (4) we have also that $g_2 \neq 0$ on $[x_1, x_2]$. Now define $f_2 = \varphi/g_2$ on $[x_1, x_2]$. One can easily check that $f_1(x_1) = f_2(x_1)$, $f_1^{(j)}(x_1^-) = f_2^{(j)}(x_1^+)$ for $j = 1, \ldots, n$. Observe that

$$\begin{aligned} ||f_2 - f||_2 &\leq 2^n||1/g_2||_2 \cdot ||\varphi - f \cdot g_2||_2 \\ &\leq 2^n||1/g_2||_2 \cdot ||\varphi - f \cdot g + f \cdot g - f \cdot g_2||_2 \\ &< 2^n||1/g_2||_2 \cdot (\varepsilon + 2^n||f|| \cdot e\beta_1\varepsilon) \\ &= 2^n||1/g_2||_2 \cdot \varepsilon(1 + 2^n e\beta_1||f||). \end{aligned} \tag{5}$$

Since $g_2 \in B(g, e\beta_1\mu_2)_{[x_1,x_2]}$, there exists a number $M_2 = M_2(f, g)$ (depending only on functions f, g) such that $||1/g_2||_2 \leq M_2$. So, we have

$$||f_2 - f||_2 < 2^n M_2\varepsilon(1 + 2^n e\beta_1||f||). \tag{6}$$

Observe that our estimation is independent of φ. Define

$$\beta_2 = \max\{e\beta_1, 2^n M_2(1 + 2^n e\beta_1||f||\}.$$

Then we get $||g_2 - g||_2 < \beta_2\varepsilon$, $||f_2 - f||_2 < \beta_2\varepsilon$ and of course $\varphi|_{[x_{i-1},x_i]} = f_i \cdot g_i$ for $i = 1, 2$.

Step 3. For the function φ from step 2, we have

$$f_2 = \frac{\varphi}{g_2}|_{[x_1,x_2]} \in B(f, \beta_2\mu_2)_{[x_1,x_2]}.$$

We want to extend our consideration to $[x_2, x_3]$, so we have to modify μ_2 as follows:

$$e\beta_2\mu_2 < \min_{x\in[x_2,x_3]} |f(x)|. \tag{7}$$

Let $\mu_3 \in (0, \mu_2)$, where μ_2 fulfills the condition (7) (by assumption we have $f \neq 0$ on $[x_2, x_3]$). Once more fix $\varepsilon \in (0, \mu_2]$ and $\varphi \in B(f \cdot g, \varepsilon)$. Analogous to steps 1 and 2, define functions f_1, g_1, f_2, g_2. Of course, conditions (3), (5), (6) hold. By Lemma 1 and Remark 2, there exists an extension f_3 of the function

f_2 to the interval $[x_2, x_3]$ such that $||f - f_3||_3 < e\beta_2\varepsilon$. Moreover, by (7) we have also that $f_3 \neq 0$ on $[x_2, x_3]$. Now define $g_3 = \varphi/f_3$ on $[x_2, x_3]$. One can easily check that $g_2(x_2) = g_3(x_2)$, $g_2^{(j)}(x_2^-) = g_3^{(j)}(x_2^+)$ for $j = 1, \ldots, n$. Analogous to step 2 we get the following estimation:

$$||g_3 - g||_3 < 2^n ||1/f_3||_3 \cdot \varepsilon(1 + 2^n ||g|| e\beta_2).$$

Observe that the estimation is independent of φ. In a similar fashion, since $f_3 \in B(f, \beta_2\mu_3)_{[x_2,x_3]}$ there exists a number $M_3 = M_3(f, g)$ (M_3 depends only on functions f, g) such that $||1/f_3||_3 \leq M_3$. Define now $\beta_3 = \max\{e\beta_2, 2^n M_3(1 + 2^n ||g|| e\beta_2)\}$. Then we have $||f - f_3||_3 < \beta_3\varepsilon$, $||g - g_3||_3 < \beta_3\varepsilon$ and of course $\varphi|_{[x_{i-1},x_i]} = f_i \cdot g_i$ for $i = 1, 2, 3$.

The next steps are analogous. Continuing in this way we can define the required numbers $\mu_1, \ldots, \mu_m$ (finally, we put $\mu = \mu_m$), $\beta_1, \ldots, \beta_m$, $M_2, \ldots, M_m$, and functions $f_1, \ldots, f_m, g_1, \ldots, g_m$. More precisely, for numbers $\beta_1, \ldots, \beta_m$ we have

$$\beta_1 = 2^n ||1/f||_1 \text{ and } \beta_i = \max\{e\beta_{i-1}, 2^n M_i(1 + 2^n ||g|| e\beta_{i-1})\}$$

if $i \in \{3, \ldots, m\}$ is odd, and

$$\beta_i = \max\{e\beta_{i-1}, 2^n M_i(1 + 2^n ||f|| e\beta_{i-1})\}$$

if $i \in \{1, \ldots, m\}$ is even. We define functions ξ, ψ in an obvious way:

$$\xi = f_i \text{ on } [x_{i-1}, x_i] \text{ for } i = 1, \ldots, m$$

and

$$\psi = g_i \text{ on } [x_{i-1}, x_i] \text{ for } i = 1, \ldots, m.$$

Then $\xi, \psi \in C^{(n)}$, $\varphi = \xi \cdot \psi$ on $[0,1]$ and $||f - \xi||_i < \beta_i\varepsilon$, $||g - \psi||_i < \beta_i\varepsilon$ for $i = 1, \ldots, m$. $\square$

Theorem 1. *If $B(f,r)$ and $B(g,r)$ are open balls in $C^{(n)}$ then their algebraic product $B(f,r) \cdot B(g,r)$ has non-empty interior in $C^{(n)}$.*

PROOF. We may assume that f, g are such functions as in Remark 1. By Lemma 2 there exist positive numbers $\mu, \beta_1, \ldots, \beta_m$ corresponding to f, g. For

$$\varepsilon = \min\left\{\mu, \frac{r}{\max\{\beta_1, \ldots, \beta_m\}}\right\}$$

we shall prove that $B(f \cdot g, \varepsilon) \subset B(f,r) \cdot B(g,r)$. Fix $\varphi \in B(f \cdot g, \varepsilon)$. Since $\varepsilon \leq \mu$ then by Lemma 2 there exist $\xi, \psi \in C^{(n)}$ such that $\varphi = \xi \cdot \psi$ and $||f - \xi||_i < \beta_i\varepsilon$, $||g - \psi||_i < \beta_i\varepsilon$ for $i = 1, \ldots, m$. Since $\varepsilon \cdot \max\{\beta_1, \ldots, \beta_m\} \leq r$ then $||f - \xi||_i < r$ and $||g - \psi||_i < r$. $\square$

In [4, Prop. 1] (see also [3, Th. 3]) we proved the following

Theorem 2. *Let X, Z be topological spaces and let $E \subset X$ be a residual set. If $\Phi : Z \to X$ is a continuous mapping such that the image $\Phi(U)$ is not nowhere dense for any nonempty open set $U \subset Z$, then $\Phi^{-1}(E)$ is a residual set.*

Theorems 1 and 2 immediately imply the following corollary.

Corollary 1. *If $\Phi : C^{(n)} \times C^{(n)} \to C^{(n)}$ is the operation of multiplication, then $\Phi^{-1}(E)$ is a residual set in $C^{(n)} \times C^{(n)}$ whenever $E \subset C^{(n)}$ is residual.*

Remark 3. *It is also worth reminding the reader that Theorem 1 does not hold if we replace the space $C^{(n)}$ by $C^{(n)}([-1,1]^2)$ of all functions $f : [-1,1] \times [-1,1] \to \mathbb{R}$ with continuous all partial derivatives of n-th order. For example, if we define $f(x,y) = x$, $g(x,y) = y$, $(x,y) \in [-1,1]^2$, then $B(f,1) \cdot B(g,1)$ is a nowhere dense set in $C^{(n)}([-1,1]^2)$ (see [2, Th. 2]). We obtain an analogous result replacing the square $[-1,1]^2$ by k-dimensional cube $[-1,1]^k$ $(k > 2)$ or even by the Hilbert cube $[-1,1]^\infty$, and using analogous functions f, g-projections on first and second coordinates, respectively.*

The interval $[-1,1]$ is used here only for simplicity of definitions of f and g. One can give analogous examples for any nondegenerate interval $[a,b]$.

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