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ON OPENNESS OF DENSITY POINTS UNDER MAPPINGS

Abstract

Basu and Ganguly recently proved a theorem connected to the classical theorem of Steinhaus which states that $A-B$ has nonempty interior if A and B are Lebesgue measurable subsets of the real line, each having positive measure. The Basu and Ganguly paper deals with a particular 2-place function, namely $f(x,y)=x/y$. There is nothing special about ratios. We will extend their results to functions satisfying simple conditions on their partial derivatives. An n dimensional analogue is also presented.

1 Introduction

In a recent article [2] Basu and Ganguly proved that if $A_1, A_2 \subseteq \mathbb{R}$ (the real line) are Lebesgue measurable and each has positive Lebesgue measure, then $R[A_1^* : A_2^*]$ is an open set, where $A_i^* = \{x \in \mathbb{R}\backslash\{0\} : A_i \text{ has density 1 at } x\}$ and $R[A_1^* : A_2^*]$ denotes the collection of all numbers $\frac{x}{y}$ and $\frac{y}{x}$ where $x \in A_1^*$, $y \in A_2^*$. This result is a contribution to the collection of analogues and extensions of a theorem of Steinhaus [5], that goes back to 1920. The purpose of this note is to show that there is nothing special about ratios and that analogous statements in $\mathbb{R}$ and $\mathbb{R}^n$ hold for 2-place functions satisfying simple conditions on their partial derivatives. That the Steinhaus theorem can be extended using general functions is not a new idea – for example see [1], [3] and [4].

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2 Results

We first consider the $n = 1$ dimensional extension of the result of Basu and Ganguly mentioned in the introduction.

Theorem 1. *Suppose that* $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, A *and* B *are Lebesgue measurable subsets of* $\mathbb{R}$ *and* $x_0 \in \widetilde{A}$, $y_0 \in \widetilde{B}$ *where* $\widetilde{A}$ *and* $\widetilde{B}$ *are the density points of* A *and* B*, respectively. Assume further that* f_x, f_y *(partial derivatives) exist and are continuous in a neighborhood of* (x_0, y_0) *and that* $f_x(x_0, y_0) \neq 0$ *and* $f_y(x_0, y_0) \neq 0$. *Then* $t_0 := f(x_0, y_0)$ *is an interior point of* $f(\widetilde{A}, \widetilde{B}) := \left\{ f(a,b) : a \in \widetilde{A}, b \in \widetilde{B} \right\}$.

PROOF. Given $\delta > 0$, there exist $0 < \delta_1, \delta_2, \delta_3 < \delta$ such that for each $c \in N_3 := (t_0 - \delta_3, t_0 + \delta_3)$ and each $x \in N_1 := (x_0 - \delta_1, x_0 + \delta_1)$ there exists a unique $y \in N_2 := (y_0 - \delta_2, y_0 + \delta_2)$ such that $f(x, y) = c$ and f_x, f_y exist and are continuous and non-zero on $N_1 \times N_2$. For each $c \in N_3$, let $g_c : N_1 \to N_2$ be defined as above, i.e., $g_c(x) = y$ and $f(x, g_c(x)) = f(x, y) = c$. For each $c \in N_3$, $g'_c(x) = \frac{-f_x(x,g_c(x))}{f_y(x,g_c(x))}$ for every $x \in N_1$. Set $M := \frac{-f_x(x_0,g_{t_0}(x_0))}{f_y(x_0,g_{t_0}(x_0))} = \frac{-f_x(x_0,y_0)}{f_y(x_0,y_0)}$.

There exists a d_1, $0 < d_1 < \delta_1$, such that $\frac{|M|}{2} d_1 < \delta_2$ and $g'_c(x) = M + \varepsilon(x, c)$, where $|\varepsilon(x, c)| < \frac{|M|}{10}$ for all $c \in N_3$ satisfying $g_c(x_0) \in U_3 := \left(y_0 - \frac{|M|}{2} d_1, y_0 + \frac{|M|}{2} d_1 \right)$ and for all $x \in U_1 := (x_0 - d_1, x_0 + d_1)$ (notice t_0 is an interior point of these c's) and such that $\frac{m\left(\widetilde{A} \cap U_1\right)}{2d_1} > 0.95$ and $\frac{m\left(\widetilde{B} \cap U_2\right)}{4|M|d_1} > 0.95$ where $U_2 := (y_0 - 2|M|d_1, y_0 + 2|M|d_1)$.

For the c's mentioned above $g_c\left(\widetilde{A} \cap U_1\right)$is a measurable set with measure greater than $\left(\frac{9|M|}{10}\right)(2d_1)(0.95)$ and is contained in U_2 since $\frac{|M|}{2} d_1 + \frac{11|M|}{10} d_1 < 2|M|d_1$. This implies $g_c\left(\widetilde{A} \cap U_1\right) \cap \left(\widetilde{B} \cap U_2\right) \neq \emptyset$. Therefore, there exists an $x_c \in \widetilde{A} \cap U_1$ such that $g_c(x_c) \in \widetilde{B} \cap U_2$, or $f(x_c, g(x_c(x_c)) = c$ with $x_c \in \widetilde{A}$, $g_c(x_c) \in \widetilde{B}$ for each $c \in U'_3 := \{c \in N_3 : g_c(x_0) \in U_3\}$ and t_0 is an interior point of U'_3. □

We now proceed to the n dimensional extension of the Basu, Ganguly result.

Theorem 2. *Suppose that* $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, A *and* B *are Lebesgue measurable subsets of* $\mathbb{R}^n$ *and* $x_0 \in \widetilde{A}$, $y_0 \in \widetilde{B}$ *where* $\widetilde{A}$ *and* $\widetilde{B}$ *are the density points of* A *and* B *respectively. Assume further that:*

(α) *The* $2n^2$ *partial derivatives (n functions and* $2n$ *variables) exist and are continuous in some neighborhood of* (x_0, y_0).

(β)

$$\begin{vmatrix} D_1 f_1 & \cdot & \cdot & \cdot & \cdot & D_n f_1 \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ D_1 f_n & \cdot & \cdot & \cdot & \cdot & D_n f_n \end{vmatrix} (x_0, y_0) \neq 0$$

and

$$\begin{vmatrix} D_{n+1} f_1 & \cdot & \cdot & \cdot & \cdot & D_{2n} f_1 \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ D_{n+1} f_1 & \cdot & \cdot & \cdot & \cdot & D_{2n} f_n \end{vmatrix} (x_0, y_0) \neq 0$$

where $f = (f_1, f_2, \ldots, f_n)$.

Then $t_0 := f(x_0, y_0)$ *is an interior point of* $f\Big(\widetilde{A}, \widetilde{B}\Big) := \Big\{f(a, b) \colon a \in \widetilde{A}, b \in \widetilde{B}\Big\}$.

Proof. f can be viewed as an $n \times 1$ column matrix. The $n \times n$ matrices $\frac{\partial f}{\partial x} = \big[\frac{\partial f}{\partial x_1} \frac{\partial f}{\partial x_2} \cdots \frac{\partial f}{\partial x_n}\big]$ and $\frac{\partial f}{\partial y} = \big[\frac{\partial f}{\partial y_1} \frac{\partial f}{\partial y_2} \cdots \frac{\partial f}{\partial y_n}\big]$ are both invertible at (x_0, y_0) by the hypotheses of this theorem. By the implicit function theorem, there is a continuously differentiable function $g(x, t)$ defined for x near x_0 and t near $t_0 = f(x_0, y_0)$ such that $f(x, g(x, t)) = t$. By implicit differentiation, we have $\frac{\partial g}{\partial x} = -\big[\frac{\partial f}{\partial y}\big]\big[\frac{\partial f}{\partial x}\big]$. Therefore $\frac{\partial g}{\partial x}$ is invertible at the point (x_0, t_0). Hence, given $\delta > 0$, there exist $\delta_1, \delta_2, \delta_3$, with $0 < \delta_1, \delta_2, \delta_3 < \delta$ such that for each $c \in N_3 := N(t_0, \delta_3)$, the open ball in $\mathbb{R}^n$ with center t_0 and radius δ_3, and each $x \in N_1 := N(x_0, \delta_1)$, there exists a unique $y \in N_2 := N(y_0, \delta_2)$ such that $f(x, y) = c$. For each $c \in N_3$, let $g_c \colon N_1 \to N_2$ be defined as above, i.e., $g_c(x) = y$ with $f(x, g_c(x)) = f(x, y) = c$.

Set $J_{g_{t_0}}(x_0) = \det\Big(\frac{\partial g_{t_0}}{\partial x}\Big)(x_0) = M_1$, which, by the above, is not zero.

By the assumption of the continuity of the $2n^2$ partial derivatives there exists a d_1, $0 < d_1 < \delta$, such that $\frac{|M_1|}{2} d_1 < \delta_2$ and $J_{g_c}(x) = M_1 + \varepsilon(x, c)$ where $|\varepsilon(x, c)| < \frac{|M_1|}{10}$ for all $x \in U_1 := N(x_0, d_1)$ and for all $c \in C := \Big\{c \in N_3; g_c(x_0) \in U_3 := N\Big(y_0, \frac{|M_1|}{2} d_1\Big)\Big\}$ and $\|g_c(x) - g_c(x_0)\| < M_2 \|x - x_0\|$ for every $x \in U_1$ and every $c \in C$ and $\frac{m\big(\widetilde{A} \cap U_1\big)}{m(U_1)} > \frac{1}{q}$ and $\frac{m\big(\widetilde{B} \cap U_2\big)}{m(U_2)} > \frac{1}{q}$ where $U_2 := N\big(y_0, M_2 d_1 + \frac{|M_1|}{2} d_1\big)$ and where $q = \big[\frac{9}{10} |M_1| \frac{m(U_1)}{m(U_2)} + 1\big]$.

Notice that $t_0 = f(x_0, y_0)$ is an interior point of C. By the definitions of U_1, U_2, U_3 and C, $g_c\Big(\widetilde{A} \cap U_1\Big) \subseteq U_2$ for every $c \in C$ and $m\Big(g_c\Big(\widetilde{A} \cap U_1\Big)\Big) = \int\limits_{\widetilde{A} \cap U_1} |J_{g_c}(x)|dx > \frac{9}{10}|M_1|m\Big(\widetilde{A} \cap U_1\Big) > \frac{9}{10}|M_1|\frac{1}{q}m(U_1)$ for each $c \in C$.

Furthermore $\frac{9}{10}|M_1|\frac{1}{q}m(U_1) + \frac{1}{q}m(U_2) = \frac{1}{q}\Big[\frac{9}{10}|M_1|\frac{m(U_1)}{m(U_2)} + 1\Big]m(U_2) = m(U_2)$ and therefore $g_c\Big(\widetilde{A} \cap U_1\Big) \cap \Big(\widetilde{B} \cap U_2\Big) \neq \emptyset$ for every $c \in C$. Hence, for each $c \in C$, there exist $a_c \in \widetilde{A}$ and $b_c \in \widetilde{B}$ with $g_c(a_c) = b_c$ or $f(a_c, b_c) = c$. Therefore, t_0 is an interior point of $\Big\{f(x,y) \colon x \in \widetilde{A}, y \in \widetilde{B}\Big\}$. $\square$

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