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# ON THE DINI DERIVATES OF A PARTICULAR FUNCTION

**Abstract**

We construct a continuous strictly increasing function such that at each point one of its right Dini derivates is 0 or $\infty$, and at each point one of its left Dini derivates is 0 or $\infty$. Thus at no point can it have a positive real unilateral derivative.

In [1, (18.8)] there is discussed a continuous strictly increasing function $F$ (attributed chiefly to Riesz-Nagy) that has no real positive derivative at any point. Consequently $F' = 0$ almost everywhere.

Put another way, $F$ satisfies the condition:

($*$) there are no positive real number $y$ and point $x$ such that

$$D^+F(x) = D_+F(x) = D^-F(x) = D_-F(x) = y$$

where $D^+$, $D_+$, $D^-$, $D_-$ denote the usual Dini derivates.

But $F$ may not satisfy the stronger condition:

($**$) at each point $x$, either $D^+f(x) = +\infty$ or $D_+f(x) = 0$, and at each point $x$, either $D^-f(x) = \infty$ or $D_-f(x) = 0$.

In this note we will construct a strictly increasing continuous function $f$ satisfying condition ($**$). Thus $f$ cannot have a positive real unilateral derivative at any point.

It is worth comparing $f$ with a nondifferentiable function $p$ constructed in [2]. At each point $x$ either $D^+p(x)$ ($D^-p(x)$) is as large as possible, $\infty$, or $D_+p(x)$ ($D_-p(x)$) is as small as possible, $-\infty$. For our continuous increasing function $f$, at each point $x$ either $D^+f(x)$ ($D^-f(x)$) is as large as possible, $\infty$, or $D_+f(x)$ ($D_-f(x)$) is as small as possible, 0.

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The construction of $f$ begins with the construction of two sequences of sets $(A_n)$ and $(B_n)$ such that each $A_n$ and $B_n$ is the union of finitely many compact intervals.

Among other things, $A_n \cup B_n$ will be $[0,1]$, and $A_n \cap B_n$ will be a finite set. We will proceed by induction on $n$. Let $A_1 = [0,1/2]$ and $B_1 = [1/2,1]$. To form $A_2$ delete from each component $I$ of $A_1$ an open symmetric subinterval $J$ of $I$ such that

$$2^2(\text{length } J) = (\text{length } I)\,.$$

Make $B_2$ the closure of $[0,1] \setminus A_2$. To form $B_3$ delete from each component $I$ of $B_2$ an open symmetric subinterval $J$ of $I$ such that

$$2^3(\text{length } J) = (\text{length } I)\,.$$

Make $A_3$ the closure of $[0,1] \setminus B_3$. If $A_1, \dots, A_{n-1}$ and $B_1, \dots, B_{n-1}$ have been constructed and if $n$ is even, form $A_n$ by deleting from each component $I$ of $A_{n-1}$, the open symmetric subinterval $J$ of $I$ with

$$2^n(\text{length } J) = (\text{length } I)\,,$$

and make $B_n$ the closure of $[0,1] \setminus A_n$. If $n$ is odd, form $B_n$ by deleting from each component $I$ of $B_{n-1}$ the open symmetric subinterval $J$ of $I$ such that

$$2^n(\text{length } J) = (\text{length } I)\,,$$

and make $A_n$ the closure of $[0,1] \setminus B_n$. By inductive construction, $A_n$ and $B_n$ have been constructed for all indices $n$. Note that the lengths of the components of $A_n$ and $B_n$ tend to 0 as $n \to \infty$.

Put

$$A = \cap_{k=1}^{\infty} \cup_{j=k}^{\infty} A_j \quad \text{and} \quad B = \cap_{k=1}^{\infty} \cup_{j=k}^{\infty} B_j\,.$$

It follows that $A \cup B = [0,1]$. (The set $A \cap B$ is nonvoid, but that will not affect our argument.)

**Lemma 1.** *Let $[a,b]$ be a component interval of $A_n$ and $a \le x < b$. Let $m$ denote the Lebesgue measure. Then*

$$m\big([x,b] \cap B\big) \le 2^{1-n}(b-x) \quad \textit{and} \quad m\big([x,b] \cap A\big) \ge (1-2^{1-n})(b-x)\,.$$

Proof. Either $B_{n+1} \setminus B_n$ is void or $[a,b] \cap (B_{n+1} \setminus B_n)$ consists of one subinterval of $[a,b]$ depending on whether $n$ is even or odd. It follows from the construction that the length of this interval is not greater than $2^{-n}(b-x)$.

Now $b$ is the right endpoint of a component of $A_{n+k}$ for $k = 1,2,3,\dots$. Thus $[x,b] \cap A_{n+k}$ consists of finitely many components of $A_{n+k}$ and/or a compact interval containing $x$. Repeated applications of the principle in the preceding paragraph and $B_{n+k} \setminus B_{n+k-1} \subset A_{n+k-1}$ show that

$$m\Big([x,b] \cap (B_{n+k} \setminus B_{n+k-1})\Big) = m\Big([x,b] \cap (B_{n+k} \setminus B_{n+k-1}) \cap A_{n+k-1}\Big)$$

$$\leq 2^{1-n-k} m\Big([x,b] \cap A_{n+k-1}\Big) \leq 2^{1-n-k}(b-x)\,.$$

But $m(A_n \cap B_n) = 0$, and it follows that

$$m\Big([x,b] \cap (\cup_{k=1}^{\infty} B_{n+k})\Big) = m\Big([x,b] \cap \big(\cup_{k=1}^{\infty} (B_{n+k} \setminus B_{n+k-1})\big)\Big)$$

$$\leq \sum_{k=1}^{\infty} 2^{1-n-k}(b-x) = 2^{1-n}(b-x)\,.$$

Consequently $m\big([x,b] \cap B\big) \leq 2^{1-n}(b-x)$. But $A \cup B = [0,1]$, so $m\big([x,b] \cap A\big) \geq \big(1 - 2^{1-n}\big)(b-x)$. This proves Lemma 1. $\square$

Let the function $h$ be the indefinite integral of the characteristic function of $A$. Now, if $x$ lies in $[a_n, b_n)$ for components $[a_n, b_n]$ of infinitely many sets $A_n$, then from

$$m\big([x,b_n] \cap A\big)/(b_n - x) \geq \big(1 - 2^{1-n}\big)$$

it follows that $D^+h(x) = 1$. By reversing the roles of the sets $A_n$ and $B_n$, we see that if $x$ lies in $[c_n, d_n)$ for components $[c_n, d_n]$ of infinitely many sets $B_n$, then from

$$m\big([x,d_n] \cap A\big)/(d_n - x) \leq 2^{1-n}$$

it follows that $D_+h(x) = 0$. Thus for $x \in [0,1)$, either $D^+h(x) = 1$ or $D_+h(x) = 0$. We reverse left and right to see that for $x \in (0,1]$, either $D_-h(x) = 0$ or $D^-h(x) = 1$.

Now any subinterval $I$ of $[0,1]$ contains component intervals of some $A_n$ and $B_n$, and from the inequalities in the preceding paragraph it follows that the functions $h(x)$ and $x - h(x)$ are strictly increasing.

Put $k = h^{-1}$ on the set $h(0,1)$. Then the function $k(y) - y$ is strictly increasing on $h(0,1)$ because $x - h(x)$ is strictly increasing on $(0,1)$. From $D^+h(x_0) = 1$ we obtain $D_+k(y_0) = 1$ where $y_0 = h(x_0)$, and from $D_+h(x_0) = 0$ we obtain $D^+k(y_0) = \infty$. Likewise from $D^-h(x_0) = 1$ we obtain $D_-k(y_0) = 1$, and from $D_-h(x_0) = 0$ we obtain $D^-k(y_0) = \infty$. At each point in $h(0,1)$, either $D^+k = \infty$ or $D_+k = 1$, and either $D^-k = \infty$ or $D_-k = 1$.

Finally, $f(y) = k(y) - y$ is a continuous strictly increasing function on $h(0,1)$ satisfying condition $(**)$.

We conclude with the observation that if all that is required is a strictly increasing continuous function whose derivative vanishes almost everywhere, one solution is well-known and easily constructed. It can be found in [3, p. 101].

## References

[1] E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, Springer Verlag, New York, 1965.

[2] A. P. Morse, *A continuous function with no unilateral derivatives*, Trans. Amer. Math. Soc. **44** (1938), 496–507.

[3] S. Saks, *Theory of the Integral*, Dover, New York, 1964.