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ON INFINITE UNILATERAL DERIVATIVES

Abstract

We prove that for any continuous real valued function f on $[a, b]$ there exists a continuous function K such that $K–f$ has bounded variation and $(K–f)' = 0$ almost everywhere on $[a, b]$ and such that in any subinterval of $[a, b]$, K has right derivative ∞ at continuum many points, K has left derivative ∞ at continuum many points, K has right derivative $-\infty$ at continuum many points, and K has left derivative $-\infty$ at continuum many points. Furthermore, functions K with these properties are dense in $C[a, b]$. We can assume the infinite derivatives of K are bilateral if f is of bounded variation on $[a, b]$ or if f satisfies Lusin's condition (N).

Let $[a, b]$ be a compact interval and let $C[a, b]$ denote the family of continuous real valued functions on $[a, b]$ endowed with the uniform metric. Here we say that a function is an s-function if in every subinterval of $[a, b]$ it has right derivative ∞ at continuum many points, left derivative ∞ at continuum many points, right derivative $-\infty$ at continuum many points, and left derivative $-\infty$ at continuum many points.

From the classical work of Stanislaw Saks [1] we infer that the s-functions form a residual subset of the complete metric space $C[a, b]$. Here we give a local companion to this global result as follows. For any f in $C[a, b]$ there is an s-function K such that $K - f$ is a singular function of bounded variation, that is $(K - f)' = 0$ almost everywhere on $[a, b]$. The idea is that K and f have the same Dini derivates at almost every point in $[a, b]$. Furthermore, the s-functions K with this property are dense in $C[a, b]$.

We say that a function in $C[a, b]$ is an s_0-function if in every subinterval of $[a, b]$ it has (bilateral) derivative ∞ at continuum many points and derivative $-\infty$ at continuum many points. We will prove that K (in the preceding paragraph) can be an s_0-function for certain kinds of functions f. This works

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when f is of bounded variation on $[a, b]$ or when f satisfies Lusin's condition (N), that is f maps sets of measure zero to sets of measure zero.

We begin with some needed lemmas. The first concerns the Dini derivates of f.

Lemma 1. *Let f be a continuous function on $[a, b]$. Then there are an uncountable compact subset S of $[a, b]$ and a countable set T such that $D_+f(x) > -\infty$ for any $x \in S \setminus T$.*

Proof. We immediately dismiss the case in which f is nonincreasing on (a, b), for if it were then f would be differentiable on a set of positive measure and on a compact subset S of positive measure. Then T could be void.

We assume then that there exist a_0 and b_0 in (a, b) such that $a_0 < b_0$ and $f(a_0) < f(b_0)$. For each y satisfying $f(a_0) < y < f(b_0)$, let $k(y)$ be the greatest point in the compact set $\{t \in (a_0, b_0) : f(t) = y\}$. Necessarily $D_+f(k(y)) \geq 0$. Let S_0 denote the set $\{k(y) : f(a_0) < y < f(b_0)\}$ and let S denote the closure of S_0. We deduce that k is a strictly increasing function from the interval $(f(a_0), f(b_0))$ into the interval (a_0, b_0) and hence S_0 is an uncountable set. It suffices to prove that $S \setminus S_0$ is a countable set.

Let $w \in S \setminus S_0$ where $w \neq a_0$, $w \neq b_0$, and w is an accumulation point of S_0 from the left. There is an increasing sequence of points (y_n) in $(f(a_0), f(b_0))$ such that $k(y_n)$ converges to w. Suppose (y_n) converges to y^*. Now $k(y^*) \neq w$, so k has a discontinuity at y^*. In this way every such point in $S \setminus S_0$ defines a point of discontinuity of k. Moreover no two w_1 and w_2 can define the same point of discontinuity of k because k is strictly increasing. The monotone function k has only countably many points of discontinuity, so there are at most countably many points in $S \setminus S_0$ that are accumulation points of S_0 from the left. The argument for accumulation points from the right is analogous. □

In the next lemma we construct a nondecreasing singular function enjoying certain desired properties.

Lemma 2. *Let S be an uncountable compact set. Then there is a continuous nondecreasing singular function g on $[a, b]$ with total variation 2 such that for any continuous function h with total variation less than 1, the set $\{x \in S : (g + h)'(x) = \infty\}$ has the power of the continuum.*

Proof. Any closed subset of the real line is the union of a countable set with a closed set all of whose points are condensation points of itself. Without loss of generality we assume that every point of S is a condensation point of S. Let $S_1 = \{x \in S : x$ is both a left and a right accumulation point of $S\}$. Routine arguments show that $S \setminus S_1$ is a countable set. Thus every point of S_1 is a condensation point of S_1 and a left and right accumulation point of S_1.

Choose points A and B in S_1 with $A < B$. We construct by induction a sequence of mutually disjoint compact subintervals of (A, B) with endpoints in S_1 as follows.

Select a_1 and b_1 in $S_1 \cap (A, B)$ such that $b_1 - a_1 > \frac{B-A}{2}$. Suppose that the intervals $[a_1, b_1]$, $[a_2, b_2]$, $\ldots$, $[a_n, b_n]$ have been selected. Let I_n be a component of $(A, B) \setminus (\cup_{j=1}^{n} [a_j, b_j])$ of maximal length. Select a_{n+1} and b_{n+1} in $S_1 \cap I_n$ such that $b_{n+1} - a_{n+1} > \frac{m(I_n)}{2}$.

Denote $X_0 = (A, B) \setminus (\cup_j [a_j, b_j])$. It follows from the construction that X_0 has measure zero. Observe that any point in X_0 is an accumulation point of the countable set $\{a_j\}_j \cup \{b_j\}_j$ and hence lies in S. Therefore $X_0 \subset S$.

Between any two intervals in the sequence there lie other members of the sequence. Thus we can assign a rational number r_n to each interval $[a_n, b_n]$ such that the sequence (r_n) is dense in $(0, 2)$, $\inf(r_n) = 0$, $\sup(r_n) = 2$, and such that $r_j < r_n$ if and only if $b_j < a_n$. Let g be the real valued function on $\cup_j [a_j, b_j]$ such that $g = r_n$ on $[a_n, b_n]$. Make $g = 0$ on $(-\infty, A)$ and $g = 2$ on (B, ∞). We extend g to a continuous nondecreasing function on the real line in the natural way with $\inf g = 0$ and $\sup g = 2$. Set $X = \{x \in X_0 : g'(x) = \infty\}$. Then $m(X) = m(X_0) = 0$. From the work of de la Vallée Poussin (consult for example [2, Theorem (9.1), Chapter IV]) it follows that the set $g\{x : g$ has a finite or infinite derivative at $x\}$ has measure 2. But the set $\{x : g'(x) > 0\}$ has measure zero and we deduce from [2, Theorem (4.5), Chapter IX] that the set $g\{x : g$ has a finite derivative at $x\}$ has measure zero. Thus it follows that

$$m(g(X)) = 2. \tag{1}$$

Now let h be a continuous function on $[a, b]$ with total variation less than 1. It suffices to prove that $(g + h)'(x) = \infty$ at continuum many $x \in S$.

Let $Y = \{y \in X : \min(D_+(g + h)(y)\,,\, D_-(g + h)(y)) < \infty\}$. Intervals of the form $[g(c), g(c + t)]$ with $t > 0$ and satisfying

$$g(c + t) - g(c) > 3((g + h)(c + t) - (g + h)(c))$$

form a Vitali covering on the y-axis of the set $g(Y)$. Observe that here

$$-\frac{g(c + t) - g(c)}{3} < -((g + h)(c + t) - (g + h)(c)). \tag{2}$$

Then from (2) we obtain

$$\begin{aligned}-(h(c+t)-h(c)) &= (g(c+t)-g(c)) - ((g+h)(c+t)-(g+h)(c)) \\ &> (g(c+t)-g(c)) - \frac{g(c+t)-g(c)}{3} \\ &= 2\cdot\frac{g(c+t)-g(c)}{3}\end{aligned}$$

and because t and $g(c+t)-g(c)$ are positive it follows that

$$|h(c+t)-h(c)| \geq 2\cdot\frac{g(c+t)-g(c)}{3}. \tag{3}$$

By the Vitali Covering Theorem there are countably many mutually disjoint such intervals $[g(c_j)\,,\,g(c_j+t_j)]$ covering almost every point in $g(Y)$. Furthermore the intervals $[c_j, c_j+t_j]$ are mutually disjoint. From (3) and the total variation of h we infer that

$$1 \geq \sum_j |h(c_j+t_j)-h(c_j)| \geq 2\cdot\sum_j \frac{g(c_j+t_j)-g(c_j)}{3} \geq 2\cdot\frac{m(g(Y)}{3}$$

and

$$m(g(Y)) \leq \frac{3}{2}. \tag{4}$$

From (1) and (4) we obtain

$$m(g(X\setminus Y)) \geq \frac{1}{2}. \tag{5}$$

It follows from (5) that the sets $g(X\setminus Y)$ and $X\setminus Y$ have the power of the continuum and because $X\subset X_0\subset S$, $\{x\in S\,:\,(g+h)'(x)=\infty\}$ has the power of the continuum. □

In the next lemma we introduce a space (BV) that has a different metric than the uniform metric.

Lemma 3. *Let S be an uncountable compact set. Let (BV) denote the family of singular functions of bounded variation on $[a,b]$ under the metric*

$$d(f,g) = |f(0)-g(0)| + V(f-g),$$

where V denotes the total variation on $[a,b]$. Let

$$W = \{f\in(BV)\,:\,f'(x)=\infty \text{ at continuum many points } x \text{ in } S\}.$$

Then (BV) is a complete metric space and the function 0 is in the closure of the interior of W.

Proof. Let (BV_1) denote the family of all functions of bounded variation on $[a,b]$ under the same metric that (BV) has. Then $(BV) \subset (BV_1)$. Routine arguments show that (BV_1) is a complete metric space.

Let $f_1 \in (BV_1) \setminus (BV)$ and $f_2 \in (BV)$. Set $\epsilon > 0$ such that the set

$$\left\{x \in [a,b] \,:\, \max(|D^+ f_1(x)|, |D_+ f_1(x)|, |D^- f_1(x)|, |D_- f_1(x)|) > \epsilon\right\}$$

has measure greater than ϵ. By a straight-forward application of the Vitali Covering Theorem, there exist mutually disjoint intervals $[x_1, x_1+t_1]$, $[x_2, x_2+t_2]$, ..., $[x_n, x_n+t_n]$ such that

$$\sum_{i=1}^{n} |f_1(x_i+t_i) - f_1(x_i)| > \epsilon \cdot \sum_{i=1}^{n} t_i > \epsilon^2.$$

It follows that $d(f_1, 0) > \epsilon^2$. By the same argument $d(f_1, f_2) = d(f_1 - f_2, 0) > \epsilon^2$ and we deduce that (BV) is a closed subset of (BV_1). But (BV_1) is a complete metric space, so (BV) is likewise a complete metric space.

We deduce that the function g in Lemma 2 lies in the interior of W, so the distance in (BV) from the 0 function to $(\textit{interior } W)$ is at most 2. For any $r > 0$, $r(\textit{interior } W) \subset \textit{interior } W$. It follows that the distance from the 0 function to $(\textit{interior } W)$ is zero. □

We are now ready for our main results.

Theorem 1. *Let f be a continuous function on $[a,b]$. Then there is an s-function K such that $K-f$ is of bounded variation and $(K-f)' = 0$ almost everywhere. Furthermore, the family of all functions K satisfying this property is dense in $C[a,b]$ under the topology of uniform convergence.*

Proof. Let I be a subinterval of $[a,b]$. For $g \in C[a,b]$ let

$$A(g) = \{G \in (BV) \,:\, (G+g)'_+(x) = \infty \text{ at continuum many } x \in I\}.$$

By Lemma 1 there is an uncountable compact set $S \subset I$ such that $D_+ g(x) > -\infty$ on a cocountable subset of S. We deduce from Lemma 3 that the 0 function is in the closure of the interior of the set $A(g)$.

Let h be any function in (BV) and let $g_0 = g + h$. Let $G_0 \in A(g_0)$. Then $(G_0 + g_0)'_+(x) = \infty$ and likewise $((G_0 + h) + (g_0 - h))'_+(x) = \infty$ at continuum many points x in I. It follows that $G_0 + h \in A(g_0 - h) = A(g)$ and furthermore $A(g_0) + h \subset A(g)$. By Lemma 3 again the 0 function is in the closure of the interior of $A(g_0)$ and h is in the closure of the interior of $A(g_0) + h$. Thus h is in the closure of the interior of $A(g)$. The choice of h is independent of g, so the interior of $A(g)$ is dense in (BV) for any $g \in C[a,b]$.

For any subinterval I of $[a,b]$, let $P(I)$ be the family $A(f)$ as described. Then $\cap_I P(I)$, where I runs over all the subintervals of $[a,b]$ with rational endpoints, is a residual subset of the complete metric space (BV).

It follows that the family P_1 of functions F in (BV) for which $(F+f)'_r(x) = \infty$ at continuum many points x in each subinterval of $[a,b]$ is a residual subset of the complete metric space (BV). Let P_2 denote the corresponding family in which we replace ∞ with $-\infty$. It follows similarly that F_2 is a residual subset of (BV). Let P_3 and P_4 be the corresponding families where left derivatives (instead of right derivatives) are employed. Then P_3 and P_4 are likewise residual subsets of (BV). Set $P = P_1 \cap P_2 \cap P_3 \cap P_4$. Then P is a residual subset of (BV). But any function in $P + f$ is an s-function and hence any function in $P + f$ suffices for K in the conclusion of Theorem 1.

Finally $(BV) + f$ is evidently dense in the space $C[a,b]$ under the topology of uniform convergence, so $P + f$ is also dense in $C[a,b]$. □

For certain kinds of functions $f \in C[a,b]$, s-functions in Theorem 1 can be replaced by s_0-functions, as we now see.

Theorem 2. *Let $f \in C[a,b]$ such that either*

(i) f is differentiable on a set of positive measure in every subinterval of $[a,b]$ or

(ii) f is differentiable at each point of a residual subset of $[a,b]$.

Then there is an s_0-function K on $[a,b]$ such that $(K-f)' = 0$ almost everywhere on $[a,b]$. Furthermore the family of all s_0-functions satisfying this property is dense in $C[a,b]$ under the topology of uniform convergence.

Proof. (i) In any subinterval I of $[a,b]$ f is differentiable on a set of positive measure that must contain an uncountable compact set. Then Lemma 1 applies to f and we continue as in the proof of Theorem 1. Observe that all the derivatives are bilateral here.

(ii) A residual subset of $[a,b]$ must contain a dense G_δ-subset of every subinterval of $[a,b]$, which in turn must contain a perfect set. Hence f is differentiable on an uncountable compact set in every subinterval of $[a,b]$. We proceed as in part (i).

□

Some corollaries are immediate.

Corollary 1. *The family of all s_0-functions in (BV) is a residual subset of (BV).*

PROOF. Apply the proof of Theorem 1 to any constant function f. $\square$

Corollary 2. *For any measurable function h on $[a,b]$, there is a continuous s_0-function F, depending on h, such that $F' = h$ almost everywhere on $[a,b]$.*

PROOF. By [2, Theorem (2.3), Chapter VII, p. 217] there is a function $F_1 \in C[a,b]$ such that $F_1' = h$ almost everywhere on $[a,b]$. Apply Theorem 2 to F_1. $\square$

Observe that any function of bounded variation in $C[a,b]$ satisfies the hypothesis of Theorem 2. Likewise any $f \in C[a,b]$ that maps sets of measure zero to sets of measure zero (Lusin's condition (N)) must be differentiable on a set of positive measure in each subinterval of $[a,b]$ (consult [2, Chapter IX, Theorem (7.9), p.286] and so satisfies the hypothesis of Theorem 2.

Corollary 3. *There is an s-function $F \in C[a,b]$ such that for any measurable function k on $[a,b]$ there is a sequence of positive numbers (t_n) converging to 0, and depending on k, such that*

$$\lim_{n\to\infty} \frac{F(x+t_n) - F(x)}{t_n} = k(x)$$

for almost every x in $[a,b]$.

PROOF. By [2, p. 118] there exists a function F_0 in $C[a,b]$ satisfying this property. Apply Theorem 1 to F_0. $\square$

Corollary 4. *Let f be a positive function in $C[a,b]$. Then there exist functions F, F_1, F_2 in (BV) such that*

(i) $F + (\log f)$ is an s-function,
(ii) $F_1 f$ is an s-function, and
(iii) f^{F_2} is an s-function, provided $f > 1$ on $[a,b]$.

PROOF. (i) The proof of (i) is in the proof of Theorem 1.

(ii) Let $p \in C[a,b]$ and $x \in (a,b)$. By the Mean Value Theorem we see that

$$\frac{\exp(p(x+t)) - \exp(p(x))}{t} = \frac{e^u(p(x+t) - p(x))}{t}$$

for some number u between $p(x+t)$ and $p(x)$. From this we deduce that p is an s-function if and only if $\exp(p)$ is an s-function.

Let G be a function in (BV) such that $G + \log f$ is an s-function. Then $\exp(G + \log f) = f(\exp G)$ is an s-function and $\exp G$ is in (BV).

(iii) Let G be a function in (BV) such that $G(\log f)$ is an s-function. Then $\exp(G \log f) = f^G$ is an s-function and G is in (BV).

$\square$

We close with the comment that if $f_1, f_2, f_3, \ldots, f_n, \ldots$ is a sequence of functions in $C[a,b]$ then there is a function F in (BV) such that $F + f_j$ is an s-function for each index j. We leave the proof.

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