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ON DISCRETE LIMITS OF SEQUENCES OF DARBOUX BILATERALLY QUASICONTINUOUS FUNCTIONS

Abstract

In this article we show that a function f, such that the complement of the set of points at which f has the Darboux property and is bilaterally quasicontinuous is nowhere dense, must be the discrete limit of a sequence of bilaterally quasicontinuous Darboux functions. Moreover, there is given a construction of a function that is the discrete limit of a sequence of bilaterally quasicontinuous Darboux functions and which does not have a local Darboux property on a dense set.

Let $\mathcal{R}$ be the set of all reals. In the article [3] the authors introduced the notion of the discrete convergence of sequences of functions and investigated the discrete limits in different families, for example in the family $\mathcal{C}$ of all continuous functions.

We will say that a sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$, $n = 1, 2, \ldots$, discretely converges to the limit f ($f = d - \lim_{n\to\infty} f_n$) if

$$\forall_x \exists_{n(x)} \forall_{n>n(x)} f_n(x) = f(x).$$

For any family $\mathcal{P}$ denote by $B_d(\mathcal{P})$ the family of all discrete limits of sequences of functions from the family $\mathcal{P}$.

In [3] the class $B_d(\mathcal{C})$ is described and the authors observe that every strictly increasing function f whose the set of discontinuity points is dense does not belong to the discrete Baire system generated by $\mathcal{C}$ and the discrete convergence.

A function $f : \mathbb{R} \to \mathbb{R}$ is quasicontinuous (bilaterally quasicontinuous) at a point x if for every positive real η there is a nonempty open set $U \subset (x-\eta, x+\eta)$

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(there are nonempty open sets $V \subset (x-\eta, x)$ and $W \subset (x, x+\eta)$) such that $f(U) \subset (f(x)-\eta, f(x)+\eta)$ ($f(V \cup W) \subset (f(x)-\eta, f(x)+\eta)$) ([5, 6]).

In [4] it is proved that

(1) A function $f : \mathbb{R} \to \mathbb{R}$ is the discrete limit of a sequence of quasicontinuous functions if and only if the set
$$D_q(f) = \{x; f \text{ is not quasicontinuous at } x\}$$
is nowhere dense.

(2) A function $f : \mathbb{R} \to \mathbb{R}$ is the discrete limit of a sequence of bilaterally quasicontinuous functions if and only if the set
$$D_{bq}(f) = \{x; f \text{ is not bilaterally quasicontinuous at } x\}$$
is nowhere dense.

Let $\mathcal{D}$ denote the class of all functions $f : \mathbb{R} \to \mathbb{R}$ having Darboux property and let Q (respectively Q_b) be the family of all quasicontinuous (bilaterally quasicontinuous) functions.

In [7] the author investigates some classes $\mathcal{P}$ of functions from $\mathbb{R}$ to $\mathbb{R}$ such that $\mathcal{P} \subset B_d(\mathcal{D} \cap \mathcal{P})$. But neither of the classes Q and Q_b satisfies the hypothesis of that general theorem from [5]. For this observe that $Q \cap \mathcal{D} \subset Q_b$ and that, for each continuous from the right hand and increasing function $f : \mathbb{R} \to \mathbb{R}$ discontinuous on a dense set, we have
$$D_q(f) = \emptyset \text{ and the set } D_{bq}(f) \text{ is dense.}$$
Consequently,
$$Q \setminus B_d(\mathcal{D} \cap Q) = Q \setminus B_d(\mathcal{D} \cap Q_b) \neq \emptyset.$$

In this article I show two theorems describing the class $B_d(\mathcal{D} \cap Q_b)$. In our considerations we will apply the following notations: Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let $x \in \mathbb{R}$ be a point. Put
$$K^+(f, x) = \{y : \exists_{(x_n)} x < x_n \to x \text{ and } y = \lim_{n\to\infty} f(x_n)\},$$
$$K^-(f, x) = \{y : \exists_{(x_n)} x > x_n \to x \text{ and } y = \lim_{n\to\infty} f(x_n)\},$$
and recall that x is a Darboux point of a function f if for every positive real r and for all reals $a \in (\min(f(x), \inf(K^+(f,x))), \max(f(x), \sup(K^+(f,x))))$ and $b \in (\min(f(x), \inf(K^-(f,x))), \max(f(x), \sup(K^-(f,x))))$ there are points $c \in (x, x+r)$ and $d \in (x-r, x)$ such that $f(c) = a$ and $f(d) = b$. It is known

([2, 1]) that a function f has the Darboux property if and only if each point x is a Darboux point of a function f.

Let

$$\mathrm{Dar}(f) = \{x : x \text{ is not a Darboux point of } f\}.$$

Theorem 1. *Let a function $f : \mathbb{R} \to \mathbb{R}$ be such that the set $\mathrm{Dar}(f) \cup D_{bq}$ is nowhere dense. Then f is the discrete limit of a sequence of Darboux bilaterally quasicontinuous functions.*

PROOF. Denote by A the closure $\mathrm{cl}(\mathrm{Dar}(f) \cup D_{bq})$ of the union $\mathrm{Dar}(f) \cup D_{bq}$. Let (I_n) be a sequence of all components of the set $\mathbb{R} \setminus A$. If $I_n = (a_n, b_n)$ and $a_n, b_n \in \mathbb{R}$ then we find two sequences of closed intervals $J_{n,k} = [c_{n,k}, d_{n,k}]$ and $L_{n,k} = [p_{n,k}, q_{n,k}]$, $k = 1, 2, \ldots$, such that

$a_n < d_{n,k+1} < c_{n,k} < d_{n,k}$ and $d_{n,1} < p_{n,k} < q_{n,k} < p_{n,k+1} < b_n$ for $k = 1, 2, \ldots$;

$a_n = \lim_{k\to\infty} d_{n,k}$ and $b_n = \lim_{k\to\infty} p_{n,k}$;

f is continuous at all points $c_{n,k}$, $d_{n,k}$, $p_{n,k}$, $q_{n,k}$, $k, n \geq 1$.

If $a_n = -\infty$ (or $b_n = \infty$) then we find only one sequence $(L_{n,k})$ (or respectively $(J_{n,k})$).

For all n, k define continuous functions $f_{n,k} : J_{n,k} \to \mathbb{R}$ and $g_{n,k} : L_{n,k} \to \mathbb{R}$ such that

$$f_{n,k}(J_{n,k}) = g_{n,k}(L_{n,k}) \supset [-k, k]$$

and

$$f_{n,k}(c_{n,k}) = f(c_{n,k}), \quad f_{n,k}(d_{n,k}) = f(d_{n,k}),$$
$$g_{n,k}(p_{n,k}) = f(p_{n,k}) \text{ and } g_{n,k}(q_{n,k}) = f(q_{n,k}).$$

For $m = 1, 2, \ldots$ let

$$f_m(x) = \begin{cases} f_{n,k}(x) & \text{for } x \in J_{n,k}, \text{ where } n \geq 1 \text{ and } \geq m \\ g_{n,k}(x) & \text{for } x \in L_{n,k}, \text{ where } n \geq 1 \text{ and } \geq m \\ f(x) & \text{otherwise on } \mathbb{R}. \end{cases}$$

Since for each $m \geq 1$, we have $A \supset \mathrm{Dar}(f_m) \cup D_{bq}(f_m)$ and

$$K^+(f_m, x) \cap K^-(f_m, x) = [-\infty, \infty] \text{ for each } x \in A,$$

every function $f_m \in \mathcal{D} \cap Q_b$.

Evidently, f is the discrete limit of the sequence (f_m). $\square$

Theorem 2. *There is a function* $f : \mathbb{R} \to \mathbb{R}$ *belonging to* $B_d(\mathcal{D} \cap Q_b)$ *such that the set* $\mathrm{Dar}(f)$ *is dense.*

PROOF. Let (I_n) be an enumeration of all open intervals with rational end-points. In the first step we find a nonempty perfect nowhere dense set $C_1 \subset I_1$ and a nowhere constant function $f_1 : \mathbb{R} \to [-1,1] \setminus \{0\}$ such that:

the restricted function $f_1/(\mathbb{R} \setminus C_1)$ is continuous;

$f_1(\mathbb{R}) = [-1,0) \cup (0,1]$;

for every $x \in C_1$ being a bilateral accumulation point of C_1 and for all reals $r > 0$ and $y \in (-1,0) \cup (0,1)$ there are points $a \in (x-r,x)$ and $b \in (x,x+r)$ such that $f_1(a) = y = f_1(b)$;

if $x \in C_1$ is unilaterally isolated in C_1 then $f_1(x)$ is an irrational number.

Next we fix a positive integer $n > 1$ and we suppose that for all positive integers $1 < k < n$ we have defined closed intervals

$$J_k \subset I_k \setminus \bigcup_{i<k} C_i,$$

nonempty perfect nowhere dense sets $C_k \subset \mathrm{Int}(J_k)$, where $\mathrm{Int}(J_k)$ denotes the interior of the interval J_k, closed intervals M_k, rationals $w_k \in \mathrm{Int}(M_k)$ and nowhere constant functions $f_k : \mathbb{R} \to [-1,1]$ such that:

the length $d(J_k)$ of the interval J_k is less then $\frac{1}{k}$;

for $k < n-1$ we have $f_k(x) = f_{k+1}(x)$ for $x \in \mathbb{R} \setminus \mathrm{Int}(J_{k+1})$;

$f_k(J_k) = M_k \setminus \{w_k\}$, where $w_0 = 0$;

$osc_{J_{k+1}} f_k < \frac{2}{8^k} \min\{\mathrm{dist}(w_i, f_k(J_{k+1})) = \inf\{|w_i - f_k(x)|; x \in J_{k+1}\}; i \leq k\}$ for $k < n-1$;

$M_1 = [-1,1]$ and for $k > 1$ the set $M_k \subset [-1,1] \setminus \{w_i; i < k\}$ is the interval containing $f_{k-1}(J_k)$ of the length less than

$$\frac{2}{8^{k-1}} \min(\{\mathrm{dist}(w_i, M_k) = \inf\{|x - w_i|; x \in M_k\}; i < k\})$$

with the same center as the center of the interval $f_{k-1}(J_k)$;

the functions f_k are continuous at all points $x \in \mathbb{R} \setminus \bigcup_{i \leq k} C_i$;

for $i \le k < n$, for each point $x \in C_i$ being a bilateral accumulation point of C_i, for each positive real r and for every point $y \in \mathrm{Int}(M_i) \setminus \{w_i\}$ there are points $a_i \in (x-r, x)$ and $b_i \in (x, x+r)$ such that $f_k(a_i) = f_k(b_i) = y$;

if a point $x \in C_k$ is isolated from the right (from the left) hand in C_k, $k < n$, then $f_k(x)$ is irrational and for each positive real $r < \mathrm{dist}(x, \mathbb{R} \setminus J_k)$ the interval $(x, x+r) \subset \mathrm{Int}(J_k) \setminus C_k$ and the image

$$f_k([x, x+r)) \subset (\min(M_k), w_k) \ \text{ or } \ f_k([x, x+r)) \subset (w_k, \max(M_k))$$

$((x-r, x) \subset \mathrm{Int}(J_k) \setminus C_k$ and

$$f_k((x-r, x]) \subset (\min(M_k), w_k) \ \text{ or } \ f_k((x-r, x]) \subset (w_k, \max(M_k)).$$

Now, in the step n we find a closed interval J_n such that

$$J_n \subset I_n \setminus \bigcup_{k<n} C_k, \ \text{ and } \ d(J_n) < \frac{1}{n},$$

and

$$\mathrm{osc}_{J_n} f_{n-1} < s_n = \frac{2}{8^{n-1}} \min\{\mathrm{dist}(w_i, f_{n-1}(J_n)); i < n\}.$$

Let $M_n \subset [-1, 1] \setminus \{w_i; i < n\}$ be a closed interval of the length $d(M_n)$ such that $d(f_{n-1}(J_n)) < d(M_n) < s_n$ with the same center as the interval $f_{n-1}(J_n)$ and let $C_n \subset \mathrm{Int}(J_n)$ be a nonempty nowhere dense perfect set. Fix a rational point $w_n \in \mathrm{Int}(M_n)$ and let

$$\min(M_n) = v_0 < v_1 = w_n < v_2 = \max(M_n).$$

The family $\{T_k\}_k$ of all components of the set $\mathrm{Int}(J_n) \setminus C_n$ is the union of pairwise disjoint subfamilies $\{T_{i,j}\}_j$, $i \le 2$, such that

$$\forall_{i \le 2} C_n \subset \mathrm{cl}(\bigcup_j T_{i,j}),$$

where $\mathrm{cl}(X)$ denotes the closure of the set X.

For $i \le 2$ and $j = 1, 2, \ldots$ we define nowhere constant continuous functions $f_{n,i,j} : T_{i,j} \to (v_{i-1}, v_i)$ such that

$f_{n,i,j}(T_{i,j}) = (v_{i-1}, v_i)$ for all $i \le 2$ and $j \ge 1$;

if $T_{i,j} = (a_{i,j}, b_{i,j})$ and $a_{i,j}$ (or resp. $b_{i,j}$) is an endpoint of the interval J_n then

$$\lim_{x \to a_{i,j}+} f_{n,i,j}(x) = f_{n-1}(a_{i,j})$$

(or resp.

$$\lim_{x \to b_{i,j}-} f_{n,i,j}(x) = f_{n-1}(b_{i,j}));$$

if $T_{i,j} = (a_{i,j}, b_{i,j})$ and $a_{i,j} \in C_n$ (or resp. $b_{i,j} \in C_n$) then for every $y \in (v_{i-1}, v_i)$ and for each positive real r there is a point $c \in (a_{i,j}, \min(a_{i,j} + r, b_{i,j}))$ (or resp. $d \in (\max(a_{i,j}, b_{i,j} - r), b_{i,j})$) such that $f_{n,i,j}(c) = y$ (or resp. $y = f_{n,i,j}(d)$).

Let $f_n : \mathbb{R} \to [-1, 1]$ be a function such that

f_n is equal f_{n-1} on the set $\mathbb{R} \setminus \mathrm{Int}(J_n)$ and is equal $f_{n,i,j}$ on the intervals $T_{i,j}$, $i \le 2$, $j \ge 1$;

if $T_{i,j} = (a_{i,j}, b_{i,j})$ and $a_{i,j} \in C_n$ (or resp. $b_{i,j} \in C_n$) then $f_n(a_{i,j}) \in (v_{i-1}, v_i)$ (or resp. $f_n(b_{i,j}) \in (v_{i-1}, v_i)$) is irrational;

$f_n(C_n) = M_n \setminus \{w_n\}$.

Finally we define $f = \lim_{n\to\infty} f_n$. Since

$$|f_{n+1} - f_n| \le s_n < \frac{2}{8^{n-1}} \text{ for } n \ge 1,$$

the sequence (f_n) uniformly converges to f. From the construction follows that the functions f_n, $n = 1, 2, \ldots$, are bilaterally quasicontinuous. So f is also a bilaterally quasicontinuous function. Since the images $f(J_n)$ of all intervals $J_n \subset I_n$ are not intervals (w_n is not in $f(J_n)$), the set $\mathrm{Dar}(f)$ is dense.

We will prove that $f \in B_d(\mathcal{D} \cap Q_b)$. For this observe that every set

$$E_n = \{x \in C_n; x \text{ is a bilateral accumulation point of } C_n\},$$

$n = 1, 2, \ldots$, is the union of pairwise disjoint sets $E_{n,k}$, $k = 1, 2, \ldots$, which are c-dense in C_n, **i.e.** for each open interval I with $I \cap C_n \ne \emptyset$ and for each $k \ge 1$ the cardinality of the intersection $E_{n,k} \cap I$ is equal continuum.

For $n, k \ge 1$ there are functions $g_{n,k} : E_{n,k} \to M_n$ such that for each interval I with $I \cap E_n \ne \emptyset$ the equality $g_{n,k}(I \cap E_{n,k}) = M_n$ is true. For $k \ge 1$ let

$$g_k(x) = \begin{cases} g_{n,i}(x) & \text{for } x \in E_{n,i}, \text{ where } i \ge k \text{ and } n = 1, 2, \ldots \\ f(x) & \text{otherwise on } \mathbb{R}. \end{cases}$$

Evidently, $f = d - \lim_{k\to\infty} g_k$. From the construction of f follows that every function g_k, $k = 1, 2, \ldots$, is bilaterally quasicontinuous. We will prove that they have also the Darboux property. For this fix a positive integer k and observe that the functions f and consequently g_k are continuous at all points $x \in \mathbb{R} \setminus \bigcup_n C_n$. So,

$$\mathrm{Dar}(f) \subset \bigcup_n C_n.$$

If $x \in E_n$ for some positive integer n then from the construction of the function g_k and the properties of the functions $g_{n,k}$ follows that x is not in $\mathrm{Dar}(g_k)$. So, we suppose that there is a positive integer n such that x belong to the difference $C_n \setminus E_n$. Then x is isolated in C_n from the right or from the left hand.

Suppose that x is isolated in C_n from the left hand. Fix a positive real r. If

$$y \in \mathrm{Int}(K^+(g_k, x)) \ni g_k(x)$$

then, by the construction of g_k on E_n, follows that there is a decreasing sequence of points $x_j \in E_n \cap (x, x+r)$ such that

$$\lim_{j\to\infty} x_j = x \text{ and } g_k(x_j) = y \text{ for } j \geq 1.$$

Moreover,

$$g_k(x) = f(x) = f_n(x) \in \mathrm{Int}(M_n) \setminus \{w_i; i \geq 1\},$$

so for any

$$y \in \mathrm{Int}(K^-(g_k, x)) = \mathrm{Int}(K^-(f, x)) = \mathrm{Int}(K^-(f_n, x)) \ni g_k(x)$$

there is an increasing sequence of points

$$t_j \in (x-r, x) \cap (J_n \setminus C_n)$$

such that

$$\lim_{j\to\infty} t_j = x \text{ and } f_n(t_j) = y \text{ for } j \geq 1.$$

If there is a positive integer j with

$$y = f_n(t_j) = f(t_j) = g_k(t_j)$$

then in the considered case the proof is completed.

If not, there are positive integers $i > n$ and j such that

$$J_i \subset (x-r, x) \cap J_n \text{ and } t_j \in \mathrm{Int}(J_i).$$

Consequently,

$$y = f_n(t_j) \in \mathrm{Int}(M_i) \subset f_i(C_i) = f(C_i) \subset g_k(C_i),$$

and there is a point

$$z \in (x - r, x) \ \text{ with } \ g_k(z) = y.$$

So, x is not in $\mathrm{Dar}(g_k)$.

The proof that any point $x \in C_n$ which is isolated in C_n from the right hand belongs to $\mathbb{R} \setminus \mathrm{Dar}(g_k)$ is analogous. So, $\mathrm{Dar}(g_k) = \emptyset$ for all $k = 1, 2, \ldots$ and the proof is completed.

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