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ON H_1–INTEGRABLE FUNCTIONS

Abstract

In this note we deal with some problems related to the H_1-integral introduced recently by Garces, Lee, and Zhao. We give a new definition of that integral and characterize H_1-integrable functions almost everywhere equal to zero. We also discuss some results stated in the original paper on the H_1-integral.

1 Basic Notation

Let $\langle a, b\rangle$ denote compact subinterval of $\mathbb{R}$. By a *partial tagged partition* of $\langle a, b\rangle$ we understand any collection $\mathcal{P}$ of pairs (I, x) where I is a compact subinterval of $\langle a, b\rangle$ and $x \in I$ satisfying the following condition

- for every $(I, x), (J, y) \in \mathcal{P}$ we have $\operatorname{int} I \cap J = \emptyset$ or $(I, x) = (J, y)$.

If moreover

- $\bigcup_{(I,x)\in\mathcal{P}} I = \langle a, b\rangle$,

then $\mathcal{P}$ is called a *tagged partition* of $\langle a, b\rangle$. Partial tagged partitions will be denoted by the letter $\mathcal{P}$, tagged partitions by π.

If $f\colon \langle a, b\rangle \to \mathbb{R}$ and $\langle c, d\rangle = I \subset \langle a, b\rangle$, by $\Delta f(I)$ we mean $f(d) - f(c)$ and by $|I|$ the length of I. If $\mathcal{P}$ is a partial tagged partition of $\langle a, b\rangle$ then we denote

- $\sigma(\mathcal{P}, f) = \sum_{(I,x)\in\mathcal{P}} f(x)|I|$,
- $\Delta f(\mathcal{P}) = \sum_{(I,x)\in\mathcal{P}} \Delta f(I)$.

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Any positive function δ defined on $\langle a,b\rangle$ we call a *gauge*. We say that a partial tagged partition $\mathcal{P}$ is δ-fine if for every $(I,x)\in\mathcal{P}$ we have $I\subset(x-\delta(x),x+\delta(x))$.

Finally let χ_E denote the characteristic function of a set E and let $\mu(E)$ denote the Lebesgue measure of a measurable set E.

2 Introduction

In [2], Garces, Lee, and Zhao remarked that the Riemann integral may be defined using Moore-Smith limits. In the set Θ of all tagged partitions of $\langle a,b\rangle$ they introduced two relations:

- $\pi_1\sqsupseteq\pi_2$ if and only if for every $(I,x)\in\pi_2$, I is the union of some intervals from π_1,
- $\pi_1\geq\pi_2$ if and only if $\pi_1\sqsupseteq\pi_2$ and the set of all tags from π_2 is a subset of the set of tags from π_1.

Both $(\Theta,\sqsupseteq)$ and $(\Theta,\geq)$ are nets. It was remarked that the R-integral may be obtained as a limit of Riemann sums in the sense of $(\Theta,\sqsupseteq)$.

Definition 2.1. *We call a function $f\colon\langle a,b\rangle\to\mathbb{R}$, R-integrable to $\mathbf{I}\in\mathbb{R}$ if for any $\varepsilon>0$ there is a tagged partition π_1 of $\langle a,b\rangle$ such that for every $\pi\sqsupseteq\pi_1$*

$$|\sigma(\pi,f)-\mathbf{I}|<\varepsilon.$$

The H_1-integral was defined using the relation $\geq$:

Definition 2.2. *We call a function $f\colon\langle a,b\rangle\to\mathbb{R}$, H_1-integrable to $\mathbf{I}\in\mathbb{R}$ if there exists a gauge δ defined on $\langle a,b\rangle$ such that for any $\varepsilon>0$ one can find a tagged partition π_1 of $\langle a,b\rangle$ such that for every δ-fine $\pi\geq\pi_1$*

$$|\sigma(\pi,f)-\mathbf{I}|<\varepsilon.$$

(This last definition is a little bit different from the one given in [2], but, of course equivalent – here we do not demand π_1 to be δ-fine.)

However, using the relation $\geq$ in Definition 2.2 is redundant too. It is enough to use $\sqsupseteq$ – the same one which is used in the Moore-Smith Definition 2.1 of the Riemann integral. We define a new integral and prove its equivalence to the H_1-integral.

Definition 2.3. *We call a function $f\colon\langle a,b\rangle\to\mathbb{R}$, H_0-integrable to $\mathbf{I}\in\mathbb{R}$ if there exists a gauge δ defined on $\langle a,b\rangle$ such that for any $\varepsilon>0$ one can find a tagged partition π_0 of $\langle a,b\rangle$ such that for every δ-fine $\pi\sqsupseteq\pi_0$*

$$|\sigma(\pi,f)-\mathbf{I}|<\varepsilon.$$

Of course every H_0-integrable function is H_1-integrable and the integrals coincide. The opposite is also true.

Theorem 2.4. *Every H_1-integrable function is H_0-integrable.*

PROOF. Suppose $f\colon \langle a,b\rangle \to \mathbb{R}$ is H_1-integrable using the gauge δ. Fix an $\varepsilon > 0$, and let π_1 be a tagged partition of $\langle a,b\rangle$ such that for any δ-fine $\pi \geq \pi_1$,

$$\left|\sigma(\pi,f) - (H_1)\int_a^b f\right| < \frac{\varepsilon}{3}.$$

Let $\pi_0 \geq \pi_1$ be a tagged partition having the same tags as π_1 and let all intervals from π_0 have tags at their ends.

Now take any δ-fine tagged partition $\pi' \sqsupseteq \pi_0$. Denote $Z = \{z : (I,z) \in \pi_0\}$. For every $z \in Z$ choose $(I_z, x_z) \in \pi'$ such that $z \in I_z$. We may assume that for $z_1 \neq z_2$, the intervals I_{z_1} and I_{z_2} are different. By $\mathcal{P}$ denote family of all (I_z, x_z) for which $x_z \neq z$. For every $(I_z, x_z) \in \mathcal{P}$ let J_z be a closed interval such that

(i) $x_z \notin J_z \subset I_z$,

(ii) $z \in J_z$.

It is seen that intervals I_z and J_z have one common end in z. It is always possible to take J_z sufficiently short so that

(iii) $|f(x_z)||J_z| < \frac{\varepsilon}{3l}$ and $|f(z)||J_z| < \frac{\varepsilon}{3l}$, (where $l = \operatorname{card} Z$) and

(iv) $J_z \subset (x_z - \delta(x_z), x_z + \delta(x_z))$.

Note that

$$\pi'' = (\pi' \setminus \mathcal{P}) \cup \bigcup_{(I_z,x_z)\in\mathcal{P}} \big\{(J_z,z), (\operatorname{cl}(I_z \setminus J_z), x_z)\big\}$$

is tagged partition of the interval $\langle a,b\rangle$. Note also that from (iv) π'' is δ-fine and $\pi'' \geq \pi_1$. We may evaluate

$$\left|\sigma(\pi',f) - (H_1)\int_a^b f\right| \leq |\sigma(\pi'',f) - \sigma(\pi',f)| + \left|\sigma(\pi'',f) - (H_1)\int_a^b f\right|.$$

Of course $\left|\sigma(\pi'', f) - (H_1)\int_a^b f\right| < \frac{\varepsilon}{3}$ and from (iii)

$$\begin{aligned}
&|\sigma(\pi'', f) - \sigma(\pi', f)| \\
&\qquad = \Big| \sum_{(I_z,x_z)\in\mathcal{P}} \big(f(z)|J_z| + f(x_z)|I_z\backslash J_z|\big) + \sigma(\pi'\backslash\mathcal{P}, f) - \sigma(\pi', f)\Big| \\
&\qquad = \Big| \sum_{(I_z,x_z)\in\mathcal{P}} \big(f(z)|J_z| + f(x_z)|I_z\backslash J_z|\big) - \sum_{(I_z,x_z)\in\mathcal{P}} f(x_z)|I_z|\Big| \\
&\qquad \le \sum_{(I_z,x_z)\in\mathcal{P}} |f(z)||J_z| + \sum_{(I_z,x_z)\in\mathcal{P}} |f(x_z)||J_z| < \frac{2}{3}\varepsilon.
\end{aligned}$$

So

$$\left|\sigma(\pi', f) - (H_1)\int_a^b f\right| < \varepsilon$$

and f is H_0-integrable. □

Corollary 2.5. *A function $f\colon \langle a, b\rangle \to \mathbb{R}$ is H_1-integrable to $\mathbf{I} \in \mathbb{R}$ if and only if there exists a gauge δ on $\langle a, b\rangle$ such that for any $\varepsilon > 0$ one can find a tagged partition π_1 of $\langle a, b\rangle$ such that for any δ-fine $\pi \sqsupseteq \pi_1$*

$$|\sigma(\pi, f) - \mathbf{I}| < \varepsilon.$$

Comparing Definitions 2.1 and 2.3 we see that every Riemann integrable function is H_0-integrable. However, there are R-nonintegrable functions which are H_0-integrable – for example the classical Dirichlet function (see *Example 2* in [2]). It is nice that using the same relation as in Definition 2.1 we may obtain – having only at the start of integrating some suitable gauge which is independent of ε – an essentially wider class of integrable functions.

3 Some Facts from [2]

The following theorems were proved in [2] (respectively *Lemma 4* and *Lemma 6*).

Theorem 3.1. *Let $f\colon \langle a, b\rangle \to \mathbb{R}$ be H_1-integrable on a closed set $X_2 \subset \langle a, b\rangle$, and – using gauge δ_1 – on another closed set $X_1 \subset X_2$. Assume $f(x) = 0$ for $x \in \langle a, b\rangle \setminus X_2$. If the H_1-primitive F of f is absolutely continuous on $\langle a, b\rangle$, then f is H_1-integrable on X_2 using gauge δ equal to δ_1 on X_1.*

It was asserted in [2] that Theorem 3.1 holds even if $X_1 \not\subset X_2$ (with X_2 replaced by $X_1 \cup X_2$), but the proof was done only for the situation described above. It is not clear that the assumption of absolute continuity of F is enough for H_1-integrability of f on $X_1 \cup X_2$ (look Section 6 of this paper).

Theorem 3.2. *Let X be a closed subset of $\langle a,b\rangle$. If function $f\colon \langle a,b\rangle \to \mathbb{R}$ is H_1-integrable and bounded on $\langle a,b\rangle$, then it is H_1-integrable on X.*

4 H_0-Integrability on Null Sets

Lemma 4.1. *Let $f\colon \langle a,b\rangle \to \mathbb{R}$ be H_0-integrable on closed sets $X_1, X_2, X_3, \dots$ all of measure zero and such that $X_1 \subset X_2 \subset X_3 \subset \dots$. Then f is H_0-integrable on $X = \bigcup_{i=1}^{\infty} X_i$.*

PROOF. There exists a decreasing sequence $(\delta_i)_{i=1}^{\infty}$ of gauges on $\langle a,b\rangle$ such that for δ_i-fine partial tagged partition $\mathcal{P}$,

$$|\sigma(\mathcal{P}, f\chi_{X_i})| < \frac{1}{2^i}.$$

(This follows from the ordinary Saks-Henstock Lemma.) Inductively using Theorem 3.1 we obtain a gauge δ' on X such that for every i there is a positive δ'_i on $\langle a,b\rangle$ equal to δ' on X_i such that for arbitrary $\varepsilon > 0$ there exist respective tagged partitions $\pi_{i,\varepsilon}$ of $\langle a,b\rangle$ such that for any δ'_i-fine tagged partition $\pi \sqsupseteq \pi_{i,\varepsilon}$ the following inequality holds,

$$|\sigma(\pi, f\chi_{X_i})| < \varepsilon.$$

(This follows from the H_0-integrability of f on X_i.) Put

$$\delta(x) = \begin{cases} \min\{\delta'(x), \delta_i(x)\} & \text{if } x \in X_i \setminus X_{i-1} \\ \text{anything} & \text{if } x \notin X. \end{cases}$$

where $X_0 = \emptyset$. Fix an $\varepsilon > 0$. Let N be a positive integer such that $\sum_{i=N+1}^{\infty} \frac{1}{2^i} < \frac{\varepsilon}{2}$. For any δ-fine $\pi \sqsupseteq \pi_{N,\frac{\varepsilon}{2}}$ we use the Saks-Henstock Lemma for the H_0-integral to obtain

$$|\sigma(\pi, f\chi_X)| \le |\sigma(\pi, f\chi_{X_N})| + \sum_{i=N+1}^{\infty} |\sigma(\pi, f\chi_{X_i \setminus X_{i-1}})| < \varepsilon.$$

□

The statement of Lemma 4.1 for positive f, any ascending $X_1, X_2, X_3, \dots$ and with the assumption of the integrals' convergence appeared in [2] as *Theorem 5.* But in this general form it is false – as we shall soon show. However, the technique we use to prove Lemma 4.1 is the same technique Garces, Lee, and Zhao use. The same scheme of proving convergence of H_0-integrals will be used to prove Theorems 5.3 and 5.5.

Theorem 4.2. *Suppose* $f\colon \langle a,b\rangle \to \mathbb{R}$ *is equal to zero almost everywhere in* $\langle a,b\rangle$. *Then,* f *is* H_0*-integrable if and only if the set*

$$E = \{x \in \langle a,b\rangle :\ f(x) \neq 0\}$$

is contained in an $\mathcal{F}_\sigma$ *null set.*

PROOF. ($\Rightarrow$) We will prove $f^+ = \max\{0, f\}$ is H_0-integrable. There is an increasing sequence of sets $(E_n)_{n=1}^{\infty}$ such that $\bigcup_{n=1}^{\infty} E_n = E$ and for all n $\mu(\operatorname{cl} E_n) = 0$. For fixed $n \in \mathbb{N}$ put $D_n = E_n \cap \{x \in \langle a,b\rangle :\ 0 < f^+(x) \leq n\}$. For every tagged partition π of $\langle a,b\rangle$ we have

$$0 \leq \sigma(\pi, f^+\chi_{\operatorname{cl} D_n}) \leq \sigma(\pi, n\chi_{\operatorname{cl} E_n}). \tag{1}$$

From Theorem 3.2, the function $n\chi_{\operatorname{cl} E_n}$ is H_0-integrable (to zero). From (1), f^+ is H_0-integrable on $\operatorname{cl} D_n$. Applying Lemma 4.1 we conclude f^+ is H_0-integrable on $\bigcup_{n=1}^{\infty} \operatorname{cl} D_n$, so f^+ is H_0-integrable on $\langle a,b\rangle$. In the same way we prove that f^- is H_0-integrable.

($\Leftarrow$) There exists positive number M for which the set

$$D = \{x \in \langle a,b\rangle :\ |f(x)| > M\}$$

is not contained in an $\mathcal{F}_\sigma$ set of measure zero. For any fixed gauge δ on $\langle a,b\rangle$ there is a positive integer n such that the closure of the set

$$E_n = \left\{x \in D :\ \delta(x) > \frac{1}{n}\right\}$$

has measure $m > 0$.

Consider any tagged partition π_0 of $\langle a,b\rangle$ and put

$$\mathcal{P} = \{(I,x) \in \pi_0 :\ I \cap E_n \neq \emptyset\}.$$

Then $\mu(\mathcal{P}) \geq m$, because the sum of intervals from $\mathcal{P}$ contains $\operatorname{cl} E_n$. Consider the family of closed intervals

$$\mathcal{A} = \left\{J = I \cap \left\langle \frac{z}{n}, \frac{z+1}{n} \right\rangle :\ (I,x) \in \mathcal{P},\ z \in \mathbb{Z},\ J \cap E_n \neq \emptyset\right\}.$$

It follows that

$$\sum_{J \in \mathcal{A}} |J| \geq m.$$

For every $J \in \mathcal{A}$ take an $x_J \in J \cap E_n$. Note that the partial partition $\{(J, x_J) :\ J \in \mathcal{A}\}$ is δ-fine and

$$\sum_{J \in \mathcal{A}} |f(x_J)||J| \geq Mm,$$

so the Saks-Henstock Lemma for the H_0-integral is not valid for f. $\square$

Corollary 4.3. *Let $E \subset \langle a, b\rangle$ be a set of measure zero. If E is contained in an $\mathcal{F}_\sigma$ set of measure zero, then χ_E is H_0-integrable (on $\langle a, b\rangle$).*

Using Theorem 4.2 it is easy to obtain examples of almost everywhere equal to zero functions which are not H_0-integrable (thus solving the problem stated at the end of [2]). These can be characteristic functions of generic null sets or even 2nd category null sets. From these examples we see that not every characteristic function of an $\mathcal{F}_\sigma$ set is H_0-integrable. This implies that Levi's theorem does not hold for the H_0-integral. Let D be a $\mathcal{G}_\delta$ null set containing $\mathbb{Q} \cap \langle 0, 1\rangle$. It is well known that D is generic, so its characteristic function is H_0-nonintegrable. The complement of D is an $\mathcal{F}_\sigma$ set, and $\chi_{\langle 0,1\rangle \setminus D}$ is H_0-nonintegrable since χ_D is.

The last example shows that *Theorem 5* in [2] is false as it is stated. From this same example we see that the proof of *Theorem 10* in [2] is also not correct – because, for every subset $E \subset \langle 0, 1\rangle \setminus D$ with measure $\mu(E) = \mu(\langle 0, 1\rangle \setminus D)$, the function χ_E is H_0-nonintegrable. The statement of *Theorem 10*, however, holds in this situation. In both proofs in [2], the δ-variation of F on $\langle a, b\rangle \setminus \bigcup_{n=1}^{\infty} X_n$ was neglected. (For the definition of δ-variation look [3].)

Problem 4.4. *Is Theorem 10 from* [2] *true?*

Using Lemma 4.3 and Theorem 4.2 we have the following result.

Corollary 4.5. *The characteristic function of a set E of measure zero is H_0-integrable if and only if E is a countable union of sets whose closures are of measure zero.*

It is well known that:

Theorem 4.6. *The characteristic function of a set E of measure zero is Riemann integrable if and only if the closure of E is of measure zero.*

Combining these two we obtain a nice characterization of the relationship between these integrals.

Corollary 4.7. *The characteristic function of a set E of measure zero is H_0-integrable if and only if E is a countable union of sets whose characteristic functions are Riemann integrable.*

Problem 4.8. *Characterize sets with H_0-integrable characteristic functions.*

5 Improper H_0-Integral and Denjoy Generalization

As was pointed out by the referee, the contents of this section have been considered in [1], however this paper is not available to the author.

By *the improper T-integral* ($\widetilde{T}$-integral), where T is some kind of integration on intervals, we mean the integral which arises from taking pointwise limits of T-integrals.

Definition 5.1. *We say that $f\colon \langle a,b\rangle \to \mathbb{R}$ is $\widetilde{T}$-integrable to $\mathbf{I} \in \mathbb{R}$ if one of the following two conditions is satisfied*

- *for every $c \in \langle a,b)$, f is T-integrable on $\langle a,c\rangle$ and*

$$\lim_{c\to b^-} (T)\int_a^c f = (\widetilde{T})\int_a^b f \in \mathbb{R},$$

- *for every $c \in (a,b\rangle$, f is T-integrable on $\langle c,b\rangle$ and*

$$\lim_{c\to a^+} (T)\int_c^b f = (\widetilde{T})\int_a^b f \in \mathbb{R}.$$

It is well known that the improper Henstock integral ($\widetilde{H}$-integral) is equivalent to the Henstock integral – taking limits we obtain nothing new. However, for the integrals of Riemann and Lebesgue we obtain essentially wider classes of integrable functions. We shall now look at the improper H_0-integral. But first we make note of the following technical observation.

Remark 5.2. *If the function f is H_0-integrable on closed intervals I_1 and I_2 with $I_2 \subset I_1$ using functions δ_1 and δ_2 respectively, then f is H_0-integrable on I_1 using δ defined as follows*

$$\delta(x) = \begin{cases} \delta_1(x) & \text{if } x \in I_1 \setminus I_2 \\ \delta_2(x) & \text{if } x \in I_2. \end{cases}$$

Lemma 5.3. *Every $\widetilde{H_0}$-integrable function is H_0-integrable and the integrals coincide.*

PROOF. Let $f\colon \langle a,b\rangle \to \mathbb{R}$ be $\widetilde{H_0}$-integrable and assume, for convenience, that $a < b-1$. Then f is Henstock integrable and we let F be a Henstock primitive of f. Consider the sequence of intervals $I_n = \langle a, b-\frac{1}{n}\rangle$. From the ordinary Saks-Henstock Lemma there are gauges δ_n on I_n respectively such that for any δ_n-fine partial tagged partition $\mathcal{P}$ of I_n, we have

$$|\sigma(\mathcal{P},f) - \Delta F(\mathcal{P})| < \frac{1}{2^n}. \tag{2}$$

Using Remark 5.2 inductively, we obtain a gauge δ' on $\langle a,b)$ such that for all n and arbitrary $\varepsilon > 0$ there are tagged partitions $\pi_{n,\varepsilon}$ of I_n such that for every

δ'-fine $\pi \sqsupseteq \pi_{n,\varepsilon}$ we have

$$\left|\sigma(\pi, f) - (H_0)\int_{I_n} f\right| < \varepsilon. \tag{3}$$

Put

$$\delta(x) = \begin{cases} \min\left\{\delta_n(x), \delta'(x), b - \frac{1}{n} - x\right\} & \text{if } b - \frac{1}{n-1} \le x < b - \frac{1}{n} \\ \text{anything} & \text{if } x = b \end{cases}$$

(here $b - \frac{1}{0} = a$). Fix an ε. There is an integer N such that

- $|(H)\int_c^b f| < \frac{\varepsilon}{6}$ for all $c \in \langle b - \frac{1}{N}, b\rangle$,
- $N\varepsilon > 6|f(b)|$,
- $\frac{1}{2^N} < \frac{\varepsilon}{3}$.

Take any δ-fine $\pi \sqsupseteq \left(\pi_{N,\frac{\varepsilon}{3}} \cup \left\{\left(\left\langle b - \frac{1}{N}, b\right\rangle, b\right)\right\}\right)$. All tagged intervals from π excluding the last one whose tag is b are contained in some I_k for $k > N$. Therefore, from (2) and (3) we have

$$\begin{aligned}\left|\sigma(\pi, f) - \int_a^b f\right| &\le \left|\sum_{(I,x)\in\pi,\ I\subset I_N} f(x)|I| - \int_{I_N} f\right| \\ &+ \left|\sum_{(I,x)\in\pi,\ I\subset \operatorname{cl}(I_k\setminus I_N)} \left(f(x)|I| - \int_I f\right)\right| + 2\frac{\varepsilon}{6} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

□

Corollary 5.4. *The H_0-integral is not absolute.*

Next we state a theorem dealing with Denjoy's generalization of the H_0-integral.

Lemma 5.5. *Let P be a nonempty perfect subset of $\langle a, b\rangle$. Let $f\colon \langle a, b\rangle \to \mathbb{R}$ be equal to zero on P and H_0-integrable on the closures $I_1, I_2, I_3, \ldots$ of intervals contiguous to P in $\langle a, b\rangle$. If F is a Henstock primitive of f and $\sum_{n=1}^{\infty} \omega(F, I_n) < +\infty$, then f is H_0-integrable and $(H_0)\int_a^b f = \sum_{n=1}^{\infty} \Delta F(I_n)$.*

Proof. It follows from the Henstock integrability of f on I_n that there exist gauges $\delta_n^{(1)}$ on I_n, $n = 1, 2, \ldots$, such that for every $\delta_n^{(1)}$-fine partial tagged partition $\mathcal{P}_n$ of I_n we have

$$|\sigma(\mathcal{P}_n, f) - \Delta F(\mathcal{P}_n)| < \frac{1}{2^n}.$$

Let gauges $\delta_n^{(2)}$ come from H_0-integrability of f on I_n and put

$$\delta(x) = \begin{cases} \min\{\delta_n^{(1)}, \delta_n^{(2)}, \rho(x, \langle a,b\rangle \setminus \operatorname{int} I_n)\} & \text{if } x \in \operatorname{int} I_n \\ \text{anything} & \text{if } x \in P. \end{cases}$$

Fix an $\varepsilon > 0$. There is an integer N such that

$$\sum_{n=N+1}^{\infty} \frac{1}{2^n} < \frac{\varepsilon}{3} \quad \text{and} \quad \sum_{n=N+1}^{\infty} \omega(F, I_n) < \frac{\varepsilon}{6}.$$

For $n = 1, 2, \dots N$, let π_n be a tagged partition of I_n such that $|\sigma(\pi, f) - \Delta F(I_n)| < \frac{\varepsilon}{3N}$ for every $\delta_n^{(2)}$-fine $\pi \sqsupseteq \pi_n$. Complete $\bigcup_{n=1}^{N} \pi_n$ to any tagged partition π_0 of $\langle a,b\rangle$ and consider an arbitrary δ-fine $\pi \sqsupseteq \pi_0$. For each n let $\mathcal{P}_n = \{(I,x) \in \pi : I \subset I_n\}$, and note that $\sigma(\pi \setminus \bigcup_{n=1}^{\infty} \mathcal{P}_n, f) = 0$. Then,

$$\left|\sigma(\pi, f) - \sum_{n=1}^{\infty} \Delta F(I_n)\right|$$

$$\leq \sum_{n=1}^{N} |\sigma(\mathcal{P}_n, f) - \Delta F(I_n)| + \sum_{n=N+1}^{\infty} |\sigma(\mathcal{P}_n, f) - \Delta F(\mathcal{P}_n)| + \left|\Delta F\Big(\pi \setminus \bigcup_{n=1}^{\infty} \mathcal{P}_n\Big)\right|$$

$$< \frac{\varepsilon}{3} + \sum_{n=N+1}^{\infty} \frac{1}{2^n} + 2 \sum_{n=N+1}^{\infty} \omega(F, I_n) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + 2\frac{\varepsilon}{6} = \varepsilon.$$

Hence, f is H_0-integrable. □

Theorem 5.6. *Suppose $f\colon \langle a,b\rangle \to \mathbb{R}$ is H_0-integrable. If $P \subset \langle a,b\rangle$ is closed and f is Henstock integrable on P, then f is H_0-integrable on P.*

PROOF. If $\phi = f\chi_{\langle a,b\rangle \setminus P}$, then ϕ is Henstock integrable on $\langle a,b\rangle$. We will prove it is H_0-integrable. Let Q be the set of points $x \in \langle a,b\rangle$ such that for every interval I with $x \in \operatorname{int} I$ the function ϕ is not H_0-integrable on I. It is easy to see that Q is closed and $Q \subset P$. Moreover, it follows from Lemma 5.3 that Q is perfect and that ϕ is H_0-integrable on the closure of every interval contiguous to Q. Suppose $Q \neq \emptyset$. As the Henstock primitive F of ϕ is ACG_* on Q, it follows from the Baire Category Theorem that there is a portion $R = Q \cap J \neq \emptyset$ of Q on which F is AC_*. But then the series of oscillations of F on intervals contiguous to R converges. Therefore, from Lemma 5.5, ϕ is H_0-integrable on $J \setminus Q$. Because $\phi = 0$ on Q it follows that ϕ is H_0-integrable on J, a contradiction.

As $Q = \emptyset$, for every $x \in \langle a,b\rangle$ there exists an interval I having x as a point of interior such that ϕ is H_0-integrable on I. But then the compactness of $\langle a,b\rangle$ implies H_0-integrability of ϕ on $\langle a,b\rangle$. □

Note that this theorem is an extension of Theorem 3.2.

6 Other Problems

In [2] it was remarked that the following is obvious:

(*) Let f be H_0-integrable on A and B. Then f is H_0-integrable on $A \cup B$.

Actually this is false. Divide the interval $\langle 0,1\rangle$ into sequence of closed nonoverlapping intervals $I_1, I_2, I_3, \dots$ having 1 as the only point of accumulation. Next divide every I_i into three nonoverlapping intervals, J_1^i, J_2^i, J_3^i and let $f : \langle 0,1\rangle \to \mathbb{R}$ be such that

$$(H_0)\int_{J_1^i} f = -\,(H_0)\int_{J_2^i} f = (H_0)\int_{J_3^i} f = \frac{1}{i}.$$

Now consider $A = \bigcup_{i=1}^{\infty}(J_1^i \cup J_2^i)$ and $B = \bigcup_{i=1}^{\infty}(J_2^i \cup J_3^i)$. Evidently f is H_0-integrable on both A and B. Moreover, it follows from Lemma 5.3 that $(H_0)\int_A f = (H_0)\int_B f = 0$. But, $(L)\int_{A\cap B} f = \sum_{i=1}^{\infty} \frac{1}{i} = +\infty$, so f cannot be H_0-integrable on $A \cap B$ and consequently not on $A \cup B$. The question is:

Problem 6.1. *Is (*) true when we add the hypothesis that f is Henstock integrable on $A \cup B$?*

Let us finally state the next problems.

Problem 6.2. *Is every Henstock integrable Baire one function H_0-integrable?*

Problem 6.3. *Is every derivative H_0-integrable?*

Problem 6.4. *Characterize H_0-integrable functions in the "Riemann manner".*

By the "Riemann manner" we mean the kind of characterization which is used in the theorem asserting that the class of Riemann integrable functions coincides with the class of bounded and almost everywhere continuous functions. After studying the nature of H_0-integrable functions the author conjectures that such a characterization is possible.

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