ON ALMOST CONTINUOUS DERIVATIONS

Abstract

It is proved that every derivation is the sum of two almost continuous (in Stallings’ sense) derivations and the limit of a sequence (of a transfinite sequence) of almost continuous derivations.

A function \( g : (a, b) \to \mathbb{R} \) is said to be almost continuous (in Stallings’ sense [5]) if for every open set \( D \subset \mathbb{R}^2 \) containing the graph \( \text{Gr}(g) \) of the function \( g \) there is a continuous function \( h : (a, b) \to \mathbb{R} \) with \( \text{Gr}(h) \subset D \).

A function \( f : \mathbb{R} \to \mathbb{R} \) is called additive ([4]) if it satisfies Cauchy’s equation

\[
f(x + y) = f(x) + f(y), \quad \text{for all } x, y \in \mathbb{R}.
\]

An additive function \( f : \mathbb{R} \to \mathbb{R} \) is called a derivation if it satisfies the equation

\[
f(xy) = xf(y) + yf(x), \quad \text{for all } x, y \in \mathbb{R}.
\]

It is well known that there exists a discontinuous additive almost continuous function \( f : \mathbb{R} \to \mathbb{R} \) ([2] and [3]) and that every additive function is the sum of two additive almost continuous functions and the limit of a sequence (of a transfinite sequence) of additive almost continuous functions ([1]). In this article I prove analogous theorems for derivations.

If \( f : \mathbb{R} \to \mathbb{R} \) is a function, by a blocking set of \( f \) we mean a closed set \( K \subset \mathbb{R}^2 \) such that \( \text{Gr}(f) \cap K = \emptyset \) and \( \text{Gr}(g) \cap K \neq \emptyset \) for every continuous function \( g : \mathbb{R} \to \mathbb{R} \). An irreducible blocking set (IBS) \( K \) of \( f \) is a blocking set of \( f \) such that no proper subset of \( K \) is a blocking set ([3]).

It is known that \( f : \mathbb{R} \to \mathbb{R} \) is almost continuous if and only if it has no blocking set. Moreover, if \( f \) is not almost continuous, then there is an
(IBS) K of f and the x-projection pr_x(K) of K is a non-degenerate interval ([3]). Let
\[ K_0, K_1, \ldots, K_\alpha, \ldots, \alpha < \omega_c, \]
be a transfinite sequence of all irreducible blocking sets in \( \mathbb{R}^2 \), with \( K_\alpha \neq K_\beta \) for \( \alpha \neq \beta, \alpha, \beta < \omega_c \) and \( \omega_c \) denoting the first ordinal of the cardinality of the continuum.

Let \( F \subset K \) be a field. An element \( a \in K \) is called algebraically dependent (or algebraic) over \( F \) if there exists a non-trivial (\( \neq 0 \)) polynomial \( p \) with the coefficients from \( F \) such that \( p(a) = 0 \).

The algebraic closure of \( F \) (in \( K \)) is the set \( \text{algcl}(F) = \{ a \in K : a \text{ is algebraic over } F \} \).

It is known that \( \mathbb{R} \neq \text{algcl}(\mathbb{Q}) \) and there exists an algebraic base of \( \mathbb{R} \) over \( \mathbb{Q} \) ([4, p. 102]).

In the proofs of the main theorems we use the following.

**Theorem 1** ([4] Th. 1, p. 352). Let \( K \) be a field of characteristic zero, let \( F \) be a subfield of \( K \), let \( X \) be an algebraic base of \( K \) over \( F \), if it exists, and let \( X = \emptyset \) otherwise. Let \( f : F \rightarrow K \) be a derivation. Then, for every function \( u : X \rightarrow K \) there exists a unique derivation \( g : K \rightarrow K \) such that \( g|_F = f \) and \( g|_X = u \).

**Theorem 2.** If \( f : \mathbb{R} \rightarrow \mathbb{R} \) is a derivation, then there are two almost continuous derivations \( g, h : \mathbb{R} \rightarrow \mathbb{R} \) such that \( f = g + h \).

**Proof.** We apply transfinite induction. Since \( \text{pr}_x(K_0) \) is a non-degenerate interval, there are algebraically independent (over \( \mathbb{Q} \)) elements \( u_0, v_0 \in \text{pr}_x(K_0)\setminus \text{algcl}(\mathbb{Q}) \).

Next, we fix an ordinal \( \alpha > 0 \) with \( \alpha < \omega_c \) and assume that for each ordinal \( \beta < \alpha \) we have defined elements \( u_\beta, v_\beta \in \text{pr}_x(K_\beta) \), such that the set
\[ S_\alpha = \{ u_\beta, v_\beta : \beta < \alpha \} \]
is algebraically independent (over \( \mathbb{Q} \)) and \( (u_\beta_1, v_\beta_1) \neq (u_\beta_2, v_\beta_2) \) for \( \beta_1 < \beta_2 \leq \beta \).

Finally, there are algebraically independent (over \( \mathbb{Q} \)) elements \( u_\alpha, v_\alpha \in \text{pr}_x(K_\alpha) \setminus \text{algcl}(\mathbb{Q} \cup S_\alpha) \).

Observe that the set \( S = A \cup B \), where \( A = \{ u_\alpha : \alpha < \omega_c \} \) and \( B = \{ v_\alpha : \alpha < \omega_c \} \) are algebraically independent (over \( \mathbb{Q} \)). Consequently, there is an algebraic base \( X \supset S \) in \( \mathbb{R} \) (over \( \mathbb{Q} \)).
For every $\alpha < \omega_c$, let $t_\alpha, z_\alpha \in \mathbb{R}$ be points such that
\[(u_\alpha, t_\alpha) \in K_\alpha \text{ and } (v_\alpha, z_\alpha) \in K_\alpha.\]

Let
\[g_1(x) = \begin{cases} t_\alpha & \text{if } x = u_\alpha, \ \alpha < \omega_c, \\ f(x) - z_\alpha & \text{if } x = v_\alpha, \ \alpha < \omega_c, \\ 0 & \text{otherwise in } X, \end{cases}\]
and let
\[h_1(x) = \begin{cases} f(x) - t_\alpha & \text{if } x = u_\alpha, \ \alpha < \omega_c, \\ z_\alpha & \text{if } x = v_\alpha, \ \alpha < \omega_c, \\ f(x) & \text{otherwise in } X, \end{cases}\]
and let $g : \mathbb{R} \to \mathbb{R}$ be an extension of $g_1$ to some derivation. Since $f - g$ is a derivation, the function $h : \mathbb{R} \to \mathbb{R}$ such that $h|_X = h_1$ and $h(x) = f(x) - g(x)$ for $x \in \mathbb{R} \setminus X$ is an extension of $h_1$ to some derivation.

Observe that for every $\alpha < \omega_c$,
\[(u_\alpha, g(u_\alpha)) = (u_\alpha, t_\alpha) \in K_\alpha \text{ and } (v_\alpha, h(v_\alpha)) = (v_\alpha, z_\alpha) \in K_\alpha.\]

So, the functions $g, h$ are almost continuous and evidently $f = g + h$. \qed

The next remark follows from Theorem 2.

**Remark 1.** There are almost continuous derivations $f : \mathbb{R} \to \mathbb{R}$ which are discontinuous.

**Proof.** It suffices to find a discontinuous derivation $\phi : \mathbb{R} \to \mathbb{R}$ ([4, Th. 2, p. 352]) and two almost continuous derivations $g, h : \mathbb{R} \to \mathbb{R}$ with $\phi = g + h$. Then, at least one derivation $g$ or $h$ is discontinuous. \qed

**Theorem 3.** If $f : \mathbb{R} \to \mathbb{R}$ is a derivation, then there is a sequence of almost continuous derivations $f_n : \mathbb{R} \to \mathbb{R}$, $n \geq 1$, such that $f = \lim_{n \to \infty} f_n$.

**Proof.** As in the proof of Theorem 2, for every $\alpha < \omega_c$ we find a sequence of points
\[x_{\alpha,n} \in \text{pr}_x(K_\alpha), \ n = 1, 2, \ldots,\]
such that the set
\[S = \{x_{\alpha,n} : \alpha < \omega_c, \ n \geq 1\}\]
is algebraically independent over $\mathbb{Q}$. Let $X \supset S$ be an algebraic basis (over $\mathbb{Q}$) in $\mathbb{R}$. For each point $x_{\alpha,n}$ there is a point $y_{\alpha,n}$ such that
\[(x_{\alpha,n}, y_{\alpha,n}) \in K_\alpha, \ \alpha < \omega_c, n \geq 1.\]
For \( n = 1, 2, \ldots \), let

\[
g_n(x) = \begin{cases} 
y_{\alpha, k} & \text{if } x = x_{\alpha, k}, \ \alpha < \omega_c, \ k \geq n, \\
f(x) & \text{otherwise in } X,
\end{cases}
\]

and let \( f_n \) be an extension of \( g_n \) to a derivation on \( \mathbb{R} \). Since

\[(x_{\alpha, n}, y_{\alpha, n}) \in K_\alpha \cap \text{Gr}(f_n) \text{ for } \alpha < \omega_c \text{ and } n \geq 1,
\]

all functions \( f_n \) are almost continuous. Moreover, if \( x = x_{\alpha, k} \), where \( \alpha < \omega_c \), and \( k \geq 1 \), then \( f_n(x) = f(x) \) for \( n > k \) and if \( x \in X \) and \( x \neq x_{\alpha, k} \) for all \( \alpha < \omega_c \) and \( k \geq 1 \), then \( f_n(x) = f(x) \) for all \( n \geq 1 \). So, \( f = \lim_{n \to \infty} f_n \) on \( X \) and consequently on \( \mathbb{R} \). Thus, the proof is completed.

Now we will consider the transfinite convergence. Recall that a transfinite sequence of functions \( f_\alpha : \mathbb{R} \to \mathbb{R} \), where \( \alpha < \omega_1 \) (\( \omega_1 \) denoting the first uncountable ordinal), converges to a function \( f : \mathbb{R} \to \mathbb{R} \) (then we write \( \lim_\alpha f_\alpha = f \)) if for each point \( x \in \mathbb{R} \) there is a countable ordinal \( \beta(x) \) such that for each countable ordinal \( \alpha > \beta(x) \) the equality \( f_\alpha(x) = f(x) \) holds ([6]).

**Theorem 4.** Assume that \( \omega_1 = \omega_c \). If \( f : \mathbb{R} \to \mathbb{R} \) is a derivation, then there is a transfinite sequence of almost continuous derivations \( f_\alpha : \mathbb{R} \to \mathbb{R}, \ \alpha < \omega_1, \) such that \( \lim_\alpha f_\alpha = f \).

**Proof.** As above we find pairwise disjoint sets \( T_\alpha, \alpha < \omega_1 = \omega_c, \) such that every set

\[
\text{pr}_x(K_\alpha) \cap T_\alpha, \ \alpha < \omega_1,
\]

is uncountable, and the union \( \bigcup_{\alpha < \omega_1} \text{pr}_x(K_\alpha) \cap T_\alpha \) is algebraically independent over \( \mathbb{Q} \) in \( \mathbb{R} \). For each \( \alpha < \omega_1 \), let \( (x_{\alpha, \beta})_{\beta < \omega_1} \) be a transfinite sequence of all points of the set \( \text{pr}_x(K_\alpha) \cap T_\alpha \), and let

\[
g_\alpha(x) = \begin{cases} 
y_{\alpha, \beta} & \text{if } x = x_{\alpha, \beta}, \ \omega_1 > \beta \geq \alpha, \\
f(x) & \text{otherwise in } X,
\end{cases}
\]

where \( y_{\alpha, \beta} \) are points such that

\[(x_{\alpha, \beta}, y_{\alpha, \beta}) \in K_\alpha, \ \alpha, \beta < \omega_1,
\]

and let \( f_\alpha \) be an extension \( g_\alpha \) to a derivation on \( \mathbb{R} \). Analogously, as in the proof of Theorem 3 we can observe that all functions \( f_\alpha \) are almost continuous and

\[
\lim_\alpha f_\alpha = f.
\]

This completes the proof.
References


