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ON THE JOHN-STRÖMBERG CHARACTERIZATION OF BMO FOR NONDOUBLING MEASURES

Abstract

A well known result proved by F. John for $0 < \lambda < 1/2$ and by J.-O. Strömberg for $\lambda = 1/2$ states that

$$\|f\|_{BMO(\omega)} \asymp \sup_Q \inf_{c\in\mathbb{R}} \inf\{\alpha > 0 : \omega\{x \in Q : |f(x) - c| > \alpha\} < \lambda\omega(Q)\}$$

for any measure ω satisfying the doubling condition. In this note we extend this result to all absolutely continuous measures. In particular, we show that Strömberg's "1/2-phenomenon" still holds in the nondoubling case. An important role in our analysis is played by a weighted rearrangement inequality, relating any measurable function and its John-Strömberg maximal function. This inequality was proved earlier by the author in the doubling case; here we show that actually it holds for all weights. Also we refine a result due to B. Jawerth and A. Torchinsky, concerning pointwise estimates for the John-Strömberg maximal function.

1 Introduction

Let ω be a weight; that is, non-negative, locally integrable function on $\mathbb{R}^n$. Given a measurable set E, let $\omega(E) = \int_E \omega(x)\,dx$. A weight (or measure) ω is doubling if there exists a constant c such that $\omega(2Q) \le c\omega(Q)$ for all cubes $Q \subset \mathbb{R}^n$. Throughout this work we shall only consider open cubes with sides parallel to the coordinate axes.

We say that f^*_ω is the weighted non-increasing rearrangement of a measurable function f with respect to ω if it is non-increasing on $(0, \omega(\mathbb{R}^n))$ and ω-equimeasurable with $|f|$; i.e., for all $\alpha > 0$,

$$|\{t \in (0, \omega(\mathbb{R}^n)) : f^*_\omega(t) > \alpha\}| = \omega\{x \in \mathbb{R}^n : |f(x)| > \alpha\}.$$

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We shall assume that the rearrangement is left-continuous. Then it is uniquely determined and can be defined by the equality

$$f^*_\omega(t) = \sup_{\omega(E)=t} \inf_{x\in E} |f(x)|.$$

A function $f \in L^1_{loc}(\omega)$ is said to belong to $BMO(\omega)$ if

$$\|f\|_{BMO(\omega)} = \sup_Q \frac{1}{\omega(Q)} \int_Q |f(x) - f_{Q,\omega}|\omega(x)\,dx < \infty,$$

where $f_{Q,\omega} = (\omega(Q))^{-1}\int_Q f\omega$ is the mean value of f over Q.

It is well known that if a weight ω is doubling, then any $f \in BMO(\omega)$ satisfies the John-Nirenberg inequality which says that for every cube Q we have (see [4, 7]):

$$\big((f - f_{Q,\omega})\chi_Q\big)^*_\omega(t) \le c\|f\|_{BMO(\omega)} \log \frac{2\omega(Q)}{t} \quad (0 < t < \omega(Q)). \tag{1}$$

(This inequality is usually formulated in terms of the distribution function but it will be a more convenient for us to use this equivalent "rearrangement" form.)

F. John [3] and J.-O. Strömberg [11] showed that a very weak condition

$$\sup_Q \inf_{c\in\mathbb{R}} \big((f - c)\chi_Q\big)^*_\omega\big(\lambda\omega(Q)\big) < \infty \ (0 < \lambda \le 1/2) \tag{2}$$

equivalent to $f \in BMO(\omega)$; so (2) implies (1). This result was obtained in the unweighted case but it can easily be extended to the case when ω is any doubling weight. In [11], the following so-called local sharp maximal function was introduced (or the John-Strömberg maximal function) which naturally connected with condition (2):

$$M^\#_{\lambda,\omega} f(x) = \sup_{Q\ni x} \inf_{c\in\mathbb{R}} \big((f - c)\chi_Q\big)^*_\omega\big(\lambda\omega(Q)\big) \quad (0 < \lambda \le 1).$$

The John-Strömberg characterization states that for $0 < \lambda \le 1/2$,

$$\lambda\|M^\#_{\lambda,\omega} f\|_\infty \le \|f\|_{BMO(\omega)} \le c\|M^\#_{\lambda,\omega} f\|_\infty. \tag{3}$$

Note that the left-hand side of (3) trivially holds, by Chebyshev's inequality, for all $0 < \lambda \le 1$. The right-hand side of (3) was proved by John [3] for $0 < \lambda < 1/2$, and a more difficult result that it also holds for $\lambda = 1/2$ was proved by Strömberg [11]. A simple argument shows (see [11]) that this inequality fails for $\lambda > 1/2$. A key ingredient in proving the right-hand side of (3) is a somewhat stronger formulation of the John-Nirenberg inequality

$$\big((f - m_{f,\omega}(Q))\chi_Q\big)^*_\omega(t) \le c\|M^{\#}_{1/2,\omega} f\|_\infty \log \frac{2\omega(Q)}{t} \ (0 < t < \omega(Q)), \qquad (4)$$

where $m_{f,\omega}(Q)$ is a weighted median value of f over Q; i.e., a, possibly nonunique, real number such that

$$\omega\{x \in Q : f(x) > m_{f,\omega}(Q)\} \le \omega(Q)/2$$

and

$$\omega\{x \in Q : f(x) < m_{f,\omega}(Q)\} \le \omega(Q)/2.$$

In a recent work [6], it is shown that actually the John-Nirenberg inequality (1) holds for any (not necessarily doubling) weight ω and the corresponding constant c in (1) depends only on n. A natural question arises whether the John-Strömberg characterization of BMO still holds for nondoubling measures. A closely related question is whether or not the "1/2-phenomenon" expressed in (4) holds in the general nondoubling case. It is known, for example, that for BMO defined in terms of local polynomial approximation the corresponding John-Strömberg characterization fails when $\lambda = 1/2$ (see [10]).

In this paper, using a covering argument presented in [6], we extend to nondoubling weights a weighted rearrangement inequality proved in [5] only in the doubling case. More precisely, we get the following theorem.

Theorem 1.1. *Let ω be any weight. Then for any measurable function f and each cube $Q \subset \mathbb{R}^n$ we have*

$$(f\chi_Q)^*_\omega(t) \le 2\big((M^{\#}_{\lambda_n,\omega} f)\chi_Q\big)^*_\omega(2t) + (f\chi_Q)^*_\omega(2t) \ (0 < t \le \lambda_n\omega(Q)), \qquad (5)$$

where a constant λ_n depends only on n.

It follows easily from this theorem that the nondoubling John-Strömberg characterization holds for $\lambda \le \lambda_n$. Next, combining geometric arguments from [6] and [11], we show that $\|M^{\#}_{\lambda_n,\omega}\|_\infty \le c_n\|M^{\#}_{1/2,\omega}\|_\infty$, which gives a positive answer to our question.

Theorem 1.2. *Inequality (4) holds for any weight ω with a constant c depending only on n.*

To state our next result, we recall that the weighted Hardy-Littlewood and Fefferman-Stein maximal functions are defined respectively by

$$M_\omega f(x) = \sup_{Q \ni x} \frac{1}{\omega(Q)} \int_Q |f(y)|\omega(y)\,dy$$

and

$$f^{\#}_{\omega}(x)=\sup_{Q\ni x}\frac{1}{\omega(Q)}\int_Q |f(y)-f_{Q,\omega}|\omega(y)\,dy.$$

In [2], a more precise result than (3) was obtained for any doubling weight ω and $\lambda\le\lambda(\omega,n)$; namely, for any $f\in L^1_{loc}(\omega)$ and all $x\in\mathbb{R}^n$,

$$c_{\lambda,\omega}M_{\omega}(M^{\#}_{\lambda,\omega}f)(x)\le f^{\#}_{\omega}(x)\le c_{\omega}M_{\omega}(M^{\#}_{\lambda,\omega}f)(x)\ (0<\lambda\le\lambda(\omega,n)). \tag{6}$$

(This was proved only in the unweighted case but the proof easily works for any doubling weight.) Clearly, (6) implies (3) for $\lambda\le\lambda(\omega,n)$. However, the method of proof shows that $\lambda(\omega,n)$ is essentially smaller than $1/2$ even in the case when ω is Lebesgue measure. We will present a different proof of (6) which yields a sharp bound for λ; namely, (6) holds for all $\lambda\le 1/2$.

Theorem 1.3. *Let ω satisfy the doubling condition. Then for any $f\in L^1_{loc}(\omega)$ and all $x\in\mathbb{R}^n$*

$$c'_{\omega}M_{\omega}(M^{\#}_{1/2,\omega}f)(x)\le f^{\#}_{\omega}(x)\le c''_{\omega}M_{\omega}(M^{\#}_{1/2,\omega}f)(x).$$

We do not know whether this theorem holds for nondoubling weights.

2 Preliminaries

We will use the following covering lemma proved in [6].

Lemma 2.1. *Let E be a subset of Q, and suppose that $\omega(E)\le\rho\omega(Q)$ for $0<\rho<1$. Then there exists a sequence $\{Q_i\}$ of cubes contained in Q such that:*

(i) $\omega(Q_i\cap E)=\rho\omega(Q_i)$;

(ii) $\bigcup_i Q_i=\bigcup_{k=1}^{B_n}\bigcup_{i\in F_k}Q_i$, *where each of the family $\{Q_i\}_{i\in F_k}$ is formed by pairwise disjoint cubes and a constant B_n depends only on n; in other words, the family $\{Q_i\}$ is almost disjoint with constant B_n;*

(iii) $E'\subset\cup_i Q_i$, *where E' is the set of ω-density points of E.*

We now make some remarks about the median value $m_{f,\omega}(Q)$. It is easy to see, by the definition of the rearrangement, that $|m_{f,\omega}(Q)|\le\big(f\chi_Q\big)^*_{\omega}\big(\omega(Q)/2\big)$. Moreover, when f is a non-negative function, one can take

$$m_{f,\omega}(Q)=\big(f\chi_Q\big)^*_{\omega}\big(\omega(Q)/2\big).$$

Next, it is clear that $m_{f,\omega}(Q) - c = m_{f-c,\omega}(Q)$ for any constant c, and thus, $|m_{f,\omega}(Q) - c| \le \big((f-c)\chi_Q\big)^*_\omega\big(\omega(Q)/2\big)$, which in turn gives

$$\big((f - m_{f,\omega}(Q))\chi_Q\big)^*_\omega\big(\lambda\omega(Q)\big) \le 2\inf_c \big((f-c)\chi_Q\big)^*_\omega\big(\lambda\omega(Q)\big) \tag{7}$$

for all $\lambda \le 1/2$.

Proposition 2.2. *Let $f \ge 0$ and let $\{Q_\varepsilon\}$ be a family of cubes, containing a cube Q, such that $Q_\varepsilon \subset Q_\delta$ when $\varepsilon < \delta$ and $Q_\varepsilon \to Q$ as $\varepsilon \to 0$. Then*

$$\limsup_{\varepsilon\to 0} |\big(f\chi_Q\big)^*_\omega\big(\omega(Q)/2\big) - \big(f\chi_{Q_\varepsilon}\big)^*_\omega\big(\omega(Q_\varepsilon)/2\big)| \le 2\inf_{x\in Q} M^{\#}_{1/2,\omega}f(x).$$

PROOF. By the above mentioned properties of the median value,

$$\begin{aligned}
&|\big(f\chi_Q\big)^*_\omega\big(\omega(Q)/2\big) - \big(f\chi_{Q_\varepsilon}\big)^*_\omega\big(\omega(Q_\varepsilon)/2\big)| \\
&\qquad \le \Big(\Big(f - \big(f\chi_Q\big)^*_\omega\big(\omega(Q)/2\big)\Big)\chi_{Q_\varepsilon}\Big)^*_\omega\Big(\omega(Q_\varepsilon)/2\Big) \\
&\qquad \le \Big(\Big(f - \big(f\chi_Q\big)^*_\omega\big(\omega(Q)/2\big)\Big)\chi_{Q_\varepsilon}\Big)^*_\omega\Big(\omega(Q)/2\Big).
\end{aligned}$$

Since $|f_k| \downarrow |f|$ implies $(f_k)^*_\omega(t) \downarrow f^*_\omega(t)$ (see [1, p. 41]), we get

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} |\big(f\chi_Q\big)^*_\omega\big(\omega(Q)/2\big) - \big(f\chi_{Q_\varepsilon}\big)^*_\omega\big(\omega(Q_\varepsilon)/2\big)| \\
&\qquad \le \Big(\Big(f - \big(f\chi_Q\big)^*_\omega\big(\omega(Q)/2\big)\Big)\chi_Q\Big)^*_\omega\Big(\omega(Q)/2\Big).
\end{aligned}$$

Now applying (7) completes the proof. □

3 Proofs of the Main Results

3.1 A Weighted Rearrangement Inequality.

Here we prove Theorem 1.1, and its corollary, the nondoubling John-Strömberg characterization for $\lambda \le \lambda_n$.

PROOF OF THEOREM 1.1. The proof follows the same lines as the one of [5, Theorem 3.1], although with some modifications. It is easy to see that for any constant c,

$$\begin{aligned}
|c| &\le \inf_{x\in Q}\big(|f(x) - c| + |f(x)|\big) \le \big((|f-c| + |f|)\chi_Q\big)^*_\omega(\omega(Q)) \\
&\le \big((f-c)\chi_Q\big)^*_\omega\big(\lambda\omega(Q)\big) + \big(f\chi_Q\big)^*_\omega\big((1-\lambda)\omega(Q)\big),\ (0 < \lambda < 1).
\end{aligned}$$

From this we get

$$\begin{aligned}\big(f\chi_Q\big)^*_\omega\big(\lambda\omega(Q)\big) &\le 2\inf_c\big((f-c)\chi_Q\big)^*_\omega\big(\lambda\omega(Q)\big)+\big(f\chi_Q\big)^*_\omega\big((1-\lambda)\omega(Q)\big)\\ &\le 2\inf_{x\in Q}M^{\#}_{\lambda,\omega}f(x)+\big(f\chi_Q\big)^*_\omega\big((1-\lambda)\omega(Q)\big).\end{aligned}\tag{8}$$

Set $\lambda_n = 1/5B_n$, where B_n is the constant from Lemma 2.1. Fix an arbitrary cube Q. Let E be an arbitrary set from Q with $\omega(E) = t$. Next, let $E_1 = \{x \in Q : |f(x)| > (f\chi_Q)^*_\omega(2t)\}$ and $\Omega = \{x \in Q : M^{\#}_{\lambda_n,\omega}f(x) > \big((M^{\#}_{\lambda_n,\omega}f)\chi_Q\big)^*_\omega(2t)\}$. Observe that $\omega(\Omega) \le 2t$ and $\omega(E_1) \le 2t$. Applying Lemma 2.1 to the set E and λ_n, we get that there exists a sequence of almost disjoint cubes $\{Q_i\}$, covering E' and such that $\omega(Q_i \cap E) = \lambda_n\omega(Q_i)$. Therefore,

$$t \le \sum_{k=1}^{B_n}\sum_{i\in F_k}\omega(Q_i\cap E) = \frac{1}{5B_n}\sum_{k=1}^{B_n}\sum_{i\in F_k}\omega(Q_i),$$

and thus there exists a family $\{Q_i\}_{i\in F_{k_0}}$ of pairwise disjoint cubes such that $\sum_{i\in F_{k_0}}\omega(Q_i) \ge 5t$.

From $\{Q_i\}_{i\in F_{k_0}}$ select a subfamily of cubes $\{Q_i\}_{i\in F'_{k_0}}$ each of which is not contained in Ω; that is, $Q_i\cap\Omega^c \ne \emptyset$ for any $i \in F'_{k_0}$. Then $\sum_{i\in F'_{k_0}}\omega(Q_i) \ge 3t$, and

$$\inf_{x\in Q_i}M^{\#}_{\lambda_n,\omega}f(x) \le \big((M^{\#}_{\lambda_n,\omega}f)\chi_Q\big)^*_\omega(2t)\tag{9}$$

whenever $i \in F'_{k_0}$. We now claim that among $\{Q_i\}_{i\in F'_{k_0}}$ there is a cube Q_{i_0} such that

$$\big(f\chi_{Q_{i_0}}\big)^*_\omega\big((1-\lambda_n)\omega(Q_{i_0})\big) \le (f\chi_Q)^*_\omega(2t).\tag{10}$$

Suppose (10) does not hold for any $i \in F'_{k_0}$. This means that $\omega(Q_i \cap E_1) \ge (1-\lambda_n)\omega(Q_i)$, and hence $3t \le \sum_{i\in F'_{k_0}}\omega(Q_i) \le 2t/(1-\lambda_n)$, which contradicts our choice of λ_n.

Combining (8) – (10), we obtain

$$\begin{aligned}\inf_{x\in E}|f(x)| \le \inf_{x\in E\cap Q_{i_0}}|f(x)| &\le \big(f\chi_{Q_{i_0}}\big)^*_\omega\big(\lambda_n\omega(Q_{i_0})\big)\\ &\le 2\inf_{x\in Q_{i_0}}M^{\#}_{\lambda_n,\omega}f(x)+\big(f\chi_{Q_{i_0}}\big)^*_\omega\big((1-\lambda_n)\omega(Q_{i_0})\big)\\ &\le 2\big((M^{\#}_{\lambda_n,\omega}f)\chi_Q\big)^*_\omega(2t)+(f\chi_Q)^*_\omega(2t).\end{aligned}$$

Taking the upper bound over all sets $E \subset Q$ with $\omega(E) = t$ completes the proof. □

Corollary 3.1. *Let ω be any weight. Then for any measurable function f and each cube $Q \subset \mathbb{R}^n$,*

$$\big((f - m_{f,\omega}(Q))\chi_Q\big)^*_\omega(t) \le \frac{2}{\log 2}\|M^{\#}_{\lambda_n,\omega}f\|_\infty \log\frac{2\omega(Q)}{t},\ (0 < t < \omega(Q)).$$

PROOF. Applying Theorem 1.1 to $f - m_{f,\omega}(Q)$, we get

$$\big((f - m_{f,\omega}(Q))\chi_Q\big)^*_\omega(t) \le 2\|M^{\#}_{\lambda_n,\omega}f\|_\infty + \big((f - m_{f,\omega}(Q))\chi_Q\big)^*_\omega(2t), \tag{11}$$

whenever $0 < t \le \lambda_n\omega(Q)$. But it follows from (7) that for $t > \lambda_n\omega(Q)$,

$$\big((f - m_{f,\omega}(Q))\chi_Q\big)^*_\omega(t) \le \big((f - m_{f,\omega}(Q))\chi_Q\big)^*_\omega\big(\lambda_n\omega(Q)\big) \le 2\|M^{\#}_{\lambda_n,\omega}f\|_\infty,$$

and so (11) holds for any $t > 0$.

Suppose now that $\omega(Q)/2^{k+1} < t \le \omega(Q)/2^k$ $(k = 0, 1, \dots)$. Iterating (11) k times yields

$$\big((f - m_{f,\omega}(Q))\chi_Q\big)^*_\omega(t) \le 2(k+1)\|M^{\#}_{\lambda_n,\omega}f\|_\infty \le \frac{2}{\log 2}\|M^{\#}_{\lambda_n,\omega}f\|_\infty \log\frac{2\omega(Q)}{t},$$

as required. In the case $k = 0$, (11) implies this result immediately.) $\square$

3.2 Proof of Theorem 1.2.

In view of Corollary 3.1, to prove the theorem, it suffices to show that

$$\|M^{\#}_{\lambda_n,\omega}\|_\infty \le c_n\|M^{\#}_{1/2,\omega}\|_\infty. \tag{12}$$

We will need the following construction from [6]. For each $x \in Q$ and for $r > 0$ satisfying $r \le \ell_Q$, where ℓ_Q denotes the sidelength of Q, define $\widetilde{Q}(x,r)$ as a unique cube with sidelength r, containing x, contained in Q and with center y closest to x. It is clear that if $\operatorname{dist}(x, Q^c) > r/2$, then $\widetilde{Q}(x,r)$ will be the cube centered at x.

Note that the bases $\{\widetilde{Q}(x,r)\}_{0<r\le\ell_Q}$ is a main tool in proving the covering Lemma 2.1. The Hardy-Littlewood maximal function with respect to this bases was considered in [8]. For our purposes it will be useful to consider the following maximal function which controls the median values of f over cubes from the bases $\{\widetilde{Q}(x,r)\}_{0<r\le\ell_Q}$.

For a measurable function f and for $x \in Q$, define the maximal function $\widetilde{m}_\omega f$ by

$$\widetilde{m}_\omega f(x) = \sup_{0<r\le\ell_Q}\big(f\chi_{\widetilde{Q}(x,r)}\big)^*_\omega\big(\omega\big(\widetilde{Q}(x,r)\big)/2\big).$$

We mention several properties of $\widetilde{m}_\omega f$. First of all, for any point $x \in Q$ of approximate continuity of f (see [9, p.132]) and for any $\varepsilon > 0$ one can find a cube $\widetilde{Q}(x,r)$ and a set $E \subset \widetilde{Q}(x,r)$ such that $\omega(E) \geq \omega\big(\widetilde{Q}(x,r)\big)/2$ and $|f(x)| \leq |f(y)| + \varepsilon$ for all $y \in E$. It follows from this that

$$|f(x)| \leq \big(f\chi_{\widetilde{Q}(x,r)}\big)^*_\omega\big(\omega\big(\widetilde{Q}(x,r)\big)/2\big) + \varepsilon \leq \widetilde{m}_\omega f(x) + \varepsilon,$$

which gives

$$|f(x)| \leq \widetilde{m}_\omega f(x) \quad a.e. \tag{13}$$

The following lemma is a variant of Strömberg's Lemma 3.6 from [11].

Lemma 3.2. *Let $f \geq 0$. For $\beta, \delta > 0$, let*

$$\Omega = \{x \in Q : M^{\#}_{1/2,\omega} f(x) > \beta\} \text{ and } E = \{x \in Q : \widetilde{m}_\omega f(x) > \delta\}.$$

Suppose that $\big(f\chi_Q\big)^*_\omega\big(\omega(Q)/2\big) \leq \delta$ and $E \backslash \Omega \neq \emptyset$. Then there exists a sequence of cubes $\{Q_i\}$ from Q, covering $E \setminus \Omega$, that are almost disjoint with constant B_n such that for any Q_i, $\delta \leq \big(f\chi_{Q_i}\big)^*_\omega\big(\omega(Q_i)/2\big) \leq \delta + 2\beta$.

Proof. For any $x \in E \setminus \Omega$ let

$$r_x = \sup\big\{r \in (0, \ell_Q] : \big(f\chi_{\widetilde{Q}(x,r)}\big)^*_\omega\big(\omega\big(\widetilde{Q}(x,r)\big)/2\big) > \delta\big\}.$$

Note that the function $\varphi(r) = \big(f\chi_{\widetilde{Q}(x,r)}\big)^*_\omega\big(\omega\big(\widetilde{Q}(x,r)\big)/2\big)$ is left-continuous since the rearrangement is. Hence, $\varphi(r_x) \geq \delta$. If $r_{x_0} = \ell_Q$ for some $x_0 \in E \setminus \Omega$, then $\big(f\chi_Q\big)^*_\omega\big(\omega(Q)/2\big) = \delta$ and we can take $Q_j \equiv Q$. So, this case is trivial. Suppose that $r_x < \ell_Q$ for any $x \in E \setminus \Omega$. Then, using Proposition 2.2, we get

$$\delta \leq \big(f\chi_{\widetilde{Q}(x,r_x)}\big)^*_\omega\big(\omega\big(\widetilde{Q}(x,r_x)\big)/2\big) \leq \delta + 2 \inf_{\xi \in \widetilde{Q}(x,r_x)} M^{\#}_{1/2,\omega}(\xi) \leq \delta + 2\beta.$$

We now proceed as in the proof of Lemma 2.1 (cf. [6]). For any $\widetilde{Q}(x,r_x)$ define the rectangle $R_x \subset \mathbb{R}^n$ as the unique rectangle centered at x such that $R_x \cap Q = \widetilde{Q}(x,r_x)$. It is easy to see that the ratio of any two sidelengths of R_x is bounded by 2. Applying the Besicovitch Covering Theorem to the family $\{R_x\}_{x \in E \setminus \Omega}$ yields a countable collection of rectangles R_j, covering $E \setminus \Omega$ that are almost disjoint with constant B_n. Replacing each R_j by its corresponding cube Q_j, we get the required sequence. □

Lemma 3.3. *For any measurable function f and each cube Q,*

$$\big((f - m_{f,\omega}(Q))\chi_Q\big)^*_\omega\big(\lambda_n \omega(Q)\big) \leq c_n\big((M^{\#}_{1/2,\omega} f)\chi_Q\big)^*_\omega\big(\lambda_n \omega(Q)/2\big).$$

PROOF. Let $\beta = \big((M^{\#}_{1/2,\omega}f)\chi_Q\big)^*_\omega\big(\lambda_n\omega(Q)/2\big)$ and $\psi(x) = |f(x) - m_{f,\omega}(Q)|$. By (13), it suffices to show that

$$\big(\widetilde{m}_\omega\psi\big)^*_\omega\big(\lambda_n\omega(Q)\big) \le c_n\beta. \tag{14}$$

Set $\Omega = \{x \in Q : M^{\#}_{1/2,\omega}\psi(x) > \beta\}$. Observe that $M^{\#}_{1/2,\omega}\psi(x) \le M^{\#}_{1/2,\omega}f(x)$ for all x, since $M^{\#}_{1/2,\omega}|f| \le M^{\#}_{1/2,\omega}f$ and $M^{\#}_{1/2,\omega}(f-c) = M^{\#}_{1/2,\omega}f$. Thus,

$$\omega(\Omega) \le \omega\{x \in Q : M^{\#}_{1/2,\omega}f(x) > \beta\} \le \lambda_n\omega(Q)/2.$$

For $k = 1, \dots, k_n$, where k_n depends only on n and will be choosen later, we consider the sets $E_k = \{x \in Q : \widetilde{m}_\omega\psi(x) > 7k\beta\}$. If $E_k = \emptyset$ for some k, then (14) holds trivially with $c_n = 7k_n$. If $E_k \setminus \Omega = \emptyset$ for some k, we get $\omega(E_k) \le \omega(\Omega) \le \lambda_n\omega(Q)/2$, and so $\big(\widetilde{m}_\omega\psi\big)^*_\omega\big((\lambda_n/2+\varepsilon)\omega(Q)\big) \le 7k\beta$, which also gives (14) with $c_n = 7k_n$.

Assume now that $E_k \setminus \Omega \ne \emptyset$ for all $k = 1, \dots, k_n$. Note that, in view of (7), $\big(\psi\chi_Q\big)^*_\omega\big(\omega(Q)\big) \le 2\beta$. Thus, we may apply Lemma 3.2 to get cubes Q^k_j almost disjoint with constant B_n, such that

$$7k\beta \le \big(\psi\chi_{Q_{jk}}\big)^*_\omega\big(\omega(Q^k_j)/2\big) \le (7k+2)\beta. \tag{15}$$

Set $A^k_j = \{x \in Q^k_j : |\psi(x) - \big(\psi\chi_{Q_{jk}}\big)^*_\omega\big(\omega(Q^k_j)/2\big)| \le 2\beta\}$ and $A_k = \bigcup_j A^k_j$. It follows from (15) that the sets A_k are pairwise disjoint sets. Next, by (7), $\omega(A^k_j) \ge \omega(Q^k_j)/2$. Therefore,

$$\sum_{k=1}^{k_n}\sum_j \omega(Q^k_j) \le 2\sum_{k=1}^{k_n}\sum_j \omega(A^k_j) \le 2B_n\sum_{k=1}^{k_n}\omega(A_k) \le 2B_n\omega(Q).$$

Taking now $k_n = [5B_n/\lambda_n] + 1$, we get that there exists a natural $k_0 \le k_n$ such that $\sum_j \omega(Q^{k_0}_j) \le \frac{2\lambda_n}{5}\omega(Q)$. Thus,

$$\omega(E_{k_n}) \le \omega(E_{k_0} \setminus \Omega) + \omega(\Omega) \le \sum_j \omega(Q^{k_0}_j) + \lambda_n\omega(Q)/2 < \lambda_n\omega(Q). \qquad \square$$

Clearly, this lemma immediately implies (12), and therefore the proof of Theorem 1.2 is complete.

3.3 Proof of Theorem 1.3.

We prove only that $f^{\#}_\omega(x) \le c_\omega M_\omega(M^{\#}_{1/2,\omega}f)(x)$, since the converse inequality can be proved exactly as in the unweighted case (see [2]). First of all, we mention the following simple corollary of Theorem 1.1.

Lemma 3.4. *For any weight ω and any $f \in L^1_{loc}(\omega)$,*

$$f^{\#}_{\omega}(x) \le 8M_{\omega}(M^{\#}_{\lambda_n,\omega}f)(x). \tag{16}$$

PROOF. Integrating (5) yields

$$\begin{aligned}\int_Q |f(x)|\omega(x)\,dx =& \int_0^{\lambda_n\omega(Q)} (f\chi_Q)^*_{\omega}(t)\,dt + \int_{\lambda_n\omega(Q)}^{\omega(Q)} (f\chi_Q)^*_{\omega}(t)\,dt\\ \le& 2\int_0^{2\lambda_n\omega(Q)} \big((M^{\#}_{\lambda_n,\omega}f)\chi_Q\big)^*_{\omega}(t)\,dt + 2\int_{\lambda_n\omega(Q)}^{\omega(Q)} (f\chi_Q)^*_{\omega}(t)\,dt\\ \le& 2\int_Q M^{\#}_{\lambda_n,\omega}f(x)\omega(x)\,dx + 2\omega(Q)(f\chi_Q)^*_{\omega}(\lambda_n\omega(Q)).\end{aligned}$$

Thus, for any constant c,

$$\begin{aligned}\int_Q |f(x) - f_{Q,\omega}|\omega(x)\,dx \le& 2\int_Q |f(x)-c|\omega(x)\,dx\\ \le& 4\int_Q M^{\#}_{\lambda_n,\omega}f(x)\omega(x)\,dx\\ &+ 4\omega(Q)\big((f-c)\chi_Q\big)^*_{\omega}(\lambda_n\omega(Q)).\end{aligned}$$

Taking the infimum over all c, we obtain

$$\begin{aligned}&\int_Q |f(x) - f_{Q,\omega}|\omega(x)\,dx\\ &\le 4\int_Q M^{\#}_{\lambda_n,\omega}f(x)\omega(x)\,dx + 4\omega(Q)\inf_c\big((f-c)\chi_Q\big)^*_{\omega}(\lambda_n\omega(Q))\\ &\le 4\int_Q M^{\#}_{\lambda_n,\omega}f(x)\omega(x)\,dx + 4\omega(Q)\inf_{x\in Q} M^{\#}_{\lambda_n,\omega}f(x)\\ &\le 8\int_Q M^{\#}_{\lambda_n,\omega}f(x)\omega(x)\,dx,\end{aligned}$$

which proves (16). □

We now define the maximal function $m_{\lambda,\omega}f$ by

$$m_{\lambda,\omega}f(x) = \sup_{Q\ni x}\big(f\chi_Q\big)^*_{\omega}\big(\lambda\omega(Q)\big),\ (0<\lambda\le 1),$$

and note that Lemma 3.3 immediately implies

$$M^{\#}_{\lambda_n,\omega}f(x) \le c_n m_{\lambda_n/2,\omega}(M^{\#}_{1/2,\omega}f)(x). \tag{17}$$

Lemma 3.5. *Let ω satisfy the doubling condition. Then for any $f \in L^1_{loc}(\omega)$ and all $x \in \mathbb{R}^n$,*

$$M_\omega(m_{\lambda,\omega} f)(x) \le \frac{c_\omega}{\lambda} M_\omega f(x) \quad (0 < \lambda \le 1).$$

PROOF. It follows from the definition of the rearrangement that for all $\alpha > 0$,

$$\{x : m_{\lambda,\omega} f(x) > \alpha\} \subset \{x : M_\omega \chi_{\{|f|>\alpha\}}(x) \ge \lambda\}.$$

Hence, by the weak type $(1,1)$ property of M_ω,

$$\omega\{x : m_{\lambda,\omega} f(x) > \alpha\} \le \frac{c_\omega}{\lambda} \omega\{x : |f(x)| > \alpha\},$$

and so

$$\|m_{\lambda,\omega} f\|_{1,\omega} \le \frac{c_\omega}{\lambda} \|f\|_{1,\omega}. \tag{18}$$

Let Q be any cube containing x. For all $y \in Q$ we get

$$\begin{aligned} m_{\lambda,\omega} f(y) &= \max\Big(\sup_{\substack{Q' \ni y, \\ Q' \subset 3Q}} \big(f\chi_{Q'}\big)^*_\omega \big(\lambda\omega(Q')\big), \sup_{\substack{Q' \ni y, \\ Q \subset 3Q'}} \big(f\chi_{Q'}\big)^*_\omega \big(\lambda\omega(Q')\big)\Big) \\ &\le \max\big(m_{\lambda,\omega}(f\chi_{3Q})(y), m_{\lambda/c'_\omega,\omega} f(x)\big) \\ &\le m_{\lambda,\omega}(f\chi_{3Q})(y) + \frac{c'_\omega}{\lambda} M_\omega f(x). \end{aligned}$$

From this and (18) we obtain

$$\begin{aligned} &\frac{1}{\omega(Q)} \int_Q m_{\lambda,\omega} f(y)\omega(y)\, dy \le \frac{1}{\omega(Q)} \|m_{\lambda,\omega}(f\chi_{3Q})\|_{1,\omega} + \frac{c'_\omega}{\lambda} M_\omega f(x) \\ &\le \frac{c_\omega}{\lambda\omega(Q)} \int_{3Q} |f(y)|\omega(y)\, dy + \frac{c'_\omega}{\lambda} M_\omega f(x) \le \frac{c_\omega}{\lambda} M_\omega f(x). \end{aligned}$$

□

Combining (16), (17) and the last lemma yields

$$\begin{aligned} f^\#_\omega(x) &\le 8 M_\omega(M^\#_{\lambda_n,\omega} f)(x) \\ &\le 8 c_n M_\omega\big(m_{\lambda_n/2,\omega}(M^\#_{1/2,\omega} f)\big)(x) \le c_{n,\omega} M_\omega(M^\#_{1/2,\omega} f)(x), \end{aligned}$$

and therefore the theorem is proved.

Remark 3.1. We note that our main results, namely Theorems 1.1 and 1.2 hold under a more general assumption on the measure ω. As in [6, 8], we can assume only that $\omega(L) = 0$ for every hyperplane L, orthogonal to one of the coordinate axes.

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