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ON THE HAHN DECOMPOSITION THEOREM

Abstract

The purpose of this article is to prove Hahn Decomposition type and Jordan Decomposition type theorems for measures on σ-semirings. These results generalize the classical theorems for measures on σ-algebras.

1 Introduction

A nonempty set $\mathcal{T}$ of subsets of a nonempty set X is called a *semiring on* X if for any given sets $A, B \in \mathcal{T}$, $A \cap B \in \mathcal{T}$ and $A \setminus B = \cup_{n=1}^{k} C_n$ for some pairwise disjoint sets $C_1, C_2, \ldots, C_k \in \mathcal{T}$. Of course, Boolean algebras and σ-rings are semirings and there are plenty of examples of semirings which are not an algebra or a σ-ring. (see [1] for semirings).

A subset A of a set X is called a σ-set with respect to a semiring $\mathcal{S}$ on X if $A = \cup_{n=1}^{\infty} A_n$ for some sequence $\{A_n\}$ in $\mathcal{S}$. It is easy to see that if $A, A_1, \ldots, A_n$ are in a semiring, then $A \setminus \cup_{i=1}^{n} A_i$ is a σ-set, but if $A \in \mathcal{S}$ and $\{A_n\}$ is a sequence in $\mathcal{S}$, then $A \setminus \cup_{n=1}^{\infty} A_n$ may not be a σ-set.

Example 1.1. i) Let $X = [0, 1)$ and $\mathcal{T} = \{[a, b) : 0 \leq a \leq b \leq 1\}$. Then $\mathcal{T}$ is a semiring on X, but $\{0\} = X \setminus \cup_n [\frac{1}{n}, 1)$ is not a σ-set in $\mathcal{T}$.
ii) Let X be a set with at least two elements, $\mathcal{T} = \{\{x\} : x \in X\} \cup \{\emptyset\}$. Although, for each $A, A_1, \dots \in \mathcal{T}$, $A \setminus \cup_n A_n$ is a σ-set while $\mathcal{T}$ is neither an algebra nor a σ-ring on X.

This observation leads us to introduce the following notion.

Definition 1.1. A semiring $\mathcal{S}$ is called a σ-semiring on a set X if for each $A \in \mathcal{S}$ and for each sequence $\{A_n\}$ in $\mathcal{S}$, the set $A \setminus \cup_n A_n$ is a σ-set.

It should be noted that for each sequences $\{A_n\}$,$\{B_n\}$ in a σ-semiring $\mathcal{S}$ there exists a disjoint sequence $\{C_n\}$ in $\mathcal{S}$ such that $\cup_n A_n \setminus \cup_n B_n = \cup_n C_n$ and if μ is a measure and $\cup_n A_n \subset \cup_n B_n$, then $\Sigma_n \mu(A_n) \leq \Sigma_n \mu(B_n)$.

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2 The Hahn Decomposition Theorem

The classical Hahn Decomposition Theorem states that if Σ is a σ-algebra (or a σ-ring), and $\mu : \Sigma \to [-\infty, \infty)$ is a signed measure, then there exist a positive set A and a negative set B in Σ such that $A \cap B = \emptyset$ and $X = A \cup B$. (See [2] for a short proof.) We prove this theorem for the semiring case. First we need the following definition.

Definition 2.1. We say that a measure μ, $\mu : \mathcal{S} \to [-\infty, \infty)$, on a semiring $\mathcal{S}$, satisfies the (*) property if $E \in \mathcal{S}$, $\{A_n\}$ a disjoint sequence in $\mathcal{S}$ satisfying $E = \cup_n A_n$, then $\mu(E) = \Sigma_{n \in K} \mu(A_n) + \Sigma_{n \in K^c} \mu(A_n)$ for each subset K of natural numbers $\mathbb{N}$.

It is obvious that all real valued measures on a σ-algebra satisfies the (*) property.

Lemma 2.1. *Let $\mathcal{S}$ be a σ-semiring on a set X and $\mu : \mathcal{S} \to [-\infty, \infty)$ be a signed measure with the (*) property, $E \in \mathcal{S}$ and $0 < \mu(E)$. Then there exists a positive subset A of E in $\mathcal{S}$ with $0 < \mu(A)$.*

Proof. For each $A \subset E$, if $A \in \mathcal{S}$, $0 \leq \mu(A)$, then there is not anything to prove. So suppose the set

$$\mathcal{F} = \{\mathcal{C} : A, B \in \mathcal{C} \Rightarrow A \subset E, A \in \mathcal{S}, \mu(A) < 0 \text{ and } A \cap B = \emptyset \text{ if } A \neq B\}$$

is nonempty. Let $\mathcal{C}_\infty$ be a maximal element of $\mathcal{F}$ with respect to inclusion. For each natural number k, the set $\mathcal{C}_k = \{A \in \mathcal{C}_\infty : \mu(A) \leq -\frac{1}{k}\}$ is finite. If this were not the case we could choose a disjoint sequence $\{A_n\}$ in $\mathcal{C}_k$ and let $\{B_n\}$ be a disjoint sequence in $\mathcal{S}$ with $E \setminus \cup_n A_n = \cup_n B_n$ Then

$$E = (\cup_n A_n) \cup (\cup_n B_n) \text{ and } \mu(E) = \Sigma_n \mu(A_n) + \Sigma_n \mu(B_n) = -\infty + \Sigma_n \mu(B_n).$$

This a contradiction. Hence $\mathcal{C}_k$ is finite. Therefore, $\mathcal{C}_\infty$ is at most countable. Let $\mathcal{C}_\infty = \{C_n : n = 1, 2, \dots\}$. Choose a disjoint sequence $\{D_n\}$ in $\mathcal{S}$ with $E \setminus \cup_n C_n = \cup_n D_n$. Since $0 < \mu(E)$ and $\mu(C_n) < 0$ for each n we have

$$\mu(E) = \Sigma_n \mu(C_n) + \Sigma_n \mu(D_n)$$

which implies that $0 < \mu(D_k)$ for some k. If there were a subset $B \subset D_k$ in $\mathcal{S}$ with $\mu(B) < 0$, then $\mathcal{C}_\infty \cup \{B\} \in \mathcal{F}$ which contradicts the maximality of $\mathcal{C}_\infty$, so D_k is required positive set. $\square$

Lemma 2.2. *Let $\mathcal{S}$ be a σ-semiring on a set X with $X \in \mathcal{S}$ and $\mu : \mathcal{S} \to [-\infty, \infty)$ be a signed measure satisfying*

$$\alpha = sup\{\Sigma_{i=1}^n \mu(A_i) : 0 \leq A_i \in \mathcal{S} \text{ and } A_i \cap A_j = \emptyset \text{ for all } i \neq j\} < \infty.$$

Then there exist a sequence $\{A_n\}$ of positive sets and a sequence $\{B_n\}$ of negative sets such that $X = (\cup_n A_n) \cup (\cup_n B_n)$ and $A_i \cap B_j = \emptyset$ for all i and j.

PROOF. We can choose an increasing sequence $\{t_n\}$ of natural numbers and a finite collection of positive sets $A_1^n, A_2^n, \ldots, A_{t_n}^n \in \mathcal{S}$ for each n satisfying $A_i^n \cap A_j^m = \emptyset$ for each$(i,n) \neq (j,m)$ and $k_n = \Sigma_{i=1}^{t_n} \mu(A_i^n) \to \alpha$. Let $\{B_n\}$ be a sequence in $\mathcal{S}$ such that $X \setminus \cup_{n=1}^{\infty} \cup_{i=1}^{t_n} A_i^n = \cup_{i=1}^{\infty} B_i$. Suppose that B_n is not negative for some n. Then there exists k and $A \in \mathcal{S}$, with $A \subset B_k$ and $0 < \mu(A)$. From the previous theorem, there exists $0 \leq E \in \mathcal{S}, E \subset A$ and $0 < \mu(E)$. We choose n with $\alpha - \epsilon \leq k_n$. Since $E \cap A_i^n = \emptyset$ for each $1 \leq i \leq t_n$,

$$\alpha - \epsilon + \mu(E) \leq k_n + \mu(E) = \Sigma_{i=1}^{t_n} \mu(A_i^n) + \mu(E) \leq \alpha.$$

Since $\alpha < \infty$, we have $\mu(E) \leq \epsilon$. Since $0 < \epsilon$ was arbitrary, we have a contradiction to $0 < \mu(E)$. Hence, B_n must be negative set for each n. □

Lemma 2.3. *Let $\mathcal{S}$ be a σ-semiring on a set X, $X \in \mathcal{S}$ and $\mu : \mathcal{S} \to [-\infty, \infty)$ a signed measure with the (*) property. Then $\alpha < \infty$, where α is as in the previous lemma.*

PROOF. Let $k_n = \Sigma_{i=1}^{t_n} \mu(A_i^n) \to \alpha$, where $\{t_n\}$ is an increasing sequence of natural numbers and $\{A_i^n : 1 \leq i \leq t_n\}$ disjoint collection of positive sets for each n. Choose a disjoint sequence $\{B_n\}$ of positive sets satisfying $\cup_n \cup_{i=1}^{t_n} A_i^n = \cup_n B_n$. and it is routine to show that for each n $\Sigma_{i=1}^{t_n} \mu(A_i^n) \leq \Sigma_n \mu(B_n)$. Let $\{C_n\}$ be a disjoint sequence in $\mathcal{S}$ with $X = (\cup_n B_n) \cup (\cup_n C_n)$. Since μ has the (*) property, we have that $\mu(X) = \Sigma_n \mu(B_n) + \Sigma_n \mu(C_n)$ which implies that $\alpha \leq \Sigma_n \mu(B_n) < \infty$. □

From the above lemmas, the proof of the following main theorem is obvious.

Theorem 2.1. *Let $\mathcal{S}$ be a σ-semiring on a set X, $X \in \mathcal{S}$ and $\mu : \mathcal{S} \to [-\infty, \infty)$ be a signed measure with (*) property. Then there exist disjoint sequences $\{P_n\}$ of positive sets and $\{N_n\}$ of negative sets such that*

$$X = (\cup_n P_n) \cup (\cup_n N_n),\ P_n \cap N_m = \emptyset \text{ for all } n, m.$$

The following example shows that the above theorem is no longer valid without "σ" condition.

Example 2.1. Let $X = [0,1)$, $\mathcal{S} = \{[x,y) : 0 \leq x, y \leq 1\}$ and choose $a, b \in \mathbb{R}, b < -1$. Let $\mu : \mathcal{S} \to \mathbb{R}$ be defined by

$$\mu([x,y)) : (y-x)\mathcal{X}_{[x,y)}(a) + (y-x-b)\mathcal{X}_{[y,1)}(a)$$

Then μ has the (*) property, $X \in \mathcal{S}$, and $\mathcal{S}$ is not a σ-semiring. It is clear that there is no positive and negative sequences as in the above theorem.

Since every signed measure on a σ-algebra has the (*) property, from the above theorem we immediately get the well known Hahn decomposition theorem.

Corollary 2.1. (Hahn Decomposition Theorem). *Let Σ be a σ-algebra on a set X and μ be a signed measure on Σ. Then there exist a positive set P and negative set N such that $X = P \cup N$ and $P \cap N = \emptyset$.*

The Jordan Decomposition Theorem states that for any signed measure $\mu : \Sigma \to [-\infty, \infty)$, Σ a σ-algebra, there exist measures μ_1, μ_2 such that $\mu = \mu_1 - \mu_2$. We can generalize this theorem as follows.

Theorem 2.2. *Let $\mathcal{S}$ be a σ-semiring on X with $X \in \mathcal{S}$. A signed measure $\mu : \mathcal{S} \to [-\infty, \infty)$ satisfies the (*) property if and only if $\mu = \mu_1 - \mu_2$ for some measures μ_1, μ_2.*

Proof. It is obvious that if $\mu = \mu_1 - \mu_2$ for some measures μ_1, μ_2, then μ satisfies the (*) property. If μ satisfies the (*) property, choose disjoint sequences $\{P_n\}$ of positive sets and $\{N_n\}$ of negative sets as in Theorem 1.1. Let $\mu_1(A) = \Sigma_n \mu(A \cap P_n)$ and $\mu_2(A) = -\Sigma_n \mu(A \cap N_n)$. It is obvious that μ_1 and μ_2 are the required measures. $\square$

Let Σ be a σ-algebra, μ a signed measure, P_1, P_2 positive sets and N_1, N_2 negative sets satisfying $P_1 \cap N_2 = P_2 \cap N_2 = \emptyset$ and $X = P_1 \cup N_1 = P_2 \cup N_2$. Then it is well known that $\mu(P_1 \Delta P_2) = \mu(N_1 \Delta N_2) = 0$. For the σ-semirings case this result reads as follows.

Theorem 2.3. *Let $\mathcal{S}$ be a σ-semiring on a set X with $X \in \mathcal{S}$ and $\mu : \mathcal{S} \to [-\infty, \infty)$ be a signed measure. Suppose that $\{P_n\}, \{Q_n\}$ are disjoint sequences of positive sets, and $\{N_n\}, \{M_n\}$ are disjoint sequences of negative sets such that $X = (\cup_n P_n) \cup (\cup_n N_n) = (\cup_n Q_n) \cup (\cup_n M_n)$. Then there exist disjoint sequences $\{R_n\}$ of positive sets and $\{S_n\}$ of negative sets such that $\cup_n R_n = (\cup_n P_n) \Delta (\cup_n Q_n)$, $\cup_n S_n = (\cup_n N_n) \Delta (\cup_n M_n)$ and $\mu(R_n) = \mu(S_n) = 0$ for each n.*

Proof. Since $\mathcal{S}$ is a σ-semiring, for each n there exist disjoint sequences $\{U_i^n\}$, $\{V_i^n\}$ of positive sets such that $P_n \setminus \cup_m Q_m = \cup_i U_i^n$ and $Q_n \setminus \cup_m P_m = \cup_i V_i^n$. Now $(\cup_n P_n) \Delta (\cup_n Q_n) = \cup_{i,n} (U_i^n \cup V_i^n)$. Since

$$U_i^n = (\cup_m (Q_m \cap U_i^n)) \cup (\cup_m (M_m \cap U_i^n))$$

and since μ has the (*) property, we have that

$$\mu(U_i^n) = \Sigma_m \mu(Q_m \cap U_i^n) + \Sigma_m \mu(M_m \cap U_i^n) = 0.$$

for each i, j. Similarly, $\mu(V_i^n) = 0$ for each i, j. Now we can set

$$\{R_n : n = 1, 2, \dots\} = \{U_i^n : i, n = 1, 2, \dots\} \cup \{V_i^n : i, n = 1, 2, \dots\}.$$

Similarly, we can construct the sequence $\{S_n\}$ of negative sets. $\square$

References

[1] C. D. Aliprantis and O. Burkinshaw, *Principles of Real Analysis*, Academic Press (third edition), 1998.

[2] R. Doss, *The Hahn decomposition theorem*, Proc. Amer. Math. Soc., **80**, (1980), 377.