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CARDINALITY OF BASES OF FAMILIES OF THIN SETS

Abstract

We construct a family of Dirichlet sets of cardinality $\mathfrak{c}$ such that the arithmetic sum of any two members of the family contains an open interval. As a corollary we obtain that every basis of many families of thin sets has cardinality at least $\mathfrak{c}$. Especially, every basis of any of trigonometric families $\mathcal{D}$, $p\mathcal{D}$, $\mathcal{B}_0$, $\mathcal{N}_0$, $\mathcal{B}$, $\mathcal{N}$, $w\mathcal{D}$ and $\mathcal{A}$ has cardinality at least $\mathfrak{c}$. Moreover, we construct an increasing tower of pseudo Dirichlet sets of cardinality $\mathfrak{t}$.

In our paper [BB] we investigated the relationship between families of thin sets obtained from different functions. The main tool for our results was a generalization of a classical lemma by J. Arbault [Ar]. As a byproduct, we have shown that any basis of any of the families $\mathcal{B}_0$, $\mathcal{N}_0$ and $\mathcal{A}$ has cardinality at least $\mathfrak{c}$. In this note, combining the idea of [BB] with an idea from J. Marcinkiewicz [Ma], we show that any basis of some other families of thin sets, including the families $\mathcal{D}$, $p\mathcal{D}$ and $\mathcal{N}$, has also cardinality greater or equal to $\mathfrak{c}$.

The classical *trigonometric families*

$$\mathcal{D},\ p\mathcal{D},\ \mathcal{B}_0,\ \mathcal{N}_0,\ \mathcal{B},\ \mathcal{N},\ \mathcal{A},\ w\mathcal{D}, \tag{1}$$

were studied e.g. in [BKR]. We recall some notions. We work with the topological group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We may identify $\mathbb{T}$ with the interval $\langle -1/2, 1/2\rangle$ identifying $-1/2$ and $1/2$ with the operation of addition mod 1. $\|x\|$ is the

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distance of the real x to the nearest integer. A subset E of $\mathbb{T}$ is called a *Dirichlet set*, a *pseudo Dirichlet set* or an A-*set*, if there exists an increasing sequence of natural numbers $\{n_k\}_{k=0}^{\infty}$ such that the sequence $\{\|n_k x\|\}_{k=0}^{\infty}$ converges uniformly, quasi–uniformly or pointwise to 0 on the set E, respectively. The families of all Dirichlet, pseudo Dirichlet and A-sets are denoted by $\mathcal{D}$, $p\mathcal{D}$ and $\mathcal{A}$, respectively. The other definitions can be found e.g. in [BKR].

A family $\mathcal{F} \subseteq \mathcal{P}(\mathbb{T})$ is called *a family of thin sets* (see [BKR], [BL]) if $\mathcal{F}$ contains every singleton $\{x\}$, $x \in \mathbb{T}$, with any $A \in \mathcal{F}$ also every subset of A belongs to $\mathcal{F}$, and $\mathcal{F}$ does not contain any (nontrivial) open interval. Each of the families (1) is a family of thin sets. The families

$$\mathcal{D}_f,\ p\mathcal{D}_f,\ \mathcal{B}_{0f},\ \mathcal{N}_{0f},\ \mathcal{B}_f,\ \mathcal{N}_f,\ \mathcal{A}_f,\ w\mathcal{D}_f \tag{2}$$

defined in [BZ] by a continuous function $f : \mathbb{T} \longrightarrow \langle 0, +\infty)$ are another examples of families of thin sets.

A family $\mathcal{G} \subseteq \mathcal{F}$ is called *a basis* of $\mathcal{F}$ if for any $A \in \mathcal{F}$ there is a set $B \in \mathcal{G}$ such that $A \subseteq B$. Everyone of families (2) has a basis consisting of Borel sets and therefore of cardinality at most $\mathfrak{c}$.

The arithmetic sum $A + B$ of two subsets of $\mathbb{T}$ is the set

$$A + B = \{z \in \mathbb{T}; z = x + y \text{ for some } x \in A \text{ and some } y \in B\}.$$

A family $\mathcal{F}$ of thin sets is called *trigonometric like*, if for every $A \in \mathcal{F}$ the arithmetic sum[1] $A + A$ also belongs to $\mathcal{F}$. All trigonometric families (1) are trigonometric like.

J. Marcinkiewicz [Ma] constructed two Dirichlet sets A, B such that the union $A \cup B$ is not an A-set. We use his idea for constructing a family of the cardinality $\mathfrak{c}$ of Dirichlet sets such that the arithmetic sum of any two of them contains an open interval. As a corollary we obtain the promised result about the cardinality of bases of corresponding families of thin sets, assuming.

Throughout the paper, $\{p_k\}_{k=0}^{\infty}$ is a fixed increasing sequence of natural numbers greater than 1. For proving the main result we shall need that

$$\text{the sequence of differences } \{p_{k+1} - p_k\}_{k=0}^{\infty} \text{ is increasing} \tag{3}$$

For an infinite subset $K \subseteq \mathbb{N}$ we denote[2] by $\mathsf{M}(K)$ the set

$$\mathsf{M}(K) = \{x \in \mathbb{T}; (\forall k \in K)\, \|2^{p_k} \cdot x\| \leq 2^{p_k - p_{k+1}}\}.$$

[1] In [BL] we considered the arithmetic difference $A - A$ instead of the sum. If $0 \in A$, then $A + A \subseteq (A - A) - (A - A)$, so our notion is weaker than that of [BL].

[2] M in honor of J. Marcinkiewicz.

If condition (3) holds, then $\lim_{k\to\infty} 2^{p_k - p_{k+1}} = 0$ and $\mathsf{M}(K)$ is a Dirichlet set (compare [Ma], [BZ]).

Two infinite subsets $K, L \subseteq \mathbb{N}$ are said to be *almost disjoint* if their intersection $K \cap L$ is finite. It is well known (see e.g. [Va]) that there exists a family $\mathcal{E} \subseteq \mathcal{P}(\mathbb{N})$ of cardinality $\mathfrak{c}$ of pairwise almost disjoint sets.

We start with a simple strengthening of well known Marcinkiewicz result.

Lemma 1. *If K, L are almost disjoint infinite subsets of $\mathbb{N}$, then the arithmetic sum $\mathsf{M}(K) + \mathsf{M}(L)$ contains an open interval.*

Proof. Let K, $L \subseteq \mathbb{N}$ be infinite, k_0 being such that $k \notin K \cap L$ for $k \geq k_0$. We show that $(0, 2^{-p_{k_0}}) \subseteq \mathsf{M}(K) + \mathsf{M}(L)$. We shall use the following simple observation. Let

$$x = \sum_{i=1}^{\infty} \frac{x_i}{2^i},\ x_i = 0, 1 \tag{4}$$

If $x_i = 0$ for every i, $p < i \leq q$, then $\|2^p \cdot x\| \leq 2^{p-q}$.

Now take arbitrary $x \in (0, 2^{-p_{k_0}})$ and assume that (4) holds true. Then $x_i = 0$ for any $i \leq k_0$. Thus for $k < k_0$ we have

$$|2^{p_k} \cdot x| \leq 2^{p_k - p_{k_0}} \leq 2^{p_k - p_{k+1}}.$$

We set

$$y = \sum_{i=1}^{\infty} \frac{y_i}{2^i},\ \text{where } y_i = \begin{cases} 0 & \text{for } p_k < i \leq p_{k+1},\ k \in K, \\ x_i & \text{otherwise.} \end{cases}$$

$$z = \sum_{i=1}^{\infty} \frac{z_i}{2^i},\ \text{where } z_i = \begin{cases} x_i & \text{for } p_k < i \leq p_{k+1},\ k \in K, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $x = y + z$.

By definition $\|2^{p_k} \cdot y\| \leq 2^{p_k - p_{k+1}}$ for $k \in K$ and therefore $y \in \mathsf{M}(K)$. On the other hand one can easily see that $z_i = 0$ for $p_k < i \leq p_{k+1}$, $k \in L$, $k \geq k_0$ and therefore $\|2^{p_k} \cdot z\| \leq 2^{p_k - p_{k+1}}$. Hence $z \in \mathsf{M}(L)$. ☐

Theorem 2. *Let $\mathcal{F}$ be a family of thin sets such that*

a) *$\mathcal{D} \subseteq \mathcal{F}$ and*

b) *there exists a trigonometric like family of thin sets $\mathcal{H}$ such that $\mathcal{F} \subseteq \mathcal{H}$.*

Then any basis of the family $\mathcal{F}$ has cardinality at least $\mathfrak{c}$.

PROOF. Let $\mathcal{G}$ be a basis of the family $\mathcal{F}$. Let $\mathcal{E}$ be a family of almost disjoint subsets of $\mathbb{N}$ of cardinality $\mathfrak{c}$. By (3) for any $K \in \mathcal{E}$, $\mathsf{M}(K)$ is a Dirichlet set.

Let $K, L \in \mathcal{E}$, $K \neq L$. Toward a contradiction assume that there exists a set $H \in \mathcal{G}$ containing both sets $\mathsf{M}(K)$ and $\mathsf{M}(L)$. By the assumption b) we have $H + H \in \mathcal{H}$. Since $\mathsf{M}(K) + \mathsf{M}(L) \subseteq H + H$, by Lemma 1 we obtain that $H + H$ contains an open interval - a contradiction.

Thus every set from the basis $\mathcal{G}$ contains at most one set $\mathsf{M}(K)$, $K \in \mathcal{E}$ and each set $\mathsf{M}(K)$, $K \in \mathcal{E}$ is contained in at least one set from $\mathcal{G}$. Consequently $|\mathcal{G}| \geq |\mathcal{E}| = \mathfrak{c}$. □

Corollary 3. *Every basis of each trigonometric family has cardinality at least* $\mathfrak{c}$.

PROOF. Any of the trigonometric families (1) contains the family $\mathcal{D}$ of Dirichlet sets as a subfamily. Since every trigonometric family (1) is trigonometric like, the assertion follows immediately. □

The cardinal $\mathfrak{t}$, the smallest cardinality of a maximal tower of subset of $\mathbb{N}$ is defined e.g. in [Va]. In [BB] we have constructed a $\mathfrak{t}$-tower of B_0-, N_0- and A-sets. We extend this result for pseudo Dirichlet sets.

Theorem 4. *There is a sequence* $\{P_\xi; \xi < \mathfrak{t}\}$ *of pseudo Dirichlet sets such that*

a) $P_\xi \subseteq P_\eta$ *for any* $\xi < \eta < \mathfrak{t}$,

b) *for any* $\xi < \eta < \mathfrak{t}$, *the set* $P_\eta \setminus P_\xi$ *contains a perfect subset,*

c) *there is no A-set containing all sets* P_ξ, $\xi < \mathfrak{t}$.

We start with an observation. Let $q_k = p_0 \cdot \ldots \cdot p_k$. For every real $x \in \langle 0, 1\rangle$ there are integers x_k, $k \in \mathbb{N}$ such that (compare [BB])

$$x = \sum_{k=0}^{\infty} \frac{x_k}{p_0 \cdot \ldots \cdot p_k}, \ |x_k| \leq \frac{p_k}{2} \text{ for } k > 0, \ x_0 = 0, \ldots, p_0.$$

One can easily see that

$$q_n x = \frac{x_{n+1}}{p_{n+1}} + \theta_n \bmod 1, \ |\theta_n| \leq 1/p_{n+1} \tag{5}$$

and therefore

$$\frac{|x_{n+1}| - 1}{p_{n+1}} \leq \|q_n x\| \leq \frac{|x_{n+1}| + 1}{p_{n+1}}.$$

More generally, if $m > n+1$ and $x_i = 0$ for $n+2 \le i \le m$, then

$$q_n x = \frac{x_{n+1}}{p_{n+1}} + \theta_n \bmod 1,\ |\theta_n| \le \frac{q_n}{q_m} \le \frac{1}{p_m}.$$

For an infinite subset $K \subseteq \mathbb{N}$ let

$$\begin{aligned}\mathsf{P}(K) &= \{x \in \mathbb{T}; (\exists n_0)(\forall n \in K, n \ge n_0)\, \|q_n \cdot x\| \le 1/p_{n+1}\},\\ \mathsf{A}(K) &= \{x \in \mathbb{T}; \lim_{n \in K} \|n \cdot x\| = 0\}.\end{aligned}$$

Evidently, $\mathsf{P}(K)$ is a pseudo Dirichlet set and $\mathsf{A}(K)$ is an A-set. Moreover, let us remark that if $K, L \subseteq \mathbb{N}$ are infinite sets, then

$$\text{if } K \setminus L \text{ is finite, then } \mathsf{P}(K) \subseteq \mathsf{P}(L) \text{ and } \mathsf{A}(K) \subseteq \mathsf{A}(L). \tag{6}$$

Moreover, one can easily check that

$$1/q_n \in \mathsf{P}(K) \text{ for any infinite } K \subseteq \mathbb{N} \text{ and any } n \in \mathbb{N}.$$

On the other side, for an infinite set $M \subseteq \mathbb{N}$, one can easily see that for any positive integer k

$$\text{if } 1/k \in \mathsf{A}(M), \text{ then } k \text{ divides all but finitely many elements of } M. \tag{7}$$

Actually, if $m = k \cdot n + r$, $0 < r < k$, then $\|m \cdot 1/k\| \ge 1/k$.

Now we can prove the easy version of Arbault's lemma (see [Ar], [BB]).

Lemma 5. *Let $M \subseteq \mathbb{N}$ be an infinite set. If $1/q_n \in \mathsf{A}(M)$ for every $n \in \mathbb{N}$, then there are sequences of natural numbers $\{s_n\}_{n=0}^{\infty}$, and $\{l_n\}_{n=0}^{\infty}$, a sequence of integers $\{r_n\}_{n=0}^{\infty}$ and a natural number n_0 such that:*

a) *$m_n = (s_n \cdot p_{l(n)+1} + r_n) q_{l(n)}$ for every $n \ge n_0$;*

b) *$0 < |r_n| \le 1/2 p_{l(n)+1}$ for every n;*

c) *the sequence $\{l(n)\}_{n=0}^{\infty}$ is unbounded.*

PROOF is easy. By (7) there exists an n_0 such that m_n is divisible by q_0 for all $n \ge n_0$. For $n \ge n_0$, let $l(n)$ be the greatest l such that m_n is divisible by q_l. Then there exist integers $s_n \ge 0$, $0 < |r_n| \le 1/2 p_{l(n)+1}$ such that

$$m_n = (s_n \cdot p_{l(n)+1} + r_n) q_{l(n)}.$$

By (7) for a given k there exists an n_1 such that every m_n, $n \ge n_1$ is divisible by q_k. Then $l(n_1) \ge k$. Thus c) holds. $\square$

Lemma 6. *Assume that* $\{s_n\}_{n=0}^{\infty}$, $\{r_n\}_{n=0}^{\infty}$ *and* $\{l_n\}_{n=0}^{\infty}$ *are sequences of natural numbers satisfying conditions* a), b), c) *of Lemma* 5. *Moreover assume that for any* $k \in \mathbb{N}$ *the inequality*

$$m_k \cdot p_{l(k)+1} \leq p_{l(k+1)} \cdot q_{l(k)} \tag{8}$$

holds. If the set $\{l_k; k \in \mathbb{N}\} \setminus K$ *is infinite, then* $\mathsf{P}(K) \nsubseteq \mathsf{A}(M)$.

PROOF. We shall follow the proof of lemma 18 of [BB]. If $i = l_k + 1$, $l_k \notin K$, take an integer $x_i < \frac{1}{2}q_i$ such that $x_i > \frac{1}{4}p_{l(k)+1}$. Otherwise set $x_i = 0$. Let $x = \sum_{i=0}^{\infty} x_i/q_i$. If $i \in K$, then $x_{i+1} = 0$ and $q_i x = \theta_i$. By (5) we have $\|q_i x\| < 2^{-p_{i+1}}$ and therefore $x \in \mathsf{P}(K)$.

If $l_k \notin K$, then we have mod 1

$$m_k x = (s_{l(k)} p_{l(k)+1} + r_k) q_{l(k)} x = r_k \frac{x_{l(k)+1}}{p_{l(k)+1}} + \frac{m_k}{q_{l(k)}} \theta_{l(k)}.$$

Since the last term is small, we obtain $\|m_k x\| \geq 1/8|r_k| \geq 1/8$ for sufficiently large k. Thus $\lim_{k\to\infty} m_k x \neq 0$ and therefore $x \notin \mathsf{A}(M)$. □

Lemma 7. *If* K, L, $K \setminus L$ *are infinite subsets of* $\mathbb{N}$, *then* $\mathsf{P}(L) \setminus \mathsf{P}(K)$ *contains a perfect subset.*

PROOF. Again, we can follow the proof of lemma 17 of [BB]. Since f is not identically equal to zero, there are reals α, β, γ such that $-1/2 < \alpha < \beta < 1/2$ and $f(x) \geq \gamma > 0$ for any $x \in \langle \alpha, \beta \rangle$. Let $N \subseteq K \setminus L$ be an infinite set such that $2/p_k < \beta - \alpha$ for any $k \in N$.

We set x_i to be an integer such that $\alpha < (x_i - 1)/p_i < (x_i + 1)/p_i < \beta$ if $i-1 \in N$. Otherwise set $x_i = 0$. Let $x(N) = \sum_{i=0}^{\infty} x_i/q_i$. For every $k \in N \subseteq K$ we have

$$q_k x(N) = x_{k+1}/p_{k+1} + \theta_k \bmod 1 \textit{ and } |\theta_k| \leq 1/p_{k+1}$$

and therefore for any $k \in N$ we have $\alpha < \|q_k x(N)\| < \beta$. Hence $x(N) \notin \mathsf{P}(K)$. On the other hand, if $k \in L$, then $x_{k+1} = 0$ and therefore $\|q_k x(N)\| \leq 1/p_{k+1}$. Thus $x(N) \in \mathsf{P}(L)$.

Since for different N's the reals $x(N)$ are different and we can find $\mathfrak{c}$ many infinite sets $N \subseteq K \setminus L$, the difference $\mathsf{P}(L) \setminus \mathsf{P}(K)$ has the power of the continuum. Being a Borel set it contains a perfect subset. □

PROOF OF THEOREM 4. Let $K_\xi; \xi < \mathfrak{t}$ be a tower of subsets of $\mathbb{N}$; i.e., for any $\xi < \eta < \mathfrak{t}$ the set $K_\eta \setminus K_\xi$ is finite, the set $K_\xi \setminus K_\eta$ is infinite, and there is no infinite set $L \subseteq \mathbb{N}$ such that $L \setminus K_\xi$ is finite for any $\xi < \mathfrak{t}$. We set $P_\xi = \mathsf{P}(K_\xi)$ for $\xi < \mathfrak{t}$. By (6) and Lemma 7 we obtain immediately the assertions a) and b) of theorem.

Toward a contradiction assume that there exists an A-set $\mathsf{A}(M)$ containing all sets P_ξ, $\xi \in \mathfrak{t}$. Since $P_0 \subseteq \mathsf{A}(M)$, there are sequences satisfying the assertions of Lemma 5. Passing to a subset of M we may achieve that condition (8) is satisfied. By the definition of a tower there exists a $\xi < \mathfrak{t}$ such that $\{l_k; k \in \mathbb{N}\} \setminus K_\xi$ is infinite. Then, by Lemma 6 we obtain $\mathsf{P}(K_\xi) \nsubseteq \mathsf{A}(M)$ - a contradiction. $\square$

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