

J. Alan Alewine, Division of Science and Mathematics, McKendree College,  
Lebanon, IL 62254. email: jaalewine@mckendree.edu

Eric Schechter, Mathematics Department, Vanderbilt University, Nashville,  
TN 37240. email: eric.schechter@vanderbilt.edu

## TOPOLOGIZING THE DENJOY SPACE BY MEASURING EQUIINTEGRABILITY

### Abstract

Basic limit theorems for the KH integral involve equiintegrable sets. We construct a family of Banach spaces  $X_\Delta$  whose bounded sets are precisely the subsets of  $\mathcal{KH}[0, 1]$  that are equiintegrable and pointwise bounded. The resulting inductive limit topology on  $\bigcup_\Delta X_\Delta = \mathcal{KH}[0, 1]$  is barreled, bornological, and stronger than both pointwise convergence and the topology given by the Alexiewicz seminorm, but it lacks the countability and compatibility conditions that are often associated with inductive limits.

### 1 Introduction.

This paper is concerned with  $\mathcal{KH}[0, 1]$ , the space of all functions  $f : [0, 1] \rightarrow \mathbb{R}$  that are KH integrable (also known as Kurzweil, Henstock, Denjoy-Perron, gauge, nonabsolute, or generalized Riemann integrable). We emphasize that we are considering individual *functions*, whereas most of the related literature deals with  $KH[0, 1]$ , the space of *equivalence classes* of KH integrable functions. Equivalence in this context means agreement outside some set of Lebesgue measure zero.

Sections 2 and 3 review basic results about the KH integral, including definitions of technical terms (gauges,  $\ll$ ,  $f(T)$ , etc.) used in this introduction. The KH integral generalizes the Lebesgue integral — we have the spaces of

$$\begin{array}{ll} \text{functions} & \mathcal{KH}[0, 1] \supseteq \mathcal{L}^1[0, 1], \text{ and} \\ \text{equivalence classes of functions} & KH[0, 1] \supseteq L^1[0, 1]. \end{array}$$

---

Key Words: barreled, convergence rate, Denjoy space, equiintegrable, Henstock integral, inductive limit, KH integral, Kurzweil integral, norm  
Mathematical Reviews subject classification: 26A39  
Received by the editors May 12, 2004  
Communicated by: Stefan Schwabik

In fact, both of those inclusions are strict;  $\mathcal{KH}[0, 1] \setminus \mathcal{L}^1[0, 1]$  contains erratic functions such as  $t^{-1} \sin(t^{-2})$ . But consequently the spaces  $\mathcal{KH}$  and  $KH$  are also somewhat erratic. Apparently they cannot be equipped with topologies as nice as that of the Banach space  $L^1[0, 1]$ .

Our notion of “niceness” is subjective, but can be formulated imprecisely as follows. A “nice” topology on a function space should be fairly simple to describe, should enjoy as many positive functional analytic properties (normability, completeness, etc.) as possible, and should be closely related to the properties (such as convergence) being studied for the functions involved. Without that requirement about convergences or other properties, we could easily devise topologies that are elegant but meaningless. Indeed, the Axiom of Choice can be used to show that every vector space is linearly isomorphic to a Hilbert space.

Arguably the “nicest” known norm on  $KH[0, 1]$  is the *Alexiewicz norm*,

$$\|f\|_A = \max_{0 \leq r \leq 1} \left| \int_0^r f \right|.$$

It is also a seminorm on  $\mathcal{KH}[0, 1]$ , and will be used as such later in this paper. Trivially,  $\|f_n - f\|_A \rightarrow 0$  implies  $\int_0^1 f_n \rightarrow \int_0^1 f$ , but that does not give us much insight into the deeper convergence theory of  $KH[0, 1]$  or  $\mathcal{KH}[0, 1]$ .

Many different convergence theorems for the KH integral (including generalizations of the Lebesgue integral’s Monotone and Dominated Convergence Theorems) can be found in the literature. Most of these convergence theorems are rather complicated, but most of them can be proved (not necessarily easily) as specializations of this one simple result, restated in 3.2.

If  $(f_n)$  is equiintegrable and  $f_n \rightarrow f$  pointwise, then  $\int f_n \rightarrow \int f$ .

Converses to this theorem can be found in Theorems 8.12 and 8.13 of [4], but they both involve replacing equiintegrability with more complicated conditions, apparently not amenable to “measurements” like that below.

Integrability and equiintegrability are generally introduced as qualitative (yes or no) properties. However, by juggling our  $\forall$ ’s and  $\exists$ ’s, we can reformulate the usual definitions to emphasize their quantitative ingredients. A function  $f : [0, 1] \rightarrow \mathbb{R}$  is *Riemann integrable*, respectively *KH integrable*, if there exists a sequence  $\Delta = (\delta_1, \delta_2, \delta_3, \dots)$  of positive numbers, respectively of positive functions, such that the number

$$\theta_\Delta(f) \stackrel{\text{def}}{=} \sup_{n \in \mathbb{N}} \sup_{T, T' \ll \delta_n} n |f(T) - f(T')|$$

is finite. (Definitions of  $f(T)$  and  $T \ll \delta$  will be given in §2 and 3.) A collection  $\mathcal{F}$  of functions is *equiintegrable* if one  $\Delta$  works uniformly for all the members of  $\mathcal{F}$ ; i.e., if there is some  $\Delta$  such that  $\sup\{\theta_\Delta(f) : f \in \mathcal{F}\}$  is finite. Thus, each  $\Delta$  gives us a numerical “measurement” of equiintegrability, or of the “rate” of convergence.

This paper is concerned with properties of the measurements  $\theta_\Delta$ . Here are a few basic results. First  $\|f\|_\Delta = \theta_\Delta(f) + |f(0)|$  defines a norm on the linear space

$$X_\Delta = \left\{ f \in \mathbb{R}^{[0,1]} : \theta_\Delta(f) < \infty \right\},$$

and that norm turns out to be complete (see 7.2). An interesting example is  $\{\text{functions of bounded variation}\} = X_{(1, \frac{1}{2}, \frac{1}{3}, \dots)}$ .

The union of the Banach spaces  $X_\Delta$  is all of  $\mathcal{KH}[0, 1]$ , and the  $X_\Delta$ 's form a directed set when ordered by inclusion. For these reasons it seems natural to try topologizing  $\mathcal{KH}[0, 1]$  as the locally convex inductive limit of the  $X_\Delta$ 's; indeed, that was the original motivation for our research. The resulting topology on  $\mathcal{KH}[0, 1]$  is barreled, bornological, stronger than pointwise convergence on  $[0, 1]$ , and stronger than the topology given by the Alexiewicz seminorm.

However, in other respects the inductive limit topology has been disappointing. We found that the  $X_\Delta$ 's lack a couple of the most important properties that make other inductive limits useful:

- $\mathcal{KH}[0, 1]$  is *not* a union of countably many of the  $X_\Delta$ 's. See 7.7.
- When  $\Delta \preceq \widehat{\Delta}$ , then  $X_\Delta \subseteq X_{\widehat{\Delta}}$ , and the inclusion  $X_\Delta \rightarrow X_{\widehat{\Delta}}$  is continuous. But in some cases the topology given by  $\|\cdot\|_\Delta$  is *strictly* stronger than the relative topology induced on  $X_\Delta$  by  $\|\cdot\|_{\widehat{\Delta}}$ . See 8.6.

These drawbacks make the inductive limit topology difficult to analyze, and we have not yet been able to answer several other questions about it; see the list of open questions at the end of this paper.

Kurzweil [8] and Thomson [15] also studied inductive limit topologies, though on  $KH$  rather than  $\mathcal{KH}$ ; we review their results briefly in 9.4.

This paper is based on results in the first author's doctoral thesis [1]. The authors are grateful for many helpful insights from the referee.

## 2 Review of Basics.

For the reader's convenience, we restate some basic definitions and known results about the KH integral that will be used later in this paper. Most of these results can be found in any of the introductory books listed in the bibliography, though the notations differ slightly in some of those books.

**Definitions 2.1.** A **division** of the interval  $[0, 1]$  is a finite partition

$$0 = t_0 < t_1 < t_2 < \cdots < t_m = 1.$$

A **tagged division** is a division, as above, together with selected points  $\tau_j \in [t_{j-1}, t_j]$ ; the number  $\tau_j$  is called the **tag** of the subinterval  $[t_{j-1}, t_j]$ . We shall denote a typical tagged division by  $T = \{(\tau_j, [t_{j-1}, t_j])\}_{j=1}^m$ . For any function  $f : [0, 1] \rightarrow \mathbb{R}$ , the **approximating Riemann sum** over the tagged division  $T$  is

$$f(T) = \sum_{j=1}^m f(\tau_j)(t_j - t_{j-1}).$$

**Definitions 2.2.** A **gauge** on an interval  $[0, 1]$  is any function  $\delta$  from  $[0, 1]$  into  $(0, +\infty)$ . Any positive number may be viewed as a constant gauge. For any gauge  $\delta$ , a tagged division  $T = \{(\tau_j, [t_{j-1}, t_j])\}_{j=1}^m$  is said to be  **$\delta$ -fine** if

$$t_j - t_{j-1} < \delta(\tau_j) \quad (j = 1, 2, \dots, m);$$

we shall abbreviate this condition as  $T \ll \delta$ .

**Lemma 2.3** (Cousin's Lemma). *Given any gauge  $\delta$  on  $[0, 1]$ , there exists a  $\delta$ -fine tagged division.*

**Lemma 2.4** (Forced tags). *If  $Q$  is any finite subset of  $[0, 1]$ , then there exists a gauge  $\delta$  with the property that any  $\delta$ -fine tagged division has all the members of  $Q$  among its tags.*

**Definition 2.5.** A number  $v$  is the **KH integral** of a function  $f : [0, 1] \rightarrow \mathbb{R}$  if for each number  $\varepsilon > 0$  there exists a gauge  $\delta$  on  $[0, 1]$  such that, whenever  $T$  is a  $\delta$ -fine tagged division, then  $|f(T) - v| < \varepsilon$ .

Clearly there is at most one such  $v$ . If it exists, we denote it by  $\int_0^1 f(t)dt$ .

**Remark.** If we add the restriction that the  $\delta$ 's must be constant gauges, we obtain the Riemann integral.

**Notation 2.6.** All integrals in this paper are KH integrals except where explicitly noted. The set of KH integrable real-valued functions on  $[0, 1]$  will be denoted by  $\mathcal{KH}[0, 1]$ . The set of *all* real-valued functions on  $[0, 1]$  will be denoted, as usual, by  $\mathbb{R}^{[0,1]}$ .

**Proposition 2.7.** *A function  $f : [0, 1] \rightarrow \mathbb{R}$  is Lebesgue integrable if and only if both  $f$  and  $|f|$  are KH integrable, in which case the two integrals give the same value for  $\int_0^1 f$ .*

**Remark.** Newcomers to the KH integral may find this imprecise analogy helpful. The relation between the KH integral and the Lebesgue integral is something like the relation between a convergent series and an absolutely convergent series.

**Proposition 2.8** (Cauchy condition for integrability). *A function  $f : [0, 1] \rightarrow \mathbb{R}$  is KH integrable (i.e., its integral exists) if and only if for each number  $\varepsilon > 0$  there exists a gauge  $\delta$  on  $[0, 1]$  such that, whenever  $T_1, T_2$  are  $\delta$ -fine tagged divisions, then  $|f(T_1) - f(T_2)| < \varepsilon$ .*

(If we use constants instead of gauges for  $\delta$ , we get Riemann integrability.)

### 3 Review of Equiintegrability.

**Definition 3.1.** A set  $\mathcal{E} \subseteq \mathcal{KH}[0, 1]$  is **equiintegrable** (or, in some articles, **uniformly integrable**) if for each number  $\varepsilon > 0$  there exists a gauge  $\delta$  on  $[0, 1]$  such that, whenever  $T$  is a  $\delta$ -fine tagged division and  $f \in \mathcal{E}$ , then  $|f(T) - \int f| < \varepsilon$ .

It is easy to see that the equiintegrable sets form a bornology — i.e., any subset of an equiintegrable set is equiintegrable, and the union of finitely many equiintegrable sets is equiintegrable. It will follow from results later in this paper that the bornology is actually a convex vector bornology — i.e., the sum, product by a scalar, or convex hull of an equiintegrable set is also equiintegrable. An introduction to bornologies can be found in [7], but is not necessary for the reading of this paper.

It is also easy to see that the union of countably many equiintegrable sets is not necessarily equiintegrable. Indeed, let  $(f_n)$  be any sequence that is not equiintegrable (e.g., as in 3.6 or 3.7); then each singleton  $\{f_n\}$  is equiintegrable but the union is not.

**Theorem 3.2** (Convergence). *Suppose the sequence  $(f_n)$  is equiintegrable and  $f_n \rightarrow f$  pointwise on  $[0, 1]$ . Then  $f \in \mathcal{KH}[0, 1]$  and  $\|f_n - f\|_A \rightarrow 0$ ; hence  $\int f_n \rightarrow \int f$  and  $\sup_n \|f_n\|_A < \infty$ .*

**Example 3.3.** *A sequence of functions may be equiintegrable and yet not be bounded in any norm or seminorm whatsoever.*

Indeed, let  $f_n$  be the constant function  $n$ . The sequence is equiintegrable since  $f_n(T) = \int f_n$  for every  $n$  and every tagged division  $T$ .

**Example 3.4.** *Equiintegrability and Alexiewicz convergence together do not imply pointwise convergence.*

Let  $f_n : [0, 1] \rightarrow \mathbb{R}$  be the characteristic function of the interval  $[0, 1/n]$ . Then  $\text{Var}(f_n) = 1$  and  $\|f_n\|_A = 1/n$ . It will follow from 5.4 and 8.1 that the sequence  $(f_n)$  is equiintegrable.

**Theorem 3.5** (Equicontinuity). *Let  $(f_n)$  be a sequence of KH integrable functions. Assume either*

- (i)  $(f_n)$  is equiintegrable and pointwise bounded — i.e.,  $\sup_n |f_n(t)| < \infty$  for each  $t$ ; or
- (ii)  $\|f_n\|_A \rightarrow 0$  as  $n \rightarrow \infty$ .

*Then the indefinite integrals  $\int_0^x f(t)dt$ , for  $f \in \mathcal{F}$ , form a uniformly equicontinuous set of functions of  $x$ . Hence the indefinite integrals form a relatively compact subset of  $C[0, 1]$ , and the equivalence classes of the members of  $\mathcal{F}$  form a relatively compact subset of the normed space  $(KH[0, 1], \|\cdot\|_A)$ .*

PROOF. The first assertion is 8.V in [4]; the second assertion follows from the Arzela-Ascoli Theorem.  $\square$

**Example 3.6.** *Equicontinuity of the  $\int f$ 's does not imply pointwise boundedness or equiintegrability of the  $f$ 's.*

Let  $f_n(0) = n$ , and  $f_n(t) = 0$  on  $(0, 1]$ .

**Example 3.7.** *A sequence may be bounded in Alexiewicz seminorm, and convergent pointwise to 0, and still not be equiintegrable.*

(This example is simplified from [13].) Define the functions

$$f_n(t) = \begin{cases} n & 0 < t < \frac{1}{2n} \\ -n & \frac{1}{2n} < t < \frac{1}{n} \\ 0 & \text{elsewhere in } [0, 1]. \end{cases}$$

The proof that this sequence has the required properties is essentially the same as the proof in [13].

#### 4 Some Technical Lemmas.

**Lemma 4.1** (Perturbed division). *Let  $\delta$  be a gauge on  $[0, 1]$ , and let  $T = \{(\tau_j, [t_{j-1}, t_j])\}_{j=1}^m$  be a tagged division of  $[0, 1]$  that is  $\delta$ -fine. Let some particular  $k \in \{1, 2, \dots, m-1\}$  be given, and suppose that  $\tau_k \neq \tau_{k+1}$ . Then there exists a number  $t'_k \in [0, 1]$  which is different from  $t_k$ , such that replacing  $t_k$  with  $t'_k$  and leaving all other ingredients of  $T$  unchanged yields a tagged division  $T'$  that is also  $\delta$ -fine.*

**Observation.** For any function  $f : [0, 1] \rightarrow \mathbb{R}$ , the difference of the resulting Riemann sums is

$$f(T) - f(T') = (t'_k - t_k) \left( f(\tau_{k+1}) - f(\tau_k) \right).$$

PROOF OF LEMMA. Since  $T$  is a tagged division, we have

$$t_{k-1} \leq \tau_k \leq t_k \leq \tau_{k+1} \leq t_{k+1}.$$

Then the interval  $[\tau_k, \tau_{k+1}] \stackrel{\text{def}}{=} P$  has positive length, since  $\tau_k \neq \tau_{k+1}$ . Note that  $t_k \in P$ . Also, since  $T$  is  $\delta$ -fine, we have

$$t_k - t_{k-1} < \delta(\tau_k) \text{ and } t_{k+1} - t_k < \delta(\tau_{k+1});$$

hence

$$t_k \in \left( t_{k+1} - \delta(\tau_{k+1}), t_{k-1} + \delta(\tau_k) \right) \stackrel{\text{def}}{=} Q.$$

Since  $P$  and  $Q$  both contain  $t_k$  and have positive length and  $Q$  is open, the interval  $P \cap Q$  also has positive length. Choose any point  $t'_k \in P \cap Q$  different from  $t_k$ . From  $t'_k \in P$  we obtain

$$t_{k-1} \leq \tau_k \leq t'_k \leq \tau_{k+1} \leq t_{k+1}.$$

Therefore the new system of intervals and points  $T'$ , obtained from  $T$  by replacing  $t_k$  with  $t'_k$ , is indeed a tagged division. Moreover, from  $t'_k \in Q$  we obtain

$$t'_k - t_{k-1} < \delta(\tau_k) \text{ and } t_{k+1} - t'_k < \delta(\tau_{k+1}),$$

so  $T'$  is also  $\delta$ -fine. □

**Lemma 4.2** (Pointwise convergence). *Let  $p$  be some given point in  $[0, 1]$ , and let  $\delta$  be some gauge on  $[0, 1]$ . Let  $(g_\alpha)$  be a net in  $\mathbb{R}^{[0,1]}$ , and let  $g \in \mathbb{R}^{[0,1]}$ . Then*

(1)  $g_\alpha(t) - g_\alpha(p) \rightarrow g(t) - g(p)$  for each  $t \in [0, 1]$

if and only if

(2)  $g_\alpha(T) - g_\alpha(T') \rightarrow g(T) - g(T')$  for each pair of tagged divisions  $T, T'$  that are  $\delta$ -fine.

Hence

(1')  $g_\alpha \rightarrow g$  pointwise on  $[0, 1]$

if and only if

(2')  $g_\alpha(T) - g_\alpha(T') \rightarrow g(T) - g(T')$  for each pair of tagged divisions  $T, T'$  that are  $\delta$ -fine, and  $g_\alpha(p) \rightarrow g(p)$ .

**Remarks.** The convergences in (1), (1') are not taken to be uniform in  $t$ , and the convergences in (2), (2') are not taken to be uniform in  $T, T'$ . Note that if we change the choice of  $p$ , then conditions (2) and (1') are unaffected, hence this also has no effect on whether conditions (1) and (2') hold.

**PROOF OF LEMMA.** The implication (1)  $\Rightarrow$  (2) is immediate, since the evaluation of each  $g_\alpha(T)$  or  $g_\alpha(T')$  depends only on the values of  $g_\alpha$  at finitely many points in  $[0, 1]$ . Thus it suffices to prove (2)  $\Rightarrow$  (1). For simplicity of notation, and without loss of generality, we may replace  $g_\alpha$  and  $g$  with  $g_\alpha - g$  and  $g - g$ , respectively; thus we may assume  $g = 0$ . Also, we may replace  $g_\alpha$  with  $g_\alpha - g_\alpha(p)$ ; thus we may assume  $g_\alpha(p) = 0$ .

Let any  $q \in [0, 1]$  be given. Thus, we are given

$$g_\alpha(p) = 0; \quad \lim_\alpha [g_\alpha(T) - g_\alpha(T')] = 0 \text{ for all } T, T \ll \delta;$$

it suffices to prove that  $g_\alpha(q) \rightarrow 0$ .

By Lemma 2.4 there is some gauge  $\delta_1$  with the property that any  $\delta_1$ -fine tagged division of  $[0, 1]$  has both the numbers  $p$  and  $q$  among its tags. Let  $\delta_2 = \min\{\delta, \delta_1\}$ . By Cousin's Lemma 2.3 there exists some tagged division  $T = \{(\tau_j, [t_{j-1}, t_j])\}_{j=1}^m$  of  $[0, 1]$  that is  $\delta_2$ -fine. (This tagged division will remain fixed throughout the remainder of this proof; in particular,  $m$  is fixed.) Then  $T$  is also  $\delta_1$ -fine and  $\delta$ -fine. Since  $T$  is  $\delta_1$ -fine,  $p$  and  $q$  are among the tags  $\tau_j$ . Since  $|g_\alpha(p)| = 0$  for all  $\alpha$ , it suffices to show that  $\lim_\alpha [g_\alpha(\tau_{k+1}) - g_\alpha(\tau_k)] = 0$  for each  $k$  in  $\{1, 2, \dots, m-1\}$ . Fix any such  $k$ . We may assume  $\tau_k \neq \tau_{k+1}$ . Form a perturbed  $\delta$ -fine tagged division  $T'$  as in Lemma 4.1. Then

$$|t'_k - t_k| |g_\alpha(\tau_{k+1}) - g_\alpha(\tau_k)| = |g_\alpha(T) - g_\alpha(T')| \rightarrow 0.$$

Since  $t'_k - t_k$  is a nonzero constant,  $g_\alpha(\tau_{k+1}) - g_\alpha(\tau_k) \rightarrow 0$  as required.  $\square$

**Corollary 4.3** (Positive definiteness). *Let  $\delta$  be a gauge on  $[0, 1]$ , and let  $f : [0, 1] \rightarrow \mathbb{R}$  be some function. Then  $f$  is constant if and only if  $f(T) = f(T')$  for all  $\delta$ -fine tagged divisions  $T, T'$ .*

PROOF. Clearly, if  $f$  takes a constant value  $c$ , then  $f(T) = c$  for all tagged divisions  $T$ . Conversely, suppose that  $f(T) = f(T')$  for all  $T, T' \ll \delta$ . Pick any point  $p \in [0, 1]$ , and any directed set  $\{\alpha\}$ . Define  $g_\alpha(t) = f(t)$  for all  $\alpha, t$ , and  $g = 0$ . Apply (2)  $\Rightarrow$  (1) of the preceding lemma.  $\square$

## 5 Seminorms That Measure Equiintegrability.

**Definition 5.1.** For any function  $f : [0, 1] \rightarrow \mathbb{R}$ , any gauge  $\delta$ , and any sequence  $\Delta = (\delta_n)$  of gauges, define

$$\begin{aligned}\theta_\delta(f) &= \sup_{T, T' \ll \delta} |f(T) - f(T')|, \\ \theta_\Delta(f) &= \sup_{n \in \mathbb{N}} n \theta_{\delta_n}(f) = \sup_{n \in \mathbb{N}} \sup_{T, T' \ll \delta_n} n |f(T) - f(T')|,\end{aligned}$$

with  $\infty$  as a possible value. Also, if  $f$  is KH integrable, define

$$\begin{aligned}\psi_\delta(f) &= \sup_{T \ll \delta} \left| f(T) - \int_0^1 f \right|, \\ \psi_\Delta(f) &= \sup_{n \in \mathbb{N}} n \psi_{\delta_n}(f) = \sup_{n \in \mathbb{N}} \sup_{T \ll \delta_n} n |f(T) - \int_0^1 f|.\end{aligned}$$

Clearly, the four functions  $\theta_\delta$ ,  $\theta_\Delta$ ,  $\psi_\delta$ ,  $\psi_\Delta$  are seminorms on the linear subspaces where they are finite (since any pointwise supremum of seminorms is a seminorm).

**Remark.** The multiplier  $n$  in the definitions of  $\theta_\Delta$  and  $\psi_\Delta$  was chosen for simplicity; analogous results would be obtained if the sequence  $(n)$  were replaced by any other sequence that tends to infinity. Using the sequence  $(2^n)$  would bring this paper's style closer to that of Kurzweil's monograph [8], though the differences are still great.

**Proposition 5.2.** *If  $f$  is any KH integrable function, then*

$$\begin{aligned}\psi_\delta(f) &\leq \theta_\delta(f) \leq 2\psi_\delta(f) \\ \text{and } \psi_\Delta(f) &\leq \theta_\Delta(f) \leq 2\psi_\Delta(f).\end{aligned}$$

Hence

- the seminorms  $\psi_\delta$  and  $\theta_\delta$  are equivalent on the subspace of  $\mathcal{KH}[0, 1]$  where they are finite;

- the seminorms  $\psi_\Delta$  and  $\theta_\Delta$  are equivalent on the subspace of  $\mathcal{KH}[0, 1]$  where they are finite.

SKETCH OF PROOF. The inequality  $\theta_\delta \leq 2\psi_\delta$  is immediate from the triangle inequality. To prove  $\psi_\delta(f) \leq \theta_\delta(f)$ , hold  $T$  fixed and let  $f(T') \rightarrow \int f$ .  $\square$

**Observation and Notation 5.3.** From 2.8 we see that a function  $f$  is KH integrable if and only if  $\theta_\Delta(f) < \infty$  for some sequence  $\Delta$  of gauges. Hence by 5.2, the linear spaces

$$\left\{ f \in \mathbb{R}^{[0,1]} : \theta_\Delta(f) < \infty \right\} \text{ and } \left\{ f \in \mathcal{KH}[0, 1] : \psi_\Delta(f) < \infty \right\}$$

are the same. Hereafter we shall denote that space by  $X_\Delta$ . The seminorms  $\theta_\Delta$  and  $\psi_\Delta$  are equivalent on that space. Again restating 2.8, we have  $\mathcal{KH}[0, 1] = \bigcup_\Delta X_\Delta$ , where the union is over all sequences  $\Delta$  of gauges.

**Proposition 5.4.** *These three conditions on a set  $\mathcal{E} \subseteq \mathcal{KH}[0, 1]$  are equivalent:*

- $\mathcal{E}$  is equiintegrable;
- there is some  $\Delta$  such that  $\sup\{\theta_\Delta(f) : f \in \mathcal{E}\} < \infty$ ;
- the members of  $\mathcal{E}$  are KH integrable and there is some  $\Delta$  such that  $\sup\{\psi_\Delta(f) : f \in \mathcal{E}\} < \infty$ .

PROOF. Immediate from 3.1.  $\square$

## 6 Pointwise Convergence and Boundedness.

**Proposition 6.1.** *Let functions  $(g_k)$  and gauge  $\delta$  satisfy  $\lim_{k \rightarrow \infty} \theta_\delta(g_k) = 0$ . Suppose that  $\lim_{k \rightarrow \infty} g_k(t) = 0$  for at least one  $t \in [0, 1]$ . Then  $\lim_{k \rightarrow \infty} g_k(t) = 0$  for every  $t \in [0, 1]$ .*

PROOF. Apply (2')  $\Rightarrow$  (1') in 4.2.  $\square$

**Corollary 6.2.** *Suppose  $\mathcal{F}$  is an equiintegrable set of functions — or more generally, suppose that  $\sup\{\theta_\delta(f) : f \in \mathcal{F}\} < \infty$  for some gauge  $\delta$ . Suppose that  $\mathcal{F}$  is that is pointwise bounded at some point  $p \in [0, 1]$ . Then  $\mathcal{F}$  is pointwise bounded at each point in  $[0, 1]$ .*

PROOF. Suppose not. Then there is some point  $q \in [0, 1]$  and some sequence  $(f_k)$  in  $\mathcal{F}$  such that  $|f_k(q)| > k$ . Define  $g_k = f_k/k$ . Then  $\theta_\delta(g_k) \rightarrow 0$  and  $|g_k(p)| \rightarrow 0$  but  $|g_k(q)| > 1$ , contradicting 6.1.  $\square$

**Example 6.3.** *The conditions of 6.2 do not imply that  $\mathcal{F}$  is uniformly bounded on  $[0, 1]$ .*

This function is KH integrable.

$$f(t) = \begin{cases} 0 & \text{when } t = 0, \\ 1/\sqrt{t} & \text{when } t \in (0, 1]. \end{cases}$$

Hence the singleton  $\{f\}$  is equiintegrable, but not uniformly bounded.  $\square$

**Proposition 6.4.** *Let  $\delta$  be any gauge on  $[0, 1]$ , and let  $f : [0, 1] \rightarrow \mathbb{R}$  be any function. Then the following conditions are equivalent:*

- $f$  is constant
- $\theta_\delta(f) = 0$ .
- $f$  is KH integrable and  $\psi_\delta(f) = 0$ .

PROOF. Immediate from 4.3.  $\square$

## 7 Banach Spaces.

**Observation and Notation 7.1.** Let  $\Delta$  be any sequence of gauges, and let  $p$  be any point in  $[0, 1]$ . Then the following functions of  $f$  are equivalent norms on  $X_\Delta$ :

$$\theta_\Delta(f) + |f(0)|, \quad \psi_\Delta(f) + |f(0)|, \quad \theta_\Delta(f) + |f(p)|, \quad \psi_\Delta(f) + |f(p)|.$$

Hereafter we shall denote  $\|f\|_\Delta = \theta_\Delta(f) + |f(0)|$ .

**Proposition 7.2.** *The normed space  $(X_\Delta, \|\cdot\|_\Delta)$  defined in 5.3, 7.1 is complete.*

INDICATION OF PROOF. The proof of completeness is along the same lines as the usual proof of completeness of  $C[0, 1]$  or  $BV[0, 1]$  or  $Lip[0, 1]$ , which is familiar to all analysts. Alternatively one may apply §22.17 of [12], an abstract theorem that simultaneously establishes completeness in all such examples. The hypotheses of that abstract theorem (when it is applied to  $X_\Delta$ ) are the conclusions of this paper's Lemma 4.2.  $\square$

**Theorem 7.3.** *Let  $\Delta = (\delta_n)$  be any sequence of gauges. Then the map*

$$i : (X_\Delta, \|\cdot\|_\Delta) \xrightarrow{\subset} (KH[0, 1], \|\cdot\|_A),$$

*sending each function to its equivalence class, is a compact linear operator.*

PROOF. Immediate from 3.5 and 6.2.  $\square$

**Definition 7.4.** For sequences  $\Delta = (\delta_1, \delta_2, \delta_3, \dots)$  and  $\widehat{\Delta} = (\widehat{\delta}_1, \widehat{\delta}_2, \widehat{\delta}_3, \dots)$  of gauges, we will write  $\Delta \preceq \widehat{\Delta}$  if

$$\delta_n(t) \geq \widehat{\delta}_n(t) \text{ for all } n \in \mathbb{N} \text{ and } t \in [0, 1].$$

(In other words, smaller gauges are considered to occur later.) This ordering makes the collection of all sequences of gauges into a directed set (since the pointwise minimum of finitely many gauges is a gauge.)

**Proposition 7.5.** *Suppose  $\Delta \preceq \widehat{\Delta}$ . Then  $X_\Delta \subseteq X_{\widehat{\Delta}}$ , and the inclusion is continuous; in fact,  $\|\cdot\|_\Delta \geq \|\cdot\|_{\widehat{\Delta}}$  on  $X_\Delta$ . Thus the topology given by  $\|\cdot\|_\Delta$  is stronger than or equal to the relative topology induced by  $\|\cdot\|_{\widehat{\Delta}}$ .*

PROOF. From  $\delta_n \geq \widehat{\delta}_n$  we obtain

$$\{(T, T') : T, T' \ll \delta_n\} \supseteq \{(T, T') : T, T' \ll \widehat{\delta}_n\}.$$

Hence  $\theta_\Delta \geq \theta_{\widehat{\Delta}}$  and  $\|\cdot\|_\Delta \geq \|\cdot\|_{\widehat{\Delta}}$ . Hence  $X_\Delta \subseteq X_{\widehat{\Delta}}$ .  $\square$

**Example 7.6.** *The preceding notions apply to any sequences of gauges, but we will be particularly interested in these two sequences of constant gauges:*

$$\Delta_1 = \left( \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \right), \quad \Delta_4 = \left( \frac{1}{1^4}, \frac{1}{2^4}, \frac{1}{3^4}, \frac{1}{4^4}, \dots \right)$$

*We have  $\Delta_1 \preceq \Delta_4$ , hence  $X_{\Delta_1} \subseteq X_{\Delta_4}$ , and  $\|\cdot\|_{\Delta_4} \leq \|\cdot\|_{\Delta_1}$  on  $X_{\Delta_1}$ . The topology given on  $X_{\Delta_1}$  by  $\|\cdot\|_{\Delta_1}$  is stronger than or equal to the topology given on that same vector space by the restriction of  $\|\cdot\|_{\Delta_4}$ .*

(We will reach a stronger conclusion in 8.6.)

**Proposition 7.7.** *The linear space  $\mathcal{KH}[0, 1]$  is not equal to the union of countably many of the  $X_\Delta$ 's.*

PROOF. Let any sequence  $\Delta(1), \Delta(2), \Delta(3), \dots$  of sequences of gauges be given; say the  $i$ th sequence is  $\Delta(i) = (\delta_1^i, \delta_2^i, \delta_3^i, \dots)$ . We shall produce a function  $f \in \mathcal{KH}[0, 1] \setminus \bigcup_{i=1}^{\infty} X_{\Delta(i)}$ . Without loss of generality we may replace each gauge  $\delta_j^i$  with a smaller gauge (as that merely makes the  $X_{\Delta(i)}$ 's larger); hence we may assume that  $\delta_j^i(t) < 3 \min\{t, 1-t\}$  for all  $t \in (0, 1)$ .

Let some enumeration

$$(i(2), j(2)), (i(3), j(3)), (i(4), j(4)), \dots$$

of  $\mathbb{N}^2$  be chosen. (Beginning the sequence with 2 rather than 1 will simplify the notation in the subsequent argument.) Thus  $\delta_{j(2)}^{i(2)}, \delta_{j(3)}^{i(3)}, \delta_{j(4)}^{i(4)}, \dots$  is a sequence that includes all the given  $\delta$ 's. Define a function  $f : [0, 1] \rightarrow \mathbb{R}$  by

$$f(t) = \begin{cases} \frac{2}{\delta_{j(p)}^{i(p)}(1/p)} & \text{if } t = 1/p \text{ for some integer } p > 1 \\ 0 & \text{for all other } t \in [0, 1]. \end{cases}$$

Then  $f$  is KH integrable, with  $\int_0^1 f(t)dt = 0$ , by 2.7.

Fix any  $\widehat{i} \in \mathbb{N}$ ; it suffices to show that  $f \notin X_{\Delta(\widehat{i})}$ . Fix any  $\widehat{j} \in \mathbb{N}$ , and let  $\widehat{\delta} = \delta_{\widehat{j}}^2$ ; it suffices to show that  $\sup\{f(T) : T \ll \widehat{\delta}\} \geq 1$ . That is, it suffices to show  $f(T) \geq 1$  for some  $\widehat{\delta}$ -fine tagged division.

Let  $\widehat{p}$  be the integer greater than 1 that satisfies  $(i(\widehat{p}), j(\widehat{p})) = (\widehat{i}, \widehat{j})$ . Let  $\sigma = \widehat{\delta}(1/\widehat{p})/3$ . We shall construct  $T$  by combining tagged divisions formed separately on the three subintervals

$$\left[0, \frac{1}{\widehat{p}} - \sigma\right], \left[\frac{1}{\widehat{p}} - \sigma, \frac{1}{\widehat{p}} + \sigma\right], \left[\frac{1}{\widehat{p}} + \sigma, 1\right].$$

For the left and right parts, we use any tagged division that is  $\widehat{\delta}$ -fine; the existence of such tagged divisions is guaranteed by Cousin's Lemma. For the middle part, we just use the one interval  $[(1/\widehat{p}) - \sigma, (1/\widehat{p}) + \sigma]$  with tag  $1/\widehat{p}$ . That interval is contained in  $[0, 1]$ , since  $\frac{1}{\widehat{p}} \in (0, 1)$  and therefore  $\sigma < \min\{1/\widehat{p}, 1 - (1/\widehat{p})\}$ . Also, that interval has length  $2\sigma$ , which is strictly less than  $\widehat{\delta}(1/\widehat{p})$ . Thus, the resulting tagged division  $T$  of  $[0, 1]$  is  $\widehat{\delta}$ -fine. Let us denote it by  $T = \{(\tau_n, [t_{n-1}, t_n])\}_{n=1}^N$ .

We have chosen  $T$  so that  $1/\widehat{p}$  is one of the tags; say  $1/\widehat{p} = \tau_{\widehat{n}}$ . Since  $f$  is nonnegative,

$$f(T) = \sum_{n=1}^N f(\tau_n)(t_n - t_{n-1}) \geq f(\tau_{\widehat{n}})(t_{\widehat{n}} - t_{\widehat{n}-1}) = 2\sigma f(1/\widehat{p}) = \frac{4}{3}.$$

This completes the proof. □

## 8 Equiintegrability and Bounded Variation.

**Proposition 8.1.** *If we consider only positive constants  $\delta$  rather than functions  $\delta$ , then*

$$\sup_{\delta > 0} \frac{\psi_{\delta}(f)}{\delta} \leq \text{Var}(f) \leq \liminf_{\delta \downarrow 0} \frac{\theta_{\delta}(f)}{\delta}$$

for any  $f \in \mathcal{KH}[0, 1]$ , whether  $f$  has bounded variation or not.

PROOF. See [2]. □

**Corollary 8.2.** *With  $\Delta_1$  as in 7.6, we have  $X_{\Delta_1} = BV[0, 1]$ . More generally, suppose  $\Delta = (\delta_1, \delta_2, \delta_3, \dots)$  is a sequence of gauges that converges to 0 at least as fast as the sequence  $(1/n)$ , in the sense that*

$$\sup_{n \in \mathbb{N}} \sup_{t \in [0, 1]} n\delta_n(t) < \infty.$$

Then  $X_{\Delta} \supseteq BV[0, 1]$ .

PROOF. We have  $\psi_{\Delta_1}(f) \leq \text{Var}(f) \leq \theta_{\Delta_1}(f)$  for any function  $f \in \mathcal{KH}[0, 1]$ . The second assertion is immediate from 7.5. □

**Corollary 8.3.**  $X_{(1/\sqrt{1}, 1/\sqrt{2}, 1/\sqrt{3}, \dots)} = \{\text{constant functions}\}$ . *More generally, suppose that  $\Delta = (\delta_1, \delta_2, \delta_3, \dots)$  is a sequence of gauges that converges to 0 more slowly than  $(1/n)$ , in the sense that the numbers  $\mu_n = \inf\{\delta_n(t) : 0 \leq t \leq 1\}$  satisfy  $\mu_n \rightarrow 0$  and  $n\mu_n \rightarrow \infty$ . Then  $X_{\Delta} = \{\text{constant functions}\}$ .*

PROOF. The constant functions are members of every  $X_{\Delta}$ . Conversely, suppose that  $\Delta$  satisfies the condition above and  $f \in X_{\Delta}$ ; we shall show  $f$  is constant. Since  $\mu_n \leq \delta_n(t)$  for all  $t$ , we have  $\{T : T \ll \mu_n\} \subseteq \{T : T \ll \delta_n\}$ . Hence

$$\frac{\theta_{\mu_n}(f)}{\mu_n} \leq \frac{\theta_{\delta_n}(f)}{\mu_n} \leq \frac{\|f\|_{\Delta}}{n\mu_n}.$$

By 8.1 it follows that  $\text{Var}(f) = 0$ , so  $f$  is a constant. □

**Example 8.4.** *Even if a sequence is  $\|\cdot\|_A$ -convergent, uniformly convergent, and  $\|\cdot\|_{\Delta}$ -bounded for some  $\Delta$ , it is not necessarily  $\|\cdot\|_{\Delta}$ -convergent for that  $\Delta$ .*

For instance, let  $f_n(t) = \frac{1}{n} \sin(2\pi nt)$ . Then  $\|f_n\|_{\text{sup}} = 1/n$ ,  $\|f_n\|_A = 1/(\pi n^2)$ , and  $\text{Var}(f_n) = 4$ . □

**Example 8.5** (Unbounded variation). *Let  $p$  be a positive integer. Define  $g_p : [0, 1] \rightarrow \mathbb{R}$  by*

$$g_p(t) = \begin{cases} 0 & \text{if } t \in \{0\} \cup \left\{ \frac{1}{\sqrt{n}} : n \in \mathbb{N} \right\} \cup \left[ \frac{1}{\sqrt{p}}, 1 \right] \\ \sqrt{p+1} - \sqrt{p} & \text{if } t \in \left( \frac{1}{\sqrt{p+1}}, \frac{1}{\sqrt{p}} \right) \\ \sqrt{p+2} - \sqrt{p+1} & \text{if } t \in \left( \frac{1}{\sqrt{p+2}}, \frac{1}{\sqrt{p+1}} \right) \\ \sqrt{p+3} - \sqrt{p+2} & \text{if } t \in \left( \frac{1}{\sqrt{p+3}}, \frac{1}{\sqrt{p+2}} \right) \\ \dots & \dots \end{cases}$$

*Then  $g_p$  has unbounded variation but is Riemann integrable. Moreover, if  $\delta$  is a number in  $(0, 1]$  and  $T \ll \delta$ , then  $|g_p(T) - \int g_p| \leq 3\delta^{1/4}p^{-1/2}$ . Hence*

$$\psi_{\Delta_4}(g_p) \leq 3p^{-1/2} \text{ and } \theta_{\Delta_4}(g_p) = \|g_p\|_{\Delta_4} \leq 6p^{-1/2}$$

*with  $\Delta_4$  as in 7.6. (This example will be used in 8.6.)*

PROOF. The function  $g_p$  is Riemann integrable, since it is bounded and has discontinuities in a set of measure 0. If  $q \geq p$ , then the variation of  $g_p$  on the interval  $\left[ \frac{1}{\sqrt{q}}, 1 \right]$  is  $2(\sqrt{q} - \sqrt{p})$ , which tends to  $\infty$  as  $q \rightarrow \infty$ .

Let any  $\delta \in (0, 1]$  be given. We will use the inequalities

$$0 < \delta \leq \delta^{1/4} \leq \delta^{3/4} \leq 1 \leq \delta^{-1/2}.$$

Let  $q = p + \llbracket \delta^{-1/2} \rrbracket$ , where  $\llbracket x \rrbracket$  denotes the greatest integer less than or equal to  $x$ . Since  $p$  is a positive integer,

$$\delta^{-1/2} \leq p - 1 + \delta^{-1/2} < p + \llbracket \delta^{-1/2} \rrbracket = q.$$

Hence  $q^{-1/2} < \delta^{1/4}$ . Also,  $q \geq p$ , so the previous paragraph's computation of the variation is applicable. Note that

$$\sqrt{q} - \sqrt{p} = \sqrt{p + \llbracket \delta^{-1/2} \rrbracket} - \sqrt{p} < \frac{\llbracket \delta^{-1/2} \rrbracket}{2\sqrt{p}} \leq \frac{\delta^{-1/2}}{2\sqrt{p}} = \frac{1}{2\sqrt{\delta p}}.$$

Now we shall estimate  $|g_p(T) - \int g_p|$ , where  $T = \{(\sigma_i, [s_{i-1}, s_i])\}_{i=1}^m$  is any  $\delta$ -fine tagged division of  $[0, 1]$ . Choose the smallest value of  $k$  that satisfies  $\frac{1}{\sqrt{q}} < s_k$ ; then  $s_{k-1} \leq \frac{1}{\sqrt{q}}$ . Since  $T$  is  $\delta$ -fine, we have  $s_k - \frac{1}{\sqrt{q}} \leq s_k - s_{k-1} < \delta$ .

Now compute

$$\begin{aligned} |g_p(T) - \int g_p| &= \left| \sum_{i=1}^m g_p(\sigma_i)(s_i - s_{i-1}) - \int_0^1 g_p(s) ds \right| \\ &\leq \left| \sum_{i=1}^k g_p(\sigma_i)(s_i - s_{i-1}) - \int_0^{s_k} g_p(s) ds \right| + \left| \sum_{i=k+1}^m g_p(\sigma_i)(s_i - s_{i-1}) - \int_{s_k}^1 g_p(s) ds \right|. \end{aligned}$$

For  $\sum_{i=1}^k - \int_0^{s_k}$  we shall use the fact that  $|g_p(t)| \leq \sqrt{p+1} - \sqrt{p} < \frac{1}{2\sqrt{p}}$  for all  $t \in [0, 1]$ ; hence *a fortiori* for all  $t \in [0, s_k]$ . For  $\sum_{i=k+1}^m - \int_{s_k}^1$  we shall apply the first inequality of 8.1, but on the interval  $[s_k, 1]$  rather than  $[0, 1]$ . Thus we obtain

$$\begin{aligned} |g_p(T) - \int g_p| &\leq 2s_k \cdot \frac{1}{2\sqrt{p}} + \delta \text{Var}(g_p; [s_k, 1]) \\ &\leq 2\left(\frac{1}{\sqrt{q}} + \delta\right) \cdot \frac{1}{2\sqrt{p}} + \delta \text{Var}\left(g_p; \left[\frac{1}{\sqrt{q}}, 1\right]\right) \\ &= \frac{1}{\sqrt{p}} \left(q^{-1/2} + \delta\right) + 2\delta(\sqrt{q} - \sqrt{p}) \\ &< \frac{1}{\sqrt{p}} \left(\delta^{1/4} + \delta^{1/4}\right) + 2\delta^{3/4} \cdot \frac{1}{2\sqrt{\delta p}} = \frac{3\delta^{1/4}}{\sqrt{p}}. \end{aligned}$$

Finally, we have  $\theta \leq 2\psi$  in 5.2, and  $\theta_{\Delta_4}(g_p) = \|g_p\|_{\Delta_4}$  since  $g_p(0) = 0$ .  $\square$

**Corollary 8.6.** *As noted in 7.6, we have the set inclusion  $X_{\Delta_1} \subseteq X_{\Delta_4}$  without regard to topologies. However,*

- (i)  $X_{\Delta_1}$  is not a closed subset of the Banach space  $(X_{\Delta_4}, \|\cdot\|_{\Delta_4})$ , and
- (ii) the  $\|\cdot\|_{\Delta_1}$ -topology is strictly stronger than the relative topology induced on  $X_{\Delta_1}$  by  $\|\cdot\|_{\Delta_4}$ .

PROOF. The space  $X_{\Delta_1}$  is just  $BV[0, 1]$ , as we noted in 8.2. Define functions  $g_1, g_2, g_3, \dots$  with unbounded variation as in 8.5. Define  $f_p = g_1 - g_p$ ; then  $f_p$  is a step function, hence an element of  $BV[0, 1]$ . We have  $\|f_p - g_1\|_{\Delta_4} = \|g_p\|_{\Delta_4} \leq \frac{6}{\sqrt{p}} \rightarrow 0$  as  $p \rightarrow \infty$ , so  $g_1$  is in the  $\|\cdot\|_{\Delta_4}$ -closure of  $BV[0, 1]$ . Thus  $(X_{\Delta_1}, \|\cdot\|_{\Delta_4})$  is not complete.

Results 7.2 and 7.5 showed that  $(X_{\Delta_1}, \|\cdot\|_{\Delta_1})$  is complete and that  $\|\cdot\|_{\Delta_1}$  is stronger than or equivalent to  $\|\cdot\|_{\Delta_4}$  on the set  $X_{\Delta_1}$ . If the two norms were equivalent, they would also be uniformly equivalent, since the identity map is linear; hence  $(X_{\Delta_1}, \|\cdot\|_{\Delta_4})$  would be complete. But that would contradict the result of the preceding paragraph.  $\square$

## 9 Topologizing $\mathcal{KH}[0, 1]$ .

Inductive limits are often used in functional analysis to topologize a linear space that is represented as a union of subspaces,  $V = \bigcup_{\alpha} V_{\alpha}$ , if those subspaces  $V_{\alpha}$  already have nice topologies — e.g., if each is a complete metrizable locally convex space. Then the *locally convex inductive limit* is the strongest locally convex topology on  $V$  that makes all the inclusions  $i_{\alpha} : V_{\alpha} \hookrightarrow V$  continuous. Basic properties of such a topology can be found in functional analysis books — for instance:

- such a strongest topology does indeed exist;
- a linear map  $g : V \rightarrow Z$ , into another locally convex space, is continuous if and only if each of the restrictions  $g \circ i_{\alpha} : V_{\alpha} \rightarrow Z$  is continuous;
- if each  $V_{\alpha}$  is barreled or bornological, then  $V$  is too.

(For analysts unfamiliar with nonmetrizable topological vector spaces, we remark that barreledness — not completeness — is the crucial assumption for Uniform Boundedness and Closed Graph properties; for instance, see §27.26–27.27 of [12].)

Additional properties, including simple characterizations of bounded sets and convergent sequences, can be obtained if

the  $V_{\alpha}$ 's form an increasing sequence, and each space's topology is the relative topology induced on it as a subset of the next space.

The resulting topology on  $V$  is then called a *strict* inductive limit. That additional assumption is satisfied in the most useful applications of inductive limits, but it is not satisfied in the topologies on  $\mathcal{KH}[0, 1]$  and  $KH[0, 1]$  discussed below.

**Observations 9.1.** Let  $\lambda$  denote the topology obtained on  $\mathcal{KH}[0, 1] = \bigcup_{\Delta} X_{\Delta}$  as the inductive limit of the Banach spaces  $X_{\Delta}$ . Then:

- (a)  $\lambda$  is barreled and bornological.
- (b) Any equiintegrable, pointwise bounded set is  $\lambda$ -bounded.
- (c)  $\lambda$  is stronger than the topology of pointwise convergence on  $[0, 1]$ , and stronger than the topology determined by the Alexiewicz seminorm  $\|\cdot\|_A$ .
- (d)  $\lambda$  is not a strict inductive limit of the  $X_{\Delta}$ 's. (This follows from either of 7.7 or 8.6.)

**Open questions 9.2.** Let  $(f_n)$  be a sequence in  $\mathcal{KH}[0,1]$ . Consider the following five conditions:

- (a)  $f_n \rightarrow 0$  pointwise and  $\|f_n\|_A \rightarrow 0$
- (b)  $f_n \rightarrow 0$  in the inductive limit topology  $\lambda$
- (c)  $(f_n)$  is bounded in the inductive limit topology  $\lambda$
- (d)  $\|f_n\|_\Delta \rightarrow 0$  for some  $\Delta$
- (e)  $\sup_n \|f_n\|_\Delta < \infty$  for some  $\Delta$ ; that is,  $(f_n)$  is equiintegrable

Then the following relations are known:

$$\begin{array}{ccc} & (d) \Rightarrow (e) & \\ & \Downarrow & \Downarrow \\ (a) \Leftarrow (b) & \Rightarrow (c) & \end{array}$$

Also, it is easy to see that either of (c), (e) does not imply either of (d), (b): just take all the  $f_n$ 's to be the same. But we have not yet been able to determine other relations in the diagram.

A particularly interesting question is whether (a) implies (e). The answer is yes under the additional assumption that each  $f_n$  is Lebesgue integrable; that is shown in [13]. We would guess that the answer is no in general, since a theorem of Gordon (see [6], or 8.12 in [4]) states that, if we assume pointwise convergence, then integral convergence is equivalent to a condition that is implied by equiintegrability. But is Gordon's condition strictly weaker than equiintegrability? It certainly appears so, but appearances can be deceptive. No example is given in Gordon's paper, nor anywhere else in the literature that we are aware of. The example given for this purpose in 8.14 of [4] is incorrect. Gordon's condition has five nested quantifiers; unraveling them to produce an example will not be easy.

**Open questions 9.3.** We have not yet been able to answer any of the following questions. Some of them are quite simple to state, so we are surprised that the answers apparently have not already appeared in the literature.

- (a) If  $(f_n)$  is equiintegrable and  $\|f_n\|_A \rightarrow 0$ , does it follow that  $f_n \rightarrow 0$  pointwise almost everywhere? (This is related to 3.4.)
- (b) Is the inductive limit topology  $\lambda$  metrizable? complete?

- (c) What classes of functions are dense in  $X_\Delta$ ? in  $(\mathcal{KH}[0, 1], \lambda)$ ? What are the continuous linear functionals on those spaces?
- (d) Can some other interesting classes of functions, besides  $BV[0, 1]$  and  $\{\text{constants}\}$ , be identified as  $X_\Delta$ 's, in a fashion analogous to 8.2 and 8.3?

**Remarks 9.4** (Comparison with other known topologies). Our results should be compared with at least a couple of other topologies in the literature, though both are on  $KH$  (the space of equivalence classes) rather than  $\mathcal{KH}$  (the space of functions). Those topologies are investigated partly in terms of the indefinite integrals  $F(x) = \int_0^x f(t)dt$ , rather than the KH integrable functions  $f$ . Each  $F$  is continuous, hence the indefinite integrals form a linear subspace  $P$  of  $C[0, 1]$ .

The investigations of Thomson [15] were motivated by the  $L^1$  norm, which can be reformulated as the variation of the indefinite integral — i.e.,  $\|f\|_1 = \text{Var}(F)$ . (The studies of the present paper are closer to the variation of  $f$  itself; see 8.1, and the definition of  $\theta_\delta$  in 5.1.) Thomson uses a generalized notion of variation, and the fact that for each  $F \in P$  there exists a sequence  $(E_n)$  of closed sets, with union equal to  $[0, 1]$ , such that  $\text{Var}(F, E_n)$  is finite for each  $n$ . Each sequence  $(E_n)$  gives rise to a Fréchet topology on a subspace of  $P$ ; then  $P$  can be topologized as the inductive limit of those Fréchet spaces. The resulting inductive limit topology turns out to be identical to the Alexiewicz norm topology.

Kurzweil [8] investigates topologies  $\tau$  on the space of indefinite integrals having the property that

if  $f_j \rightarrow f$  pointwise and  $(f_j)$  is equiintegrable, then the corresponding indefinite integrals satisfy  $F_j \xrightarrow{\tau} F$ .

(In other words, the sequential convergence determined by  $\tau$  is weaker than or equal to pointwise, equiintegrable convergence.) Kurzweil shows that

the Alexiewicz norm topology is the strongest locally convex topology on  $P$  that satisfies this convergence requirement.

His proof is by way of a convex vector bornology on  $P$ , somewhat more complicated than the bornology of equiintegrable sets studied in the present paper. He also shows that there exists a complete topological vector space topology on  $P$  satisfying the convergence requirement given above, but that there does not exist one that is both complete and locally convex. All of his constructions use sequences of gauges, analogous to the  $\Delta$ 's in this paper (but with different notations throughout).

## References

- [1] J. Alan Alewine, *An inductive limit topology on the Denjoy space*, Ph.D. thesis, Vanderbilt University, 2003.
- [2] J. Alan Alewine and Eric Schechter, *Rates of uniform convergence for Riemann integrals*, in preparation.
- [3] A. Alexiewicz, *Linear functionals on Denjoy-integrable functions*, *Colloquium Math.*, **1** (1948), 289–293.
- [4] Robert G. Bartle, *A modern theory of integration*, *Graduate Studies in Mathematics* **32**, American Mathematical Society, Providence, RI, 2001.
- [5] Russell A. Gordon, *The integrals of Lebesgue, Denjoy, Perron, and Henstock*, *Graduate Studies in Mathematics*, **4**, American Mathematical Society, Providence, RI, 1994.
- [6] Russell A. Gordon, *An iterated limits theorem applied to the Henstock integral*, *Real Anal. Exchange*, **21(2)** (1995/96), 774–781.
- [7] Henri Hogbe-Nlend, *Bornologies and functional analysis*, North-Holland Publishing Co., Amsterdam, 1977.
- [8] Jaroslav Kurzweil, *Henstock-Kurzweil integration: its relation to topological vector spaces*, *Series in Real Analysis*, **7**, World Scientific Publishing Co. Inc., River Edge, NJ, 2000.
- [9] Peng Yee Lee, *Lanzhou lectures on Henstock integration*, *Series in Real Analysis*, **2** World Scientific Publishing Co. Inc., Teaneck, NJ, 1989.
- [10] Peng Yee Lee and Rudolf Výborný, *Integral: an easy approach after Kurzweil and Henstock*, *Australian Mathematical Society Lecture Series*, **14**, Cambridge University Press, Cambridge, 2000.
- [11] Balmohan Vishnu Limaye, *Functional analysis*, John Wiley & Sons Inc., New York, 1981, A Halsted Press Book.
- [12] Eric Schechter, *Handbook of analysis and its foundations*, Academic Press Inc., San Diego, CA, 1997.
- [13] Štefan Schwabik and Ivo Vrkoč, *On Kurzweil-Henstock equiintegrable sequences*, *Mathematica Bohemica*, **121(2)** (1996), 189–207.

- [14] Charles Swartz, *Norm convergence and uniform integrability for the Henstock-Kurzweil integral*, Real Anal. Exchange, **24(1)** (1998/99), 423–426.
- [15] Brian S. Thomson, *The space of Denjoy-Perron integrable functions*, Real Anal. Exchange, **25(2)** (1999/00), 711–726.

