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WEIGHTED INEQUALITIES FOR MULTIVARIABLE DYADIC PARAPRODUCTS

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Abstract

Using Wilson's Haar basis in $\mathbb{R}^n$, which is different than the usual tensor product Haar functions, we define its associated dyadic paraproduct in $\mathbb{R}^n$. We can then extend "trivially" Beznosova's Bellman function proof of the linear bound in $L^2(w)$ with respect to $[w]_{A_2}$ for the 1-dimensional dyadic paraproduct. Here trivial means that each piece of the argument that had a Bellman function proof has an n-dimensional counterpart that holds with the same Bellman function. The lemma that allows for this painless extension we call the good Bellman function Lemma. Furthermore the argument allows to obtain dimensionless bounds in the anisotropic case.

1. Introduction and main results

The name *Paraproduct* was coined by Bony, in 1981 (see [**2**]), who used paraproducts to linearize the problem in the study of singularities of solutions of semilinear partial differential equations. After his work, the paraproducts have played an important role in harmonic analysis because they are examples of singular integral operators which are not translation-invariant. Also, every singular integral operator which is bounded on L^2 decomposes into a paraproduct, an adjoint of a paraproduct, and an almost convolution operator. Moreover they arise as building blocks for more general operators such as multipliers.

For the locally integrable functions b and f, the dyadic paraproduct is defined by

$$\pi_b f := \sum_{I\in\mathcal{D}} \langle b, h_I\rangle \langle f\rangle_I h_I,$$

on the real line. Here the $\mathcal{D}$ denotes the collection of all dyadic intervals. $\{h_I\}_{I\in\mathcal{D}}$ is the Haar basis in $L^2_{\mathbb{R}}$, $\langle\cdot,\cdot\rangle$ stands for the standard inner

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product in $L^2_{\mathbb{R}}$, and $\langle\cdot\rangle$ denotes the average over the interval I. It is now well known fact that the dyadic paraproduct is bounded on L^p if $b \in BMO^d$ (see [**13**]). Thus, after we fix b in BMO^d, we consider $\pi_b f$ as a linear operator acting on f.

We say the positive almost everywhere and locally integrable function w, a weight, satisfies the A_p condition if:

$$[w]_{A_p} := \sup_I \langle w\rangle_I \langle w^{-1/(p-1)}\rangle_I^{p-1} < \infty, \tag{1.1}$$

where the supremum is taken over all intervals. The class A_p was first presented in [**11**]. The Hardy-Littlewood maximal operator is bounded on $L^p(w)$ for $1 < p < \infty$ if and only if the weight w belongs to the class A_p. If we take the supremum over all dyadic intervals in (1.1), then we call it the dyadic A_p-characteristic and denote it by $[w]_{A_p^d}$. Beznosova proved in [**1**] that the following linear estimate holds for the dyadic paraproduct in $L^2(w)$.

Theorem 1.1 (O. Beznosova)**.** *The norm of dyadic paraproduct on the weighted Lebesgue space $L^2_{\mathbb{R}}(w)$ is bounded from above by a constant multiple of the product of the A_2^d-characteristic of the weight w and the BMO^d norm of b, that is*

$$\|\pi_b f\|_{L^2(w)} \le C[w]_{A_2^d}\|b\|_{BMO^d}\|f\|_{L^2(w)}.$$

In fact, the linear bound in $L^2_{\mathbb{R}^n}(w)$ of the n-dimensional dyadic paraproducts are recovered in [**9**], [**5**] using different methods. However, in this paper, to prove the linear bound of the dyadic paraproduct in $L^2_{\mathbb{R}^n}(w)$ we use the Bellman function arguments as in [**1**].

One of the main purposes in this paper is an estimate for the n-dimensional analog of the dyadic paraproduct and to establish a linear bound with $[w]_{A_2^d}$ and $\|b\|_{BMO^d}$. Thus, throughout the paper, we will be concerned with a class of weights, A_p^d, on $\mathbb{R}^n$. If

$$[w]_{A_p^d} := \sup_{Q\in\mathcal{D}^n} \langle w\rangle_Q \langle w^{-1/(p-1)}\rangle_Q^{p-1} < \infty,$$

then we say the weight w belongs to the class of A_p^d weights. Here $\mathcal{D}^n$ denotes the collection of all dyadic cubes in $\mathbb{R}^n$.

In order to extend Theorem 1.1 to the multivariable setting in the spirit of [**1**], we are using Bellman function arguments. This allows us to establish the dimension free estimates in terms of anisotropic weight characteristic. Thus we need to consider the class of an anisotropic A_2-weights and the class of anisotropic BMO functions which are defined as follows.

Definition 1.2. A locally integrable and positive almost everywhere function w on the space $\mathbb{R}^n$ belongs to class of A_p^R weights, $1 < p < \infty$ if

$$[w]_{A_p^R} := \sup_R \langle w\rangle_R \langle w^{-1/(p-1)}\rangle_R^{p-1} < \infty,$$

where the supremum is taken over all rectangles $R \subset \mathbb{R}^n$ with sides parallel to the coordinate axes.

Definition 1.3. A locally integrable function on $\mathbb{R}^n$ belongs to BMO^R if

$$\|b\|_{BMO^R} := \sup_R \frac{1}{|R|}\int_R |b(x) - \langle b\rangle_R|\, dx < \infty,$$

where the supremum runs over all rectangles $R \subset \mathbb{R}^n$ with sides parallel to the coordinate axes.

Since a cube is a particular case of a rectangle, it is easy to observe that $\|b\|_{BMO} \leq \|b\|_{BMO^R}$. In [**10**], one can find the example which is in $BMO_{\mathbb{R}^2}$ but not in $BMO^R_{\mathbb{R}^2}$. Thus, $BMO \supset BMO^R$ when $n \geq 2$. It is a well known fact that the weight $|x|^\alpha \in A_2$ if and only if $|\alpha| < n$. Thus, $|x| \in A_2$ in $\mathbb{R}^2$. However, one can see that $\langle |x|\rangle_{R_t}\langle |x|^{-1}\rangle_{R_t}$ behaves like $\log t$, where $R_t = [0,t]\times[0,1]$. Even if $[w]_{A_2} = [w]_{A_2^R}$ in $\mathbb{R}$, we can see that the A_2^R weight class belongs strictly to A_2, when $n \geq 2$. We now state our main results. Here π_b is the dyadic paraproduct associated to Wilson's Haar basis in $\mathbb{R}^n$ and defined in Section 2.

Theorem 1.4. *For $1 < p < \infty$ there exists constants $C(n,p)$ only depending on p and dimension n and C which does not depend on the dimensional constant such that*

$$\|\pi_b\|_{L^p_{\mathbb{R}^n}(w)\to L^p_{\mathbb{R}^n}(w)} \leq C(n,p)[w]_{A_p^d}^{\max\{1,\frac{1}{p-1}\}}\|b\|_{BMO^d_{\mathbb{R}^n}}$$

and for all weights $w \in A_p^d$ and $b \in BMO^d_{\mathbb{R}^n}$, and

$$\|\pi_b\|_{L^2_{\mathbb{R}^n}(w)\to L^2_{\mathbb{R}^n}(w)} \leq C\,[w]_{A_2^R}\|b\|_{BMO^R_{\mathbb{R}^n}}$$

for all weight $w \in A_2^R$ and $b \in BMO^R_{\mathbb{R}^n}$.

Throughout the paper, we denote a constant by C, which may change line by line, and we indicate its dependence on parameters using a parenthesis, for example $C(n,p)$. In Section 2 we discuss some n-dimensional Haar systems and introduce notations. In Section 3 we introduce the multivariable dyadic paraproducts. In Section 4 we introduce certain embedding theorems and weight inequalities which are extended to the several variable setting. In Section 5 we prove the main results which provide the linear bounds for the dyadic paraproduct in $L^2_{\mathbb{R}^n}(w)$, and dimension free estimates. We remark in Section 6 that a similar method recovers known estimates for martingale transform and obtains dimension free estimates as well.

2. Wilson's Haar system in $\mathbb{R}^n$

First of all, we need to introduce the appropriate n-dimensional Haar systems. For any $Q \in \mathcal{D}^n$, we set $\mathcal{D}_1^n(Q) \equiv \{Q' \in \mathcal{D}^n : Q' \subset Q,\, \ell(Q') = \ell(Q)/2\}$, the class of 2^n dyadic sub-cubes of Q, where we denote the side length of cubes by $\ell(Q)$. We will also denote the class of all dyadic sub-cubes of Q by $\mathcal{D}^n(Q)$. Then we can write $\mathcal{D}^n(Q) = \bigcup_{j=0}^{\infty} \mathcal{D}_j^n(Q)$, where $\mathcal{D}_j^n(Q) = \{Q' \in \mathcal{D}^n(Q) : \ell(Q') = \ell(Q)/2^j\}$. We refer to [**18**] for the following lemma.

Lemma 2.1. *Let $Q \in \mathcal{D}^n$. Then, there are $2^n - 1$ pairs of sets $\{(E_{j,Q}^1, E_{j,Q}^2)\}_{j=1}^{2^n-1}$ such that:*

1. *for each j, $\left|E_{j,Q}^1\right| = \left|E_{j,Q}^2\right|$;*
2. *for each j, $E_{j,Q}^1$ and $E_{j,Q}^2$ are non-empty unions of cubes from $\mathcal{D}_1^n(Q)$;*
3. *for each j, $E_{j,Q}^1 \cap E_{j,Q}^2 = \emptyset$;*
4. *for every $j \neq k$, one of the following must hold:*
   - (a) *$E_{j,Q}^1 \cup E_{j,Q}^2$ is entirely contained in either $E_{k,Q}^1$ or $E_{k,Q}^2$;*
   - (b) *$E_{k,Q}^1 \cup E_{k,Q}^2$ is entirely contained in either $E_{j,Q}^1$ or $E_{j,Q}^2$;*
   - (c) *$\left(E_{j,Q}^1 \cup E_{j,Q}^2\right) \cap \left(E_{k,Q}^1 \cup E_{k,Q}^2\right) = \emptyset$.*

We can construct such a set by induction on n. It is clear when $n = 1$. We assume that Lemma 2.1 is true for $n-1$ and let $\widetilde{Q}$ be the $(n-1)$-dimensional cube and $\left\{(E_{j,\widetilde{Q}}^1, E_{j,\widetilde{Q}}^2)\right\}_{j=1}^{2^{n-1}-1}$ be the corresponding pairs of sets for $\widetilde{Q}$. We can get the first pair of sets by $(E_{1,Q}^1, E_{1,Q}^2) := (\widetilde{Q} \times I_-, \widetilde{Q} \times I_+)$ where I is a dyadic interval so that $|I| = \ell(\widetilde{Q})$, and $\widetilde{Q} \times I = Q$. We also have the last $2^n - 2$ pairs of sets as follows.

$$\begin{aligned}
&\{(E_{2j,Q}^1, E_{2j,Q}^2), (E_{2j+1,Q}^1, E_{2j+1,Q}^2)\}_{j=1}^{2^{n-1}-1} \\
&\qquad := \left\{(E_{j,\widetilde{Q}}^1 \times I_-, E_{j,\widetilde{Q}}^2 \times I_-), (E_{j,\widetilde{Q}}^1 \times I_+, E_{j,\widetilde{Q}}^2 \times I_+)\right\}_{j=1}^{2^{n-1}-1}.
\end{aligned}$$

To save space, we denote $E_{j,Q}^1 \cup E_{j,Q}^2$ by $E_{j,Q}$ and, by (1) in Lemma 2.1, we have $|E_{j,Q}| = 2|E_{j,Q}^i|$ for $i = 1, 2$. Note that the sets $E_{j,Q}$ are rectangles. Also note that we assign $E_{1,Q} = Q$, $E_{2,Q} = E^1_{1,Q}$ and $E_{3,Q} = E^2_{1,Q}$ and so on. With such a choice, we have

$$\begin{aligned} Q &= E_{1,Q} = E^1_{1,Q} \cup E^2_{1,Q} = E_{2,Q} \cup E_{3,Q} = (E_{4,Q} \cup E_{6,Q}) \cup (E_{5,Q} \cup E_{7,Q}) \\ &= \cdots = E_{2^{n-1},Q} \cup E_{2^{n-1}+1,Q} \cup \cdots \cup E_{2^n-1,Q} \\ &= E^1_{2^{n-1},Q} \cup E^2_{2^{n-1},Q} \cup E^1_{2^{n-1}+1,Q} \cup E^2_{2^{n-1}+1,Q} \cup \cdots \cup E^1_{2^n-1,Q} \cup E^2_{2^n-1,Q}, \end{aligned}$$

in fact,

$$Q = \bigcup_{j=2^{k-1}}^{2^k-1} E_{j,Q} = \bigcup_{j=2^{k-1}}^{2^k-1} (E^1_{j,Q} \cup E^2_{j,Q}),$$

the sets $E_{j,Q}$ in that range of j's are disjoint, and

$$\mathcal{D}^n_1(Q) = \{E^1_{2^{n-1},Q}, E^2_{2^{n-1},Q}, E^1_{2^{n-1}+1,Q}, E^2_{2^{n-1}+1,Q}, \ldots, E^1_{2^n-1,Q}, E^2_{2^n-1,Q}\}.$$

As a consequence of Lemma 2.1, we can introduce the proper weighted Wilson's Haar system for $L^2_{\mathbb{R}^n}(w)$, $\{h^w_{j,Q}\}_{1\le j\le 2^n-1,\, Q\in\mathcal{D}^n}$, where

$$h^w_{j,Q} := \frac{1}{\sqrt{w(E_{j,Q})}} \left[\sqrt{\frac{w(E^1_{j,Q})}{w(E^2_{j,Q})}} \chi_{E^2_{j,Q}} - \sqrt{\frac{w(E^2_{j,Q})}{w(E^1_{j,Q})}} \chi_{E^1_{j,Q}} \right].$$

When $w \equiv 1$ we denote the Wilson's Haar functions by $h_{j,Q}$. Then, every function $f \in L^2_{\mathbb{R}^n}(w)$ can be written as

$$f = \sum_{Q\in\mathcal{D}^n} \sum_{j=1}^{2^n-1} \langle f, h^w_{j,Q}\rangle_w h^w_{j,Q}.$$

Moreover, $\|f\|^2_{L^2_{\mathbb{R}^n}(w)} = \sum_{Q\in\mathcal{D}^n} \sum_{j=1}^{2^n-1} |\langle f, h^w_{j,Q}\rangle_w|^2$. For all $Q' \in \mathcal{D}^n_1(Q)$, the $h_{j,Q}$'s and $h^w_{j,Q}$'s are constant on Q', we will also denote this constant by $h_{j,Q}(Q')$ and $h^w_{j,Q}(Q')$ respectively. We can obtain the weighted average of f over $E_{j,Q}$ for some $1 \le j \le 2^n - 1$,

$$\langle f\rangle_{E_{j,Q},w} = \sum_{Q'\in\mathcal{D}^n : Q'\supseteq Q} \; \sum_{i : E_{i,Q'} \supsetneq E_{j,Q}} \langle f, h^w_{i,Q'}\rangle h^w_{i,Q'}(E_{j,Q}). \tag{2.1}$$

Furthermore, for $j = 1$, $E_{1,Q} = Q$, we have

$$\langle f\rangle_{E_{1,Q},w} = \langle f\rangle_{Q,w} = \sum_{Q'\in\mathcal{D}^n : Q'\supsetneq Q} \; \sum_{j=1}^{2^n-1} \langle f, h^w_{j,Q'}\rangle_w h^w_{j,Q'}(Q). \tag{2.2}$$

Because it is occasionally more convenient to deal with simpler functions, it might be good to have an orthogonal system in $L^2_{\mathbb{R}^n}(w)$. Let us define

$$(2.3)\quad H^w_{j,Q} := h_{j,Q}\sqrt{|E_{j,Q}|} - A^w_{j,Q}\,\chi_{E_{j,Q}}, \text{ where } A^w_{j,Q} := \frac{\langle w\rangle_{E^2_{j,Q}} - \langle w\rangle_{E^1_{j,Q}}}{2\langle w\rangle_{E_{j,Q}}}.$$

Then, the family of functions $\{w^{1/2}H^w_{j,Q}\}_{j,Q}$ is an orthogonal system for $L^2_{\mathbb{R}^n}$ with norms satisfying the inequality

$$\|w^{1/2}H^w_{j,Q}\|_{L^2_{\mathbb{R}^n}} \le \sqrt{|E_{j,Q}|\langle w\rangle_{E_{j,Q}}}.$$

By Bessel's inequality in $L^2_{\mathbb{R}^n}$ one gets, for all $g \in L^2_{\mathbb{R}^n}$,

$$(2.4)\qquad \sum_{Q\in\mathcal{D}^n}\sum_{j=1}^{2^n-1}\frac{\langle gw^{1/2}, H^w_{j,Q}\rangle^2}{|E_{j,Q}|\langle w\rangle_{E_{j,Q}}} \le \|g\|^2_{L^2_{\mathbb{R}^n}}.$$

As well as in the one dimensional case, one can define

$$(2.5)\qquad \|b\|_{BMO^d_{\mathbb{R}^n}} := \sup_{Q\in\mathcal{D}^n}\frac{1}{|Q|}\int_Q |b(x) - \langle b\rangle_Q|\,dx,$$

for a locally integrable function on $\mathbb{R}^n$. The function b is said to have dyadic bounded mean oscillation if $\|b\|_{BMO^d_{\mathbb{R}^n}} < \infty$, and we denote the class of all locally integrable functions b on $\mathbb{R}^n$ with dyadic bounded mean oscillation by $BMO^d_{\mathbb{R}^n}$. Notably one can replace (2.5) by

$$(2.6)\qquad \|b\|^2_{BMO^d_{\mathbb{R}^n}} = \sup_{Q\in\mathcal{D}^n}\frac{1}{|Q|}\sum_{Q\in\mathcal{D}^n(Q)}\sum_{j=1}^{2^n-1}|\langle b, h_{j,Q}\rangle|^2.$$

In the anisotropic case, it is known that the John-Nirenberg inequality holds for all $b \in BMO^R$ and any rectangle $R \subset \mathbb{R}^n$ (see [**10**]),

$$(2.7)\quad |\{x\in R \mid |b(x) - \langle b\rangle_R| > \lambda\}| \le e^{1+2/e}|R|\exp\left(-\frac{2/e}{\|b\|_{BMO^R}}\lambda\right), \quad \lambda > 0.$$

Note that the John-Nirenberg inequality is dimensionless in the anisotropic case. As an easy consequence of (2.7), we have a self improving property for the anisotropic BMO class. For any rectangle $R \in \mathbb{R}^n$, there exists a constant $C(p)$ independent of the dimension n such that

$$(2.8)\qquad \left(\frac{1}{|R|}\int_R |b(x) - \langle b\rangle_R|^p\,dx\right)^{1/p} \le C(p)\|b\|_{BMO^R}.$$

Using the self improving property (2.8), we have, for $i = 1, \dots, 2^n - 1$, $Q' \in \mathcal{D}^n$

$$\frac{1}{|E_{i,Q'}|} \sum_{Q \in \mathcal{D}^n(Q')} \sum_{j : E_{j,Q} \subseteq E_{i,Q'}} \langle b, h_{j,Q} \rangle^2 \leq C \|b\|^2_{BMO^R}. \tag{2.9}$$

3. The multivariable dyadic paraproduct associated with Wilson's Haar system

We now define the multivariable dyadic paraproduct. It is well known fact that the product of two square integrable functions can be written as the sum of two dyadic paraproducts and a diagonal term in a single variable case. Moreover, the diagonal term is the adjoint of one dyadic paraproduct i.e. for all $f, g \in L^2_{\mathbb{R}}$,

$$fg = \pi_g^*(f) + \pi_g(f) + \pi_f(g). \tag{3.1}$$

Thus, we expect to have analogous decomposition. Let us assume that $f, g \in L^2_{\mathbb{R}^n}$. Expanding f and g in Wilson's Haar system,

$$f = \sum_{Q \in \mathcal{D}^n} \sum_{j=1}^{2^n-1} \langle f, h_{j,Q} \rangle h_{j,Q}, \quad g = \sum_{Q' \in \mathcal{D}^n} \sum_{i=1}^{2^n-1} \langle g, h_{i,Q'} \rangle h_{i,Q'}$$

and multiplying these sums formally we can get

$$fg = \sum_{Q \in \mathcal{D}^n} \sum_{j=1}^{2^n-1} \sum_{Q' \in \mathcal{D}^n} \sum_{i=1}^{2^n-1} \langle f, h_{j,Q} \rangle \langle g, h_{i,Q'} \rangle h_{j,Q}(x) h_{i,Q'} = (I) + (II) + (III).$$

Here, (I) is the diagonal term $Q' = Q,\ j = i$;

$$\begin{aligned} (I) &= \sum_{Q \in \mathcal{D}^n} \sum_{j=1}^{2^n-1} \langle f, h_{j,Q} \rangle \langle g, h_{j,Q} \rangle h^2_{j,Q} \\ &= \sum_{Q \in \mathcal{D}^n} \sum_{j=1}^{2^n-1} \langle f, h_{j,Q} \rangle \langle g, h_{j,Q} \rangle \frac{\chi_{E_{j,Q}}}{|E_{j,Q}|}. \end{aligned} \tag{3.2}$$

The second term (II) is the upper triangle term corresponding to those $Q' \supsetneq Q$, all i, j and $Q' = Q$ so that $E_{i,Q'} \supsetneq E_{j,Q}$.

$$(II) = \sum_{Q \in \mathcal{D}^n} \sum_{j=1}^{2^n-1} \langle f, h_{j,Q} \rangle \langle g \rangle_{E_{j,Q}} h_{j,Q}, \tag{3.3}$$

where we used formula (2.1) for the average of g on $E_{j,Q}$. Similarly, the third term is the lower triangle corresponding to those $Q' \subsetneq Q$, all i, j

and $Q' = Q$ so that $E_{i,Q} \subsetneq E_{j,Q}$.

$$(III) = \sum_{Q \in \mathcal{D}^n} \sum_{j=1}^{2^n-1} \langle g, h_{j,Q} \rangle \langle f \rangle_{E_{j,Q}} h_{j,Q}. \tag{3.4}$$

If we consider the sum (3.2) as an operator acting on f, then we can easily check that (III) is its adjoint operator. We now can define the multivariable dyadic paraproduct by pairing the dyadic *BMO* function. In $\mathbb{R}^n$, the dyadic paraproduct associated with Wilson's Haar system is an operator π_b, given by

$$\pi_b f(x) = \sum_{Q \in \mathcal{D}^n} \sum_{j=1}^{2^n-1} \langle f \rangle_{E_{j,Q}} \langle b, h_{j,Q} \rangle h_{j,Q}(x). \tag{3.5}$$

Note that the constructions of the Haar systems are not unique. One can actually construct different Haar systems [**7**]. Furthermore, the dyadic paraproduct depends on the choice of the Haar functions. Thus, one can establish the different dyadic paraproducts associated with different Haar functions. But the decomposition (3.1) holds for all of them. We will finish this section by including a comparison to the standard tensor product Haar basis in $\mathbb{R}^n$, $\{h^s_{\sigma,Q}\}$, with Wilson's Haar basis introduced in Section 2.2 and associated paraproducts. Let us denote the Haar function associated with a dyadic interval $I \in \mathcal{D}$ by $h^0_I = |I|^{-1/2}(\chi_{I_+} - \chi_{I_-})$ and normalized characteristic functions $h^1_I = |I|^{-1/2}\chi_I$. Here 0 stands for mean value zero and 1 for the indicator. Also we consider a set of signatures $\Sigma = \{0,1\}^{\{1,\ldots,n\}} \setminus \{(1,\ldots,1)\}$ which contains $2^n - 1$ signatures. These are all n-tuples with entries 0 and 1, but excluding n-tuple whose entries are all 1. Then, for each dyadic cube $Q = I_1 \times \cdots \times I_n$, one can get the standard tensor product Haar basis in $\mathbb{R}^n$ by

$$h^s_{\sigma,Q}(x_1,\ldots,x_n) = h^{\sigma_1}_{I_1}(x_1) \times \cdots \times h^{\sigma_n}_{I_n}(x_n),$$

where $\sigma = (\sigma_1,\ldots,\sigma_n) \in \Sigma$. Notice that all $h^s_{\sigma,Q}$ are supported on Q. In this case, we have the paraproduct associated to the standard tensor product Haar basis:

$$\pi^s_b f(x) = \sum_{Q \in \mathcal{D}^n} \langle f \rangle_Q \sum_{\sigma \in \Sigma} \langle b, h^s_{\sigma,Q} \rangle h^s_{\sigma,Q}(x). \tag{3.6}$$

Observe that, for each dyadic cube $Q \in \mathcal{D}^n$,

$$\mathcal{W}(Q) = \operatorname{span}\{h^s_{\sigma,Q}\}_{\sigma \in \Sigma} = \operatorname{span}\{h_{j,Q}\}_{j=1}^{2^n-1}. \tag{3.7}$$

Hence

$$\mathrm{Proj}_{\mathcal{W}(Q)}\, b = \sum_{\sigma\in\Sigma} \langle b, h^s_{\sigma,Q}\rangle h^s_{\sigma,Q} = \sum_{j=1}^{2^n-1} \langle b, h_{j,Q}\rangle h_{j,Q}.$$

Changing the basis, we can see that two multivariable paraproducts, (3.5) and (3.6), are in general different, that is

$$\begin{aligned}\pi^s_b f(x) &= \sum_{Q\in\mathcal{D}^n} \langle f\rangle_Q \sum_{\sigma\in\Sigma}\langle b, h^s_{\sigma,Q}\rangle h^s_{\sigma,Q} = \sum_{Q\in\mathcal{D}^n} \langle f\rangle_Q \sum_{j=1}^{2^n-1} \langle b, h_{j,Q}\rangle h_{j,Q} \\ &\neq \sum_{Q\in\mathcal{D}^n} \sum_{j=1}^{2^n-1} \langle f\rangle_{E_{j,Q}} \langle b, h_{j,Q}\rangle h_{j,Q} = \pi_b f(x).\end{aligned}$$

4. Embedding theorems and weighted inequalities in $\mathbb{R}^n$

In general, once we have a Bellman function proof for a certain property in $\mathbb{R}$ then we can extend a property into $\mathbb{R}^n$ with the same Bellman function. This process is essentially trivial when we use the Haar system in $\mathbb{R}^n$ introduced in Section 2, and it allow us to do the “induction in scales argument” at once, instead of once per each $j = 1, \dots, 2^n - 1$, which then introduces a dimensional constant of order 2^n in the estimates. We present this as the lemma named the *Good Bellman Function Lemma*.

Lemma 4.1 (Good Bellman Function Lemma)**.** *Suppose there is a Bellman function, $B(X,Y)$ defined on domain $\mathfrak{D}$, that has the size property:*

$$0 \le B(X,Y) \le A\, b(X,Y), \tag{4.1}$$

and the convexity property:

$$B(X,Y) - \frac{B(X^1,Y^1)+B(X^2,Y^2)}{2} \ge C(X,Y,X^1,Y^1,X^2,Y^2,M), \tag{4.2}$$

for (X,Y) and $(X^{1,2},Y^{1,2}) \in \mathfrak{D}$, where $2X = X^1 + X^2$, and $Y = \frac{Y^1+Y^2}{2} + M$. Furthermore, suppose the given Bellman function proves a certain dyadic property in $\mathbb{R}$, that is, for all dyadic interval $I \in \mathcal{D}$,

$$\sum_{J\in\mathcal{D}(I)} C(X_J, Y_J, X^1_J, Y^1_J, X^2_J, Y^2_J, M_J) \le |I|\, A\, b(X_I, Y_I). \tag{4.3}$$

Then, the extended property (4.3) *to* $\mathbb{R}^n$*, that is*

$$(4.4)\qquad \sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}} C(X_{E_{j,Q}},Y_{E_{j,Q}},X_{E^1_{j,Q}},Y_{E^1_{j,Q}},X_{E^2_{j,Q}},Y_{E^2_{j,Q}},M_{E_{j,Q}}) \le |E_{i,Q'}|\,A\,b(X_{E_{i,Q'}},Y_{E_{i,Q'}}),$$

is proved by the same Bellman function, B*, and checking* $(X_{E_{j,Q}},Y_{E_{j,Q}})$ *and* $(X_{E^{1,2}_{j,Q}},Y_{E^{1,2}_{j,Q}})$ *belong to* $\mathfrak{D}$*, where*

$$\big(X_{E_{j,Q}},Y_{E_{j,Q}}\big)=\left(\frac{X_{E^1_{j,Q}}+X_{E^2_{j,Q}}}{2},M_{E_{i,Q}}+\frac{Y_{E^1_{j,Q}}+Y_{E^2_{j,Q}}}{2}\right).$$

Note that the variables in (4.2), X, $X^{1,2}$, Y, $Y^{1,2}$, and M can be considered as arbitrary tuples. The number of tuples depend on the given Bellman function.

Proof: Without loss of generality we assume that $i=1$. Let us assume that $(X_{E_{j,Q}},Y_{E_{j,Q}})$, and $(X_{E^{1,2}_{j,Q}},Y_{E^{1,2}_{j,Q}})$ are in the domain $\mathfrak{D}$, where

$$\big(X_{E_{j,Q}},Y_{E_{j,Q}}\big)=\left(\frac{X_{E^1_{j,Q}}+X_{E^2_{j,Q}}}{2},M_{E_{j,Q}}+\frac{Y_{E^1_{j,Q}}+Y_{E^2_{j,Q}}}{2}\right),$$

for all $Q\in\mathcal{D}^n(Q')$ and $j=1,\dots,2^n-1$. Then, using the size condition (4.1) and the convexity condition (4.2) we have, for a fixed dyadic cube Q' and index i,

$$\begin{aligned}
A|E_{1,Q'}|b(X_{E_{1,Q'}},Y_{E_{1,Q'}}) &\ge |E_{1,Q'}|B(X_{E_{1,Q'}},Y_{E_{1,Q'}})\\
&\ge \frac{|E_{1,Q'}|}{2}\sum_{l=1}^{2}B(X_{E^l_{1,Q'}},Y_{E^l_{1,Q'}})\\
&\quad + C(X_{E_{1,Q'}},Y_{E_{1,Q'}},X_{E^1_{1,Q'}},Y_{E^1_{1,Q'}},X_{E^2_{1,Q'}},Y_{E^2_{1,Q'}},M_{E_{1,Q'}})\\
&=\sum_{j=2}^{3}|E_{j,Q'}|B(X_{E_{j,Q'}},Y_{E_{j,Q'}})\\
&\quad + C(X_{E_{j,Q'}},Y_{E_{j,Q'}},X_{E^1_{i,Q'}},Y_{E^1_{i,Q'}},X_{E^2_{i,Q'}},Y_{E^2_{i,Q'}},M_{E_{i,Q'}}).
\end{aligned}$$

If we iterate this process $n-1$ times more, we get:

$$
\begin{aligned}
&A|Q'|b(X_{Q'},Y_{Q'})\\
&\quad\geq \sum_{j=2^{n-1}}^{2^n-1}\sum_{l=1}^{2}|E^l_{j,Q'}|B(X_{E^l_{j,Q'}},Y_{E^l_{j,Q'}})\\
&\qquad+\sum_{j=1}^{2^n-1}C(X_{E_{j,Q'}},Y_{E_{j,Q'}},X_{E^1_{j,Q'}},Y_{E^1_{j,Q'}},X_{E^2_{j,Q'}},Y_{E^2_{j,Q'}},M_{E_{j,Q'}}).
\end{aligned}
$$

Due to our construction of the Haar system, for all $j=2^{n-1},2^{n-1}+1,\ldots,2^n-1$, and $l=1,2$, $E^l_{j,Q'}$'s are mutually disjoint and $|E^l_{j,Q'}|=|Q'|/2^n$ i.e. $\{E^1_{j,Q'},E^2_{j,Q'}\}_{j=2^n-1}^{2^{n-1}}$ is the set $\mathcal{D}^n_1(Q')$ of first-generation dyadic sub-cubes of Q'. Thus,

$$
\begin{aligned}
&A|Q'|b(X_{Q'},Y_{Q'})\geq |Q'|B(X_{Q'},Y_{Q'})\\
&\quad\geq \sum_{k=1}^{2^n}|Q'_k|B(X_{Q'_k},Y_{Q'_k})\\
&\qquad+\sum_{j=1}^{2^n-1}C(X_{E_{j,Q'}},Y_{E_{j,Q'}},X_{E^1_{j,Q'}},Y_{E^1_{j,Q'}},X_{E^2_{j,Q'}},Y_{E^2_{j,Q'}},M_{E_{j,Q'}}),
\end{aligned}
$$

where Q'_k's are enumerations of 2^n dyadic sub-cubes of Q'. Iterating this procedure and using the fact $B\geq 0$ yields that

$$
\begin{aligned}
\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j=1}^{2^n-1}C(X_{E_{j,Q}},Y_{E_{j,Q}},X_{E^1_{j,Q}},Y_{E^1_{j,Q}},X_{E^2_{j,Q}},Y_{E^2_{j,Q}},M_{E_{j,Q}})&\\
\leq A|Q'|b(X_{Q'},Y_{E_{Q'}})&,
\end{aligned}
$$

which completes the proof. □

We now state several multivariable versions of embedding theorems and weight inequalities, that we will need to prove our main theorems. One can find the proof of Lemma 4.2 in [**12**].

Lemma 4.2. *The following function*

$$B(F,f,u,Y)=4A\left(F-\frac{f^2}{u+Y}\right)$$

is defined on domain $\mathfrak{D}$ *which is given by*

$$\mathfrak{D}=\big\{(F,f,u,Y)\in\mathbb{R}^4\mid F,f,u,Y>0 \text{ and } f^2\leq Fu,\, Y\leq u\big\},$$

and B satisfies the following size and convexity property in $\mathfrak{D}$: $0 \le B(F,f,u,Y) \le 4AF$, and for all (F,f,u,Y), (F_1,f_1,u_1,Y_1) and $(F_2,f_2,u_2,Y_2) \in \mathfrak{D}$,

$$B(F,f,u,Y) - \frac{B(F_1,f_1,u_1,Y_1)+B(F_2,f_2,u_2,Y_2)}{2} \ge \frac{f^2}{u^2}M,$$

where

$$(F,f,u,Y) = \left(\frac{F_1+F_2}{2}, \frac{f_1+f_2}{2}, \frac{u_1+u_2}{2}, M + \frac{Y_1+Y_2}{2}\right) \text{ and } M \ge 0.$$

Replacing

$$X_{E_{j,Q'}} = \big(\langle f^2\rangle_{E_{j,Q'}}, \langle fw^{1/2}\rangle_{E_{j,Q'}}, \langle w\rangle_{E_{j,Q}}\big),$$
$$Y_{E_{j,Q'}} = \frac{1}{|E_{j,Q'}|}\sum_{Q\in\mathcal{D}^n(Q')}\sum_{i:E_{i,Q}\subseteq E_{j,Q'}} \alpha_{i,Q}u_{j,Q}^2,$$

and

$$M_{E_{j,Q'}} = \frac{1}{|E_{j,Q'}|}\alpha_{j,Q'}u_{j,Q'}^2,$$

in Lemma 4.1 and using Lemma 4.2, we have the following theorem whose one-dimensional version can be found in [**12**].

Theorem 4.3 (Multivariable Version of Weighted Carleson Embedding Theorem)**.** *Let w be a weight and $\{\alpha_{j,Q}\}_{Q,j}$, $Q \in \mathcal{D}^n$, $j = 1,\dots,2^n-1$, be a sequence of nonnegative numbers such that for all dyadic cubes $Q' \in \mathcal{D}^n$ and a positive constant $A > 0$,*

$$\frac{1}{|E_{i,Q'}|}\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}} \alpha_{j,Q}\langle w\rangle_{E_{j,Q}}^2 \le A\langle w\rangle_{E_{i,Q'}}. \tag{4.5}$$

Then for all positive $f \in L^2_{\mathbb{R}^n}$

$$\sum_{Q\in\mathcal{D}^n}\sum_{j=1}^{2^n-1} \alpha_{j,Q}\langle fw^{1/2}\rangle_{E_{j,Q}}^2 \le CA\|f\|_{L^2_{\mathbb{R}^n}}^2 \tag{4.6}$$

holds with some constant $C > 0$.

Similarly, one can prove the following theorem with Lemma 4.1 and the Bellman function appearing in [**15**].

Theorem 4.4 (Multivariable Version of Petermichl's the Bilinear Embedding Theorem)**.** *Let w and v be weights so that $\langle w\rangle_{Q'}\langle v\rangle_{Q'} < A$ and $\{\alpha_{j,Q}\}_{Q,j}$ a sequence of nonnegative numbers such that, for all dyadic* *cubes $Q' \in \mathcal{D}^n$ and $i = 1, \dots, 2^n - 1$, the three inequalities below hold with some constant $A > 0$,*

$$\frac{1}{|E_{i,Q'}|} \sum_{Q \in \mathcal{D}^n(Q')} \sum_{j : E_{j,Q} \subseteq E_{i,Q'}} \frac{\alpha_{j,Q}}{\langle w \rangle_{E_{j,Q}}} \leq A \langle v \rangle_{E_{i,Q'}}$$

$$\frac{1}{|E_{i,Q'}|} \sum_{Q \in \mathcal{D}^n(Q')} \sum_{j : E_{j,Q} \subseteq E_{i,Q'}} \frac{\alpha_{j,Q}}{\langle v \rangle_{E_{j,Q}}} \leq A \langle w \rangle_{E_{i,Q'}}$$

$$\frac{1}{|E_{i,Q'}|} \sum_{Q \in \mathcal{D}^n(Q')} \sum_{j : E_{j,Q} \subseteq E_{i,Q'}} \alpha_{j,Q} \leq A.$$

Then for all $f \in L^2_{\mathbb{R}^n}(w)$ and $g \in L^2_{\mathbb{R}^n}(v)$

$$\sum_{Q \in \mathcal{D}^n(Q')} \sum_{j=1}^{2^n - 1} \alpha_{j,Q} \langle f \rangle_{E_{j,Q}, w} \langle g \rangle_{E_{j,Q}, v} \leq CA \|f\|_{L^2_{\mathbb{R}^n}(w)} \|g\|_{L^2_{\mathbb{R}^n}(v)}$$

holds with some constant $C > 0$.

Changing $\alpha_{j,Q}$, f and g by $\alpha_{j,Q} \langle v \rangle_{E_{j,Q}} \langle w \rangle_{E_{j,Q}} |E_{j,Q}|$, $f w^{-1/2}$ and $g v^{-1/2}$ respectively in Theorem 4.4, we can get the following corollary.

Corollary 4.5 (Multivariable Version of the Bilinear Embedding Theorem)**.** *Let w and v be weights so that $\langle w \rangle_{Q'} \langle v \rangle_{Q'} < A$ and $\{\alpha_{j,Q}\}_{Q,j}$ a sequence of nonnegative numbers such that, for all dyadic cubes $Q' \in \mathcal{D}^n$ and $i = 1, \dots, 2^n - 1$, the three inequalities below hold with some constant $A > 0$,*

$$\frac{1}{|E_{i,Q'}|} \sum_{Q \in \mathcal{D}^n(Q')} \sum_{j : E_{j,Q} \subseteq E_{i,Q'}} \alpha_{j,Q} \langle v \rangle_{E_{j,Q}} |E_{j,Q}| \leq A \langle v \rangle_{E_{i,Q'}}$$

$$\frac{1}{|E_{i,Q'}|} \sum_{Q \in \mathcal{D}^n(Q')} \sum_{j : E_{j,Q} \subseteq E_{i,Q'}} \alpha_{j,Q} \langle w \rangle_{E_{j,Q}} |E_{j,Q}| \leq A \langle w \rangle_{E_{i,Q'}}$$

$$\frac{1}{|E_{i,Q'}|} \sum_{Q \in \mathcal{D}^n(Q')} \sum_{j : E_{j,Q} \subseteq E_{i,Q'}} \alpha_{j,Q} \langle w \rangle_{E_{j,Q}} \langle v \rangle_{E_{j,Q}} |E_{j,Q}| \leq A.$$

Then for all $f, g \in L^2_{\mathbb{R}^n}$

$$\sum_{Q \in \mathcal{D}^n(Q')} \sum_{j=1}^{2^n - 1} \alpha_{j,Q} \langle f w^{1/2} \rangle_{E_{j,Q}} \langle g v^{1/2} \rangle_{E_{j,Q}} |E_{j,Q}| \leq CA \|f\|_{L^2_{\mathbb{R}^n}} \|g\|_{L^2_{\mathbb{R}^n}}$$

holds with some constant $C > 0$.

We now state several propositions including the multidimensional analogues to the corresponding one-dimensional results in [**1**] and [**15**] for both regular and anisotropic cases.

One can find the 1-dimensional analogue of the following proposition and the associated Bellman function Lemma in [**1**]. Repeating the proof of Lemma 4.1 with $X_{j,Q'} = \big(\langle w\rangle_{E_{j,Q'}}, \langle w^{-1}\rangle_{E_{j,Q'}}\big)$, $Y_{E_{j,Q'}} = \frac{1}{A|E_{j,Q'}|}\sum_{Q\in\mathcal{D}^n(Q')}\sum_{i:E_{i,Q}\subseteq E_{j,Q'}}\alpha_{i,Q}$, and $M_{j,Q'} = \frac{\alpha_{j,Q'}}{A}$, and the corresponding Bellman function Lemma will return the following proposition.

Proposition 4.6. *Let w be a weight, so that w^{-1} is also a weight. Let $\alpha_{j,Q}$ be a Carleson sequence of nonnegative numbers i.e., there is a constant $A>0$ such that, for all $Q'\in\mathcal{D}^n$ and $i=1,\dots,2^n-1$,*

$$\frac{1}{|E_{i,Q'}|}\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}}\alpha_{j,Q}\le A. \tag{4.7}$$

Then, for all $Q'\in\mathcal{D}^n$ and $i=1,\dots,2^n-1$,

$$\frac{1}{|E_{i,Q'}|}\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}}\frac{\alpha_{j,Q}}{\langle w^{-1}\rangle_{E_{j,Q}}}\le 4A\langle w\rangle_{E_{i,Q'}}, \tag{4.8}$$

and if $w\in A_2^d$ then for any $Q'\in\mathcal{D}^n$ and $i=1,\dots,2^n-1$, we have

$$\frac{1}{|E_{i,Q'}|}\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}}\langle w\rangle_{E_{j,Q}}\alpha_{j,Q}\le 4\cdot 2^{2(n-1)}A[w]_{A_2^d}\langle w\rangle_{E_{i,Q'}}. \tag{4.9}$$

Furthermore, if $w\in A_2^R$ then for any $Q'\in\mathcal{D}^n$ and $i=1,\dots,2^{n-1}$, we have

$$\frac{1}{|E_{i,Q'}|}\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}}\langle w\rangle_{E_{j,Q}}\alpha_{j,Q}\le 4A[w]_{A_2^R}\langle w\rangle_{E_{i,Q'}}. \tag{4.10}$$

Observe that in the case $w\in A_2^R$ then

$$\frac{1}{\langle w^{-1}\rangle_{E_{j,Q}}}\ge\frac{\langle w\rangle_{E_{j,Q}}}{[w]_{A_2^R}}.$$

Now (4.10) follows from (4.8). Observe if $w\in A_2^d$ then

$$\begin{aligned}[w]_{A_2} &\ge \langle w\rangle_Q\langle w^{-1}\rangle_Q\\ &\ge\left(\frac{|E_{j,Q}|}{|Q|}\right)^2\langle w\rangle_{E_{j,Q}}\langle w^{-1}\rangle_{E_{j,Q}} = 2^{-2(n-1)}\langle w\rangle_{E_{j,Q}}\langle w^{-1}\rangle_{E_{j,Q}}.\end{aligned}$$

Thus, we can have (4.9) from (4.10). We refer to [**15**] for the corresponding one-dimensional result and the associated Bellman function Lemma to Proposition 4.7. Lemma 4.1 with $X_{E_{j,Q'}} = \big(\langle w\rangle_{E_{j,Q'}}, \langle w^{-1}\rangle_{E_{j,Q}}\big)$,

$Y_{E_{j,Q'}} = M_{E_{j,Q'}} = 0$ and the associated Bellman function Lemma yield Proposition 4.7.

Proposition 4.7. *There exist a positive constant C so that for all weights w such that w^{-1} is also a weight, and for all dyadic cubes $Q' \in \mathcal{D}^n$ and $i = 1, \dots, 2^n - 1$:*

$$\frac{1}{|E_{i,Q'}|}\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}} \frac{\left(\langle w\rangle_{E^1_{j,Q}} - \langle w\rangle_{E^2_{j,Q}}\right)^2}{\langle w\rangle^3_{E_{j,Q}}}|E_{j,Q}| \leq C\langle w^{-1}\rangle_{E_{i,Q'}} \tag{4.11}$$

and, if $w \in A_2^d$, the following inequality holds for all dyadic cubes $Q' \in \mathcal{D}^n$ and $i = 1, \dots, 2^n - 1$:

$$\begin{aligned}\frac{1}{|E_{i,Q'}|}\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}} &\left(\frac{\langle w\rangle_{E^1_{j,Q}} - \langle w\rangle_{E^2_{j,Q}}}{\langle w\rangle_{E_{j,Q}}}\right)^2 |E_{j,Q}|\langle w^{-1}\rangle_{E_{j,Q}} \\ &\leq C2^{2(n-1)}[w]_{A_2^d}\langle w^{-1}\rangle_{E_{i,Q'}}.\end{aligned} \tag{4.12}$$

Moreover, if $w \in A_2^R$, the following inequality holds for all dyadic cubes $Q' \in \mathcal{D}^n$ and $i = 1, \dots, 2^n - 1$:

$$\begin{aligned}\frac{1}{|E_{i,Q'}|}\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}} &\left(\frac{\langle w\rangle_{E^1_{j,Q}} - \langle w\rangle_{E^2_{j,Q}}}{\langle w\rangle_{E_{j,Q}}}\right)^2 |E_{j,Q}|\langle w^{-1}\rangle_{E_{j,Q}} \\ &\leq C[w]_{A_2^R}\langle w^{-1}\rangle_{E_{i,Q'}}.\end{aligned} \tag{4.13}$$

The similar observations in Proposition 4.6 yield (4.12) and (4.13). The following generalizes the result that appeared in [**1**] to the multi-dimensional regular and anisotropic cases. With the same changes in Proposition 4.7 and the associated Bellman function Lemma, one can prove the following.

Proposition 4.8. *There exist a positive constant C so that for all weights w such that w^{-1} is also a weight, and for all dyadic cubes $Q' \in \mathcal{D}^n$ and $i = 1, \dots, 2^n - 1$:*

(4.14)

$$\begin{aligned}\frac{1}{|E_{i,Q'}|}\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}} &\left(\frac{\langle w\rangle_{E^1_{j,Q}} - \langle w\rangle_{E^2_{j,Q}}}{\langle w\rangle_{E_{j,Q}}}\right)^2 |E_{j,Q}|\langle w\rangle^{1/4}_{E_{j,Q}}\langle w^{-1}\rangle^{1/4}_{E_{j,Q}} \\ &\leq C\langle w\rangle^{1/4}_{E_{i,Q'}}\langle w^{-1}\rangle^{1/4}_{E_{i,Q'}}\end{aligned}$$

and, if $w \in A_2^d$*, the following inequality holds for all dyadic cubes* $Q' \in \mathcal{D}^n$*:*

(4.15)
$$\frac{1}{|E_{i,Q'}|} \sum_{Q\in\mathcal{D}^n(Q')} \sum_{j:E_{j,Q}\subseteq E_{i,Q'}} \left(\frac{\langle w\rangle_{E^1_{j,Q}} - \langle w\rangle_{E^2_{j,Q}}}{\langle w\rangle_{E_{j,Q}}}\right)^2 |E_{j,Q}| \langle w\rangle_{E_{j,Q}} \langle w^{-1}\rangle_{E_{j,Q}} \leq C 2^{2(n-1)} [w]_{A_2^d}.$$

Moreover, if $w \in A_2^R$*, the following inequality holds for all dyadic cubes* $Q' \in \mathcal{D}^n$*:*

(4.16)
$$\frac{1}{|E_{i,Q'}|} \sum_{Q\in\mathcal{D}^n(Q')} \sum_{j:E_{j,Q}\subseteq E_{i,Q'}} \left(\frac{\langle w\rangle_{E^1_{j,Q}} - \langle w\rangle_{E^2_{j,Q}}}{\langle w\rangle_{E_{j,Q}}}\right)^2 |E_{j,Q}| \langle w\rangle_{E_{j,Q}} \langle w^{-1}\rangle_{E_{j,Q}} \leq C [w]_{A_2^R}.$$

The single variable version of the following proposition first appeared in [**20**]. In [**14**], one can also find a Bellman function proof of a similar result which can be extended to the doubling measure case.

Proposition 4.9 (Wittwer's sharp version of Buckley's inequality)**.** *There exist a positive constant* C *so that for all weight* $w \in A_2^d$ *and all dyadic cubes* $Q' \in \mathcal{D}^n$ *and* $i = 1, \dots, 2^n - 1$*:*

$$\frac{1}{|E_{i,Q'}|} \sum_{Q\in\mathcal{D}^n(Q')} \sum_{j:E_{j,Q}\subseteq E_{i,Q'}} \left(\frac{\langle w\rangle_{E^1_{j,Q}} - \langle w\rangle_{E^2_{j,Q}}}{\langle w\rangle_{E_{j,Q}}}\right)^2 |E_{j,Q}| \langle w\rangle_{E_{j,Q}} \leq C 2^{2(n-1)} [w]_{A_2^d} \langle w\rangle_{E_{i,Q'}}, \tag{4.17}$$

and for all weight $w \in A_2^R$ *and all dyadic cubes* $Q' \in \mathcal{D}^n$ *and* $i = 1, \dots, 2^n - 1$*:*

$$\frac{1}{|E_{i,Q'}|} \sum_{Q\in\mathcal{D}^n(Q')} \sum_{j:E_{j,Q}\subseteq E_{i,Q'}} \left(\frac{\langle w\rangle_{E^1_{j,Q}} - \langle w\rangle_{E^2_{j,Q}}}{\langle w\rangle_{E_{j,Q}}}\right)^2 |E_{j,Q}| \langle w\rangle_{E_{j,Q}} \leq C [w]_{A_2^R} \langle w\rangle_{E_{i,Q'}}. \tag{4.18}$$

The same choice of $X_{E_{j,Q'}}$, $Y_{E_{j,Q'}}$, and $M_{E_{j,Q'}}$ with Proposition 4.9 prove (4.17). The inequality (4.18) can be seen by using the domain $\mathfrak{D} = \{(u,v) \in \mathbb{R}^2 \mid u, v > 0 \text{ and } 1 \leq uv \leq [w]_{A_2^R}\}$.

5. Proof of Theorem 1.4

We are going to prove Theorem 1.4 only when $p = 2$, and following the one-dimensional proof discovered by Beznosova [**1**]. The sharp extrapolation theorem [**6**] returns immediately the other cases ($1 < p < \infty$). For the case $p = 2$ we use the duality arguments. Precisely, it is sufficient to prove the inequality

$$\langle \pi_b(fw^{-1/2}), gw^{1/2}\rangle \leq C(n)[w]_{A_2^d}\|b\|_{BMO^d_{\mathbb{R}^n}}\|f\|_{L^2_{\mathbb{R}^n}}\|g\|_{L^2_{\mathbb{R}^n}}. \tag{5.1}$$

Proof: Using the orthogonal Haar system (2.3), we can split the left hand side of (5.1) as follows.

$$\begin{aligned}
&\langle \pi_b(fw^{-1/2}), gw^{1/2}\rangle \\
&\quad= \sum_{Q\in\mathcal{D}^n}\sum_{j=1}^{2^n-1}\langle b, h_{j,Q}\rangle\langle fw^{-1/2}\rangle_{E_{j,Q}}\langle gw^{1/2}, h_{j,Q}\rangle \\
&\quad= \sum_{Q\in\mathcal{D}^n}\sum_{j=1}^{2^n-1}\langle b, h_{j,Q}\rangle\langle fw^{-1/2}\rangle_{E_{j,Q}}\langle gw^{1/2}, H^w_{j,Q}\rangle\frac{1}{\sqrt{|E_{j,Q}|}} &(5.2)\\
&\qquad+ \sum_{Q\in\mathcal{D}^n}\sum_{j=1}^{2^n-1}\langle b, h_{j,Q}\rangle\langle fw^{-1/2}\rangle_{E_{j,Q}}\langle gw^{1/2}, A^w_{j,Q}\chi_{E_{j,Q}}\rangle\frac{1}{\sqrt{|E_{j,Q}|}}. &(5.3)
\end{aligned}$$

We are going to prove that both sum (5.2) and (5.3) are bounded with a bound that depends linearly on both $[w]_{A_2^d}$ and $\|b\|_{BMO^d_{\mathbb{R}^n}}$. Using Cauchy-Schwarz inequality, for the term (5.2), we have

$$\begin{aligned}
&\left|\sum_{Q\in\mathcal{D}^n}\sum_{j=1}^{2^n-1}\langle b, h_{j,Q}\rangle\langle fw^{-1/2}\rangle_{E_{j,Q}}\langle gw^{1/2}, H^w_{j,Q}\rangle\frac{1}{\sqrt{|E_{j,Q}|}}\right| \\
&\qquad\leq \left(\sum_{Q\in\mathcal{D}^n}\sum_{j=1}^{2^n-1}\frac{\langle gw^{1/2}, H^w_{j,Q}\rangle^2}{|E_{j,Q}|\langle w\rangle_{E_{j,Q}}}\right)^{1/2} \\
&\qquad\quad\times\left(\sum_{Q\in\mathcal{D}^n}\sum_{j=1}^{2^n-1}\langle b, h_{j,Q}\rangle^2\langle fw^{-1/2}\rangle^2_{E_{j,Q}}\langle w\rangle_{E_{j,Q}}\right)^{1/2} &(5.4)\\
&\qquad\leq \|g\|_{L^2_{\mathbb{R}^n}}\left(\sum_{Q\in\mathcal{D}^n}\sum_{j=1}^{2^n-1}\langle b, h_{j,Q}\rangle^2\langle fw^{-1/2}\rangle^2_{E_{j,Q}}\langle w\rangle_{E_{j,Q}}\right)^{1/2}.
\end{aligned}$$

Here the inequality (5.4) follows from (2.4). We now claim that the sum in (5.4) is bounded by $C[w]^2_{A_2^d}\|b\|^2_{BMO^d_{\mathbb{R}^n}}\|f\|^2_{L^2_{\mathbb{R}^n}}$, which will be provided by the Multivariable Version of the Weighted Carleson Embedding Theorem 4.3. with the embedding condition: For all $Q' \in \mathcal{D}^n$,

$$\frac{1}{|E_{i,Q'}|}\sum_{Q\in\mathcal{D}^n(E_{i,Q'})}\sum_{j:E_{j,Q}\subseteq E_{i,Q}}\langle w^{-1}\rangle^2_{E_{j,Q}}\langle w\rangle_{E_{j,Q}}\langle b,h_{j,Q}\rangle^2 \\ \leq C[w]^2_{A_2^d}\|b\|^2_{BMO^d_{\mathbb{R}^n}}\langle w^{-1}\rangle_{E_{i,Q'}}. \tag{5.5}$$

Since, for all $Q\in\mathcal{D}^n$, $\langle w\rangle_{E_{j,Q}}\langle w^{-1}\rangle_{E_{j,Q}} \leq 2^{2(n-1)}[w]_{A_2^d}$, in order to see the embedding condition (5.5) it is enough to show that

$$\frac{1}{|E_{i,Q'}|}\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}}\langle w^{-1}\rangle_{E_{j,Q}}\langle b,h_{j,Q}\rangle^2 \\ \leq 4\cdot 2^{3(n-1)}[w]^2_{A_2^d}\|b\|^2_{BMO^d_{\mathbb{R}^n}}\langle w^{-1}\rangle_{E_{i,Q'}}. \tag{5.6}$$

Here the inequality (5.6) follows from the fact that $b\in BMO^d_{\mathbb{R}^n}$ and hence the sequence $\{\langle b,h_{j,Q}\rangle^2\}_{j,Q}$ is a Carleson sequence with Carleson constant $2^{n-1}\|b\|^2_{BMO^d_{\mathbb{R}^n}}$, and Proposition 4.6 applied to $\alpha_{j,Q} = \langle b,h_{j,Q}\rangle^2$, $A = 2^{n-1}\|b\|^2_{BMO^d_{\mathbb{R}^n}}$ and $v = w^{-1}$. This estimates finishes the estimate of the term (5.2) with $C(n)\approx 2^{5(n-1)/2}$.

We now turn to estimate of the term (5.3). In order to estimate it, we need to show that

$$\sum_{Q\in\mathcal{D}^n}\sum_{j=1}^{2^n-1}\langle b,h_{j,Q}\rangle\langle fw^{-1/2}\rangle_{E_{j,Q}}\langle gw^{1/2}\rangle_{E_{j,Q}}A^w_{j,Q}\sqrt{|E_{j,Q}|} \\ \leq C[w]_{A_2^d}\|b\|_{BMO^d_{\mathbb{R}^n}}\|f\|_{L^2_{\mathbb{R}^n}}\|g\|_{L^2_{\mathbb{R}^n}}, \tag{5.7}$$

and this is provided by the following three embedding conditions due to the Multivariable Version of the Bilinear Embedding Theorem 4.5: For all $Q'\in\mathcal{D}^n$ and $i=1,\dots,2^n-1$,

$$\frac{1}{|E_{i,Q'}|}\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}}\left|\langle b,h_{j,Q}\rangle A^w_{j,Q}\right|\sqrt{|E_{j,Q}|}\langle w\rangle_{E_{j,Q}}\langle w^{-1}\rangle_{E_{j,Q}} \\ \leq C(n)[w]_{A_2^d}\|b\|_{BMO^d_{\mathbb{R}^n}}, \tag{5.8}$$

$$\frac{1}{|E_{i,Q'}|}\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}}\left|\langle b,h_{j,Q}\rangle A^w_{j,Q}\right|\sqrt{|E_{j,Q}|}\langle w\rangle_{E_{j,Q}} \leq C(n)[w]_{A_2^d}\|b\|_{BMO^d_{\mathbb{R}^n}}\langle w\rangle_{E_{i,Q'}}, \tag{5.9}$$

$$\frac{1}{|E_{i,Q'}|}\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}}\left|\langle b,h_{j,Q}\rangle A^w_{j,Q}\right|\sqrt{|E_{j,Q}|}\langle w^{-1}\rangle_{E_{j,Q}} \leq C(n)[w]_{A_2^d}\|b\|_{BMO^d_{\mathbb{R}^n}}\langle w^{-1}\rangle_{E_{i,Q'}}. \tag{5.10}$$

Proposition 4.8 makes it easy to prove the embedding condition (5.8). Using the Cauchy-Schwarz inequality,

$$\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}}\left|\langle b,h_{j,Q}\rangle A^w_{j,Q}\right|\sqrt{|E_{j,Q}|}\langle w\rangle_{E_{j,Q}}\langle w^{-1}\rangle_{E_{j,Q}} \tag{5.11}$$

$$\leq \left(\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}}\langle b,h_{j,Q}\rangle^2\langle w\rangle_{E_{j,Q}}\langle w^{-1}\rangle_{E_{j,Q}}\right)^{1/2} \times\left(\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}}\left(A^w_{j,Q}\right)^2|E_{j,Q}|\langle w\rangle_{E_{j,Q}}\langle w^{-1}\rangle_{E_{j,Q}}\right)^{1/2} \tag{5.12}$$

$$\leq 2^{n-1}[w]^{1/2}_{A_2^d}\left(\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}}\langle b,h_{j,Q}\rangle^2\right)^{1/2} \times\left(\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}}\left(A^w_{j,Q}\right)^2|E_{j,Q}|\langle w\rangle_{E_{j,Q}}\langle w^{-1}\rangle_{E_{j,Q}}\right)^{1/2} \tag{5.13}$$

$$\leq C2^{2(n-1)}[w]_{A_2^d}\left|E_{i,Q'}\right|^{1/2}\left(\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}}\langle b,h_{j,Q}\rangle^2\right)^{1/2} \tag{5.14}$$

$$\leq C2^{5(n-1)/2}[w]_{A_2^d}\|b\|_{BMO^d_{\mathbb{R}^n}}\left|E_{i,Q'}\right|. \tag{5.15}$$

Here we use (4.15) for the inequality (5.14) and the fact that $b\in BMO^d_{\mathbb{R}^n}$ for the inequality (5.15). We also use the Cauchy-Schwarz inequality for the inequality (5.9). Then

$$\begin{aligned}
&\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}}\left|\langle b,h_{j,Q}\rangle A^w_{j,Q}\right|\sqrt{|E_{j,Q}|}\langle w\rangle_{E_{j,Q}}\\
&\qquad\le\left(\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}}\langle b,h_{j,Q}\rangle^2\langle w\rangle_{E_{j,Q}}\right)^{1/2}\\
&\qquad\quad\times\left(\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}}\left(A^w_{j,Q}\right)^2|E_{j,Q}|\langle w\rangle_{E_{j,Q}}\right)^{1/2}\\
&\qquad\le C2^{5(n-1)/2}\|b\|_{BMO^d_{\mathbb{R}^n}}[w]^{1/2}_{A^d_2}\langle w\rangle^{1/2}_{E_{i,Q'}}\\
&\qquad\quad\times\left(\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}}\left(A^w_{j,Q}\right)^2|E_{j,Q}|\langle w\rangle_{E_{j,Q}}\right)^{1/2} &(5.16)\\
&\qquad\le C2^{7(n-1)/2}\|b\|_{BMO^d_{\mathbb{R}^n}}[w]_{A^d_2}\langle w\rangle_{E_{i,Q'}}. &(5.17)
\end{aligned}$$

Inequality (5.16) and (5.17) follow by (4.9) and Proposition 4.9 respectively. Similarly, we can establish inequality (5.10) with Proposition 4.7. To sum up, we can establish the inequality (5.1) with a constant $C(n)\approx 2^{7(n-1)/2}$. Furthermore, if we replace $[w]_{A^d_2}$ by $[w]_{A^R_2}$ and $\|b\|_{BMO^d}$ by $\|b\|_{BMO^R}$ then we can establish proof of the dimension free estimate in Theorem 1.4. □

6. Final remarks

Remark 6.1. The martingale transform in $\mathbb{R}$ is defined by

$$T_\sigma f := \sum_{I\in\mathcal{D}([0,1])}\sigma_I\langle f,h_I\rangle h_I,$$

where $\sigma_I=\pm 1$. It is also known to be a dyadic analog of singular integral operators. In [**20**], the author presented a linear estimate of the martingale transform on the weighted Lebesgue space $L^2(w)$, that is

$$\|T_\sigma f\|_{L^2(w)}\le C[w]_{A_2}\|f\|_{L^2(w)},$$

for $w \in A_2$ and $f \in L^2(w)$. With Wilson's Haar system, we can also define the multivariable martingale transform:

$$T_\sigma f = \sum_{Q \in \mathcal{D}^n([0,1]^n)} \sum_{j=1}^{2^n-1} \sigma_{E_{j,Q}} \langle f, h_{j,Q} \rangle h_{j,Q}, \tag{6.1}$$

where $\sigma_{E_{j,Q}}$ assumes the values ± 1 only. It was also considered in [**7**] to search for the L^p estimate of the Beurling-Ahlfors operator.

In fact, we already present all the tools to extend the result of [**20**] to (6.1). In [**20**], the one dimensional analogue of the inequality:

$$\begin{aligned} \frac{1}{|E_{i,Q'}|} \sum_{Q \in \mathcal{D}^n(Q')} \sum_{j : E_{j,Q} \subseteq E_{i,Q'}} \left(\langle w^{-1} \rangle_{E^1_{j,Q}} - \langle w \rangle_{E^2_{j,Q}} \right)^2 |E_{j,Q}| \langle w \rangle_{E_{j,Q}} \\ \leq C(n) [w]^2_{A_2} \langle w^{-1} \rangle_{E_{i,Q'}} \end{aligned} \tag{6.2}$$

was proved by using the sharp estimate of the dyadic square function in $L^2(w)$. Alternatively, one can also have the inequality (6.2) using (4.9), (4.15) and $[w]_{A_2} = [w^{-1}]_{A_2}$. By the Cauchy-Schwarz inequality and Proposition 4.8 we have the following inequality:

$$\begin{aligned} \left| \sum_{Q \in \mathcal{D}^n(Q')} \sum_{j : E_{j,Q} \subseteq E_{i,Q'}} (\langle w \rangle_{E^1_{j,Q}} - \langle w \rangle_{E^2_{j,Q}})(\langle w^{-1} \rangle_{E^1_{j,Q}} - \langle w^{-1} \rangle_{E^2_{j,Q}}) |E_{j,Q}| \right| \\ \leq C(n) [w]_{A_2} |E_{i,Q'}|. \end{aligned} \tag{6.3}$$

One can find the Bellman function proof of the single variable version of the following inequality.

$$\begin{aligned} \left| \sum_{Q \in \mathcal{D}^n(Q')} \sum_{j : E_{j,Q} \subseteq E_{i,Q'}} \frac{(\langle w \rangle_{E^1_{j,Q}} - \langle w \rangle_{E^2_{j,Q}})(\langle w^{-1} \rangle_{E^1_{j,Q}} - \langle w^{-1} \rangle_{E^2_{j,Q}})}{\langle w^{-1} \rangle_{E_{j,Q}}} |E_{j,Q}| \right| \\ \leq C(n) [w]_{A_2} w(E_{i,Q'}). \end{aligned} \tag{6.4}$$

Using Lemma 4.1 we can get the inequality (6.4). However, more simply we have

$$\left|\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}}\frac{(\langle w\rangle_{E^1_{j,Q}}-\langle w\rangle_{E^2_{j,Q}})(\langle w^{-1}\rangle_{E^1_{j,Q}}-\langle w^{-1}\rangle_{E^2_{j,Q}})}{\langle w^{-1}\rangle_{E_{j,Q}}}|E_{j,Q}|\right|$$

$$\leq\left(\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}}(\langle w\rangle_{E^1_{j,Q}}-\langle w\rangle_{E^2_{j,Q}})^2|E_{j,Q}|\langle w\rangle_{E_{j,Q}}\right)^{1/2} \tag{6.5}$$

$$\times\left(\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}}\left(\frac{\langle w^{-1}\rangle_{E^1_{j,Q}}-\langle w^{-1}\rangle_{E^2_{j,Q}}}{\langle w^{-1}\rangle_{E_{j,Q}}}\right)^2|E_{j,Q}|\langle w\rangle_{E_{j,Q}}\right)^{1/2}. \tag{6.6}$$

Using inequality (4.17) for the term (6.5) and (4.12) interchanging the role of w and w^{-1} for the term (6.6), we can get the inequality (6.4). By repeating the proof presented in Section 5 in [**20**] with previous observations and the sharp extrapolation theorem [**6**], we have that for $1<p<\infty$ there exists constants $C(n,p)$ only depending on p and the dimension n and C which does not depend on the dimension n such that

$$\|T_\sigma\|_{L^p_{\mathbb{R}^n}(w)\to L^p_{\mathbb{R}^n}(w)}\leq C(n,p)[w]_{A^d_p}^{\max\{1,\frac{1}{p-1}\}}, \tag{6.7}$$

for all weights $w\in A^d_p$ and

$$\|T_\sigma\|_{L^2_{\mathbb{R}^n}(w)\to L^2_{\mathbb{R}^n}(w)}\leq C[w]_{A^R_2}, \tag{6.8}$$

for all weights $w\in A^R_2$.

Remark 6.2. Let us consider the difference between π^s_b and π_b as an operator,

$$\pi^s_b f(x)-\pi_b f(x)=\sum_{Q\in\mathcal{D}^n}\sum_{j=1}^{2^n-1}\big(\langle f\rangle_Q-\langle f\rangle_{E_{j,Q}}\big)\langle b,h_{j,Q}\rangle h_{j,Q}(x). \tag{6.9}$$

For fixed Q and $j>1$, $\langle f\rangle_Q-\langle f\rangle_{E_{j,Q}}$ can be written by

$$\sum_{i:E_{i,Q}\supsetneq E_{j,Q}}\langle f,h_{i,Q}\rangle h_{i,Q}(E_{j,Q}).$$

Thus, the difference operator (6.9) can be estimated by

$$(6.10)\quad \begin{aligned}&\left|\pi_b^s f(x) - \pi_b f(x)\right| \\ &\quad \le C(n) \sum_{Q\in\mathcal{D}^n} \sum_{j=1}^{2^n-1} \sum_{i:E_{i,Q}\supsetneq E_{j,Q}} |\langle f, h_{i,Q}\rangle|\,|\langle b, h_{j,Q}\rangle| \frac{\chi_{E_{j,Q}}(x)}{|E_{j,Q}|}.\end{aligned}$$

The right hand side of the inequality (6.10) can be considered as a finite sum of the compositions of adjoint of paraproducts and dyadic shifting operators, that is denoted by $\pi_b^* H_\tau$. It is known that the dyadic shifting operators obey linear bounds in $L^2(w)$. We have shown that the paraproduct (π_b) obey linear bounds in $L^2(w)$, so the adjoint of paraproduct also does. One can easily expect that its composition, $\pi_b^* H_\tau$ obey quadratic bounds in $L^2(w)$. However, $\pi_b^* H_\tau$ obey linear bounds in $L^2(w)$ [**4**] but $H_\tau \pi_b^*$ does not. See [**4**] for more detailed arguments in the one dimensional case. Then the difference operator $\pi_b^s - \pi_b$ obeys the linear bound in $L^2_{\mathbb{R}^n}(w)$. Furthermore, this observation and the linear bound for the paraproduct associated to the Wilson's Haar basis (π_b) yield the linear bound for the paraproduct associated to the standard tensor product Haar basis (π_b^s).

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