PACIFIC JOURNAL OF MATHEMATICS
Vol. 32, No. 3, 1970

CONJUGATE SPACE REPRESENTATIONS OF BANACH SPACES

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Let a linear homeomorphism T from a Banach space X onto the conjugate space Y^* of a Banach space Y be called a conjugate space representation of X. If $T: X \to Y^*$ and $U: X \to Z^*$ are two conjugate space representations of X, say that T and U are essentially different if there is no linear homeomorphism P from Y onto Z satisfying $P^* = T \circ U^{-1}$. It is proven here that if a nonreflexive Banach space has one conjugate space representation, it has uncountably many essentially different conjugate space representations. A Banach space X with norm p will be denoted by (X, p) when it is important to emphasize the norm. The dual of p is the norm p^* defined on the conjugate space $(X, p)^*$ of (X, p) by

$$p^*(f) = \sup\{|f(x)|: x \in X \text{ and } p(x) = 1\}.$$

It is proven here that if $T: (X, p) \to (Y, r)^*$ and $U: (X, p) \to (Z, s)^*$ are two essentially different conjugate space representations of (X, p), then there exists a norm q on X equivalent to p such that $q \circ T^{-1} = r_1^*$ for some norm r_1 on Y equivalent to r, but such that $q \circ U^{-1} \neq s_1^*$ for any norm s_1 on Z equivalent to s.

Williams has shown [7, Th. 1, p. 163] that a Banach space (X, p) is reflexive if and only if every norm q on X^* equivalent to p^* is the dual of some norm on X equivalent to p. We show here that if (X, p) is a nonreflexive Banach space, then there exists a norm q on X^* equivalent to p^* such that q is not the dual of any norm on X equivalent to p, but such that the Banach space (X^*, q) is isometrically isomorphic to a conjugate Banach space. By contrast, Klee [3, Th. 4, p. 21] has exhibited a Banach space (X, p) and a norm q on X^* equivalent to p^* such that (X^*, q) is not isometrically isomorphic to a conjugate Banach space.

We shall use the following notation. If A and B are sets, $A \backslash B$ denotes the set of elements in A but not in B. If x is an element in a linear space, $[x]$ denotes the linear span of x. If A and B are linear subspaces of a linear space X, and if $A \cap B = \{0\}$, then $A \oplus B$ denotes the linear direct sum of A and B. If A is a subset of a normed linear space (X, p), $A^{\perp}$ donotes the annihilator of A in X^*. If A is a subset of the conjugate space X^* of a normed linear space (X, p), $A_{\perp}$ denotes the set of elements in X annihilated by A. If (X, p) is a normed linear space, J_X denotes the canonical map from X into X^{**} defined by

$$(J_X x)f = f(x) \text{ for all } x \in X \text{ and } f \in X^* .$$

LEMMA. *If $T: X \to Y^*$ and $U: X \to Z^*$ are two conjugate space representations of a Banach space X, then T and U are essentially different if and only if*

$$T^*[J_Y Y] \neq U^* [J_Z Z] .$$

Proof. (i) Suppose T and U are not essentially different. Then there exists a linear homeomorphism P from Y onto Z satisfying $P^* = T \circ U^{-1}$. It is straightforward to verify that $(P^{**}(J_Y y))g = (J_Z(Py))g$ for all $y \in Y$ and $g \in Z^*$; that is, $P^{**} \circ J_Y = J_Z \circ P$. Therefore $T^*[J_Y Y] = (P^* \circ U)^*[J_Y Y] = (U^* \circ P^{**} \circ J_Y)[Y] = U^*[J_Z Z]$.

(ii) Suppose $T^*[J_Y Y] = U^*[J_Z Z]$. Let $P = J_Z^{-1} \circ U^{*-1} \circ T^* \circ J_Y = J_Z^{-1} \circ (T \circ U^{-1})^* \circ J_Y$. Then P is a linear homeomorphism from Y onto Z. It can be verified directly that $(P^* g)y = ((T \circ U^{-1})g)y$ for all $g \in Z^*$ and $y \in Y$. Therefore $P^* = T \circ U^{-1}$.

THEOREM 1. *Suppose that (X, p) is a nonreflexive Banach space which is linearly homeomorphic to the conjugate (Y^*, r^*) of a Banach space (Y, r). Then there exists an uncountable collection of essentially different conjugate space representations $U_\alpha: (X, p) \to (Z_\alpha^*, s_\alpha^*)$ such that each space (Z_α, s_α) is linearly homeomorphic to (Y, r).*

Proof. By hypothesis there exists a linear homeomorphism T from (X, p) onto (Y^*, r^*). Let $M = T^*[J_Y Y]$. Then [4, p. 577] M is a minimal total norm-closed subspace of (X^*, p^*). That is, M is total and norm-closed, and no proper subspace of M is both total and norm-closed. If L is any norm-closed subspace of X^*, let Q_L denote the canonical map from X into L^* defined by

$$(Q_L x)f = f(x) \text{ for all } x \in X \text{ and } f \in L .$$

In particular, Q_{X^*} is the canonical map J_X from X into X^{**}. By [4, p. 577], the map Q_L is a linear homeomorphism from (X, p) onto $(L^*, (p^* | L)^*)$ if and only if L is a minimal total norm-closed subspace of (X^*, p^*). Since (X, p) is not reflexive, Q_{X^*} is not a linear homeomorphism from (X, p) onto (X^{**}, p^{**}), and X^* is not a minimal total norm-closed subspace of X^*.

Let $f \in X^*$. Let us show that there is a minimal total norm-closed subspace B of X^* such that $f \in B$ and such that B is linearly homeomorphic to (Y, r). If $f \in M$, we may take $B = M$. Now suppose $f \notin M$. By a theorem of Dixmier [1, Th. 11, p. 1065] a norm-closed total subspace V of the conjugate E^* of a Banach space E is a minimal total norm-closed subspace of E^* if and only if $E^{**} = J_E E \oplus V^\perp$. Thus we have $X^{**} = J_X X \oplus M^{\perp}$. Let $H = [f]^{\perp}$. By the Hahn-Banach Theorem [6, Corollary 2, p. 67], $H \not\supseteq M^{\perp}$ so $J_X X \oplus (H \cap M^{\perp})$ is a maximal subspace of X^{**}. Now $H \not\supseteq J_X X$ since $f \neq 0$, so $J_X X \oplus (H \cap M^{\perp}) \neq H$. Since $J_X X \oplus (H \cap M^{\perp})$ and H are distinct maximal subspaces of X^{**}, there exists $G \in H$ such that $X^{**} = J_X X \oplus (H \cap M^{\perp}) \oplus [G]$. Let $D = (H \cap M^{\perp}) \oplus [G]$, and let $B = D_{\perp}$. Then [6, (x), p. 238] B is a norm-closed subspace of (X^*, p^*). The subspaces H and $M^{\perp}$ are $w(X^{**}, X^*)$-closed [6, (x), p. 238], so $H \cap M^{\perp}$ is $w(X^{**}, X^*)$-closed. Therefore [6, Corollary 5, p. 192] D is $w(X^{**}, X^*)$closed, and [6, Th. 1, p. 238] $B^{\perp} = (D_{\perp})^{\perp} = D$. Now B is a total subspace of X^* since $B^{\perp} \cap J_X X = \{0\}$. By the theorem of Dixmier mentioned above, B is a minimal total norm-closed subspace of X^*. By the Hahn-Banach Theorem, we have $f \in B$ since $B^{\perp} \subseteqq H = [f]^{\perp}$. Now observe that both B and M are maximal subspaces of $(H \cap M^{\perp})_{\perp}$, and consequently each of them is linearly homeomorphic to the topological direct sum of $B \cap M$ with a one-dimensional space. Therefore B is linearly homeomorphic to M which is in turn linearly homeomorphic to Y.

Let $\{Z_\alpha : \alpha \in \Phi\}$ be the collection of all minimal total norm-closed subspaces of X^* which are linearly homeomorphic to Y. For each $\alpha \in \Phi$ let $s_\alpha = p^* \mid Z_\alpha$. We have established that every element $f \in X^*$ is contained in some Z_α. Each Z_α is nowhere dense in X^* since each Z_α is a proper norm-closed subspace of X^*. Since X* is a complete linear metric space, the Baire Category Theorem guarantees that X^* is not a countable union of nowhere dense sets. Therefore $\{Z_\alpha : \alpha \in \Phi\}$ is an uncountable collection. For every $\alpha \in \Phi$, let $U_\alpha = Q_{Z_\alpha}$. It is straightforward to verify that $U_\alpha^* \circ J_{Z_\alpha}$ is the identity map on Z_α. Thus $U_\alpha^*[J_{Z_\alpha} Z_\alpha] = Z_\alpha$. Therefore by the lemma U_β and U_α are essentially different whenever $\beta, \alpha \in \Phi$ and $\beta \neq \alpha$.

THEOREM 2. *Suppose that* $T: (X, p) \to (Y^*, r^*)$ *and* $U: (X, p) \to (Z^*, s^*)$ *are two essentially different conjugate space representations. Then there exists a norm* q *on* X *equivalent to* p *such that* $q \circ T^{-1}$ *is the dual of some norm* r_1 *on* Y *equivalent to* r, *but such that* $q \circ U^{-1}$ *is not the dual of any norm* s_1 *on* Z *equivalent to* s.

REMARK. An interesting example may be obtained by letting X, Y and Z be the sequence spaces l, c, and c_0, respectively.

Proof of Theorem 2. Let $A = T^*[J_Y Y]$ and let $B = U^*[J_Z Z]$. Then $A \neq B$ by the lemma. By [4, p. 577], A and B are minimal total norm-closed subspaces of X^*. The map T is a vector space isomorphism from X onto Y^* and $T^*[J_Y Y] = A$; it follows that T is a $w(X, A) - w(Y^*, J_Y Y)$-homeomorphism. Let $S = T^{-1}[\{g \in Y^* : r^*(g) \leqq 1\}]$. Then S is $w(X, A)$-compact, because $\{g \in Y^* : r^*(g) \leqq 1\}$ is $w(Y^*, J_Y Y)$-

compact by the Banach-Alaoglu Theorem [6, Th. 1, p. 239]. Since $A \neq B$ and since A and B are both minimal with respect to certain properties, we must have $A \not\subseteq B$. Thus there exists $f \in A \backslash B$. Let $L = f^{-1}(0)$ and let $V = L \cap S$. The subspace L is $w(X, A)$-closed [6, Th. 3, p. 186] since $f \in A$. Thus V is $w(X, A)$-compact.

Now f is not $w(X, B)$-continuous [2, Th. 9, p. 421] since $f \notin B$, so [6, Th. 3, p. 186] L is not $w(X, B)$-closed. However, L is norm-closed [6, Th. 3, p. 186] since $f \in X^*$. Thus $U[L]$ is a norm-closed subspace of (Z^*, s^*), but $U[L]$ is not $w(Z^*, J_Z Z)$-closed. Let K be the $w(X, B)$-closure of V. Then [2, Lemma 4, p. 415] K is convex since V is convex. Now $U[K]$ is a convex $w(Z^*, J_Z Z)$-closed subset of Z^*. By a corollary of the Krein-Šmulian Theorem [2, Corollary 9, p. 429], the linear span of a convex, weak* closed set is weak* closed if and only if it is norm-closed. Therefore $U[L] \neq \operatorname{span}(U[K])$. Consequently, $L \neq \operatorname{span}(K)$. However, $\operatorname{span}(K) \supseteqq \operatorname{span}(V) = L$, so there exists an element $x_0 \in K \backslash L$. Let W be the convex balanced hull of $V \cup \{(1/2)x_0\}$. Then if co denotes convex hull and bal denotes balanced hull, we have

$$\begin{aligned} W &= \operatorname{co}\left(\operatorname{bal}\left(V \cup \left\{\frac{1}{2}x_0\right\}\right)\right) = \operatorname{co}\left(\operatorname{bal}(V) \cup \operatorname{bal}\left\{\frac{1}{2}x_0\right\}\right) \\ &= \operatorname{co}\left(V \cup \operatorname{bal}\left\{\frac{1}{2}x_0\right\}\right) \end{aligned}$$

which is $w(X, A)$-closed [2, Lemma 5, p. 415] since the sets V and $\operatorname{bal}\{(1/2)x_0\}$ are convex and $w(X, A)$-compact. The set W is norm-closed since the norm topology is stronger than the $w(X, A)$ topology. Also W is norm-bounded since V and $\operatorname{bal}\{(1/2)x_0\}$ are norm-bounded. Now $\operatorname{span}(W) = X$ since $\operatorname{span}(W)$ properly contains the maximal subspace L. Thus for any $x \in X$ there exist elements $w_1, \cdots w_N \in W$ and nonzero numbers $t_1, \cdots, t_N$ such that $x = t_1 w_1 + \cdots + t_N w_N$. Let $t = \sum_{i=1}^N |t_i|$. Then $x/t \in W$ since W is convex and balanced. Thus W is absorbing. We have shown that W is a convex, balanced, absorbing, norm-closed norm-bounded subset of the Banach space (X, p). Therefore W is a norm-neighborhood of zero since Banach spaces are barrelled. By [6, p. 58] the gauge q of W is a norm on X equivalent to p, and $W = \{x \in X: q(x) \leqq 1\}$.

Let $q_1 = q \circ T^{-1}$. Then q_1 is a norm on Y^* equivalent to r^* since T is a linear $p - r^*$ homeomorphism. Also $\{g \in Y^*: q_1(g) \leqq 1\} = T[W]$, and $\{g \in Y^*: q_1(g) \leqq 1\}$ is $w(Y^*, J_Y Y)$-closed since W is $w(X, A)$-closed. Singer has shown [5, Lemma 2, p. 450] that if (E, h) is a Banach space, and if h_1 is a norm on E^* equivalent to h^*, then h_1 is the dual of some norm on E equivalent to h if and only if the set $\{g \in E^*: h_1(g) \leqq 1\}$ is $w(E^*, J_E E)$-closed. (In one direction, of course, this is the well-known Banach-Alaoglu Theorem.) Therefore there exists a norm r_1 on Y equivalent to r such that $r_1^* = q_1 = q \circ T^{-1}$.

Let $q_2 = q \circ U^{-1}$. Then q_2 is a norm on Z^* equivalent to s^* since U is a linear $p - s^*$-homeomorphism. Also $\{g \in Z^*: q_2(g) \leqq 1\} = U[W]$. Now $x_0 \notin W$, for if $x_0 = cv + (1 - c)(d)((1/2)x_0)$ with $v \in V$, $0 \leqq c \leqq 1$, and $|d| \leqq 1$, then $(1 - 1/2(1 - c)d)x_0 = cv \in L$, so that $x_0 \in L$, contrary to the definition of x_0. However, x_0 belongs to the $w(X, B)$-closure of W since the $w(X, B)$-closure of W contains the $w(X, B)$-closure of V, namely K. Therefore W is not $w(X, B)$-closed. Thus $U[W] = \{g \in Z^*: q_2(g) \leqq 1\}$ is not $w(Z, J_Z Z)$-closed. By the Banach-Alaoglu Theorem, there is no norm s_1 on Z equivalent to s such that $s_1^* = q_2 = q \circ U^{-1}$.

COROLLARY. *If (X, p) is a nonreflexive Banach space, there is a norm q on X^* equivalent to p^* such that q is not the dual of any norm on X equivalent to p, but such that the Banach space (X^*, q) is isometrically isomorphic to a conjugate Banach space.*

Proof. Suppose that (X, p) is a nonreflexive Banach space. By Theorem 1 there exists a conjugate space representation $T: (X^*, p^*) \rightarrow (Y^*, r^*)$ such that T is essentially different from the identity map I on X^*. By Theorem 2 there exists a norm q on X^* equivalent to p^* such that $q \circ T^{-1} = r_1^*$ for some norm r_1 on Y equivalent to r, but such that $q \circ I^{-1}$ is not the dual of any norm on X equivalent to p. Now T is an isometric isomorphism from (X^*, q) onto the conjugate Banach space (Y^*, r_1^*).

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Received November 5, 1968. Most of the results of this paper are contained in the author's doctoral dissertation written at The Florida State University under the direction of Professor Ralph D. McWilliams.

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