ON SOME TRIGONOMETRIC TRANSFORMS

OTTO SZÁSZ

1. Introduction. To a given series $\Sigma_{n=1}^{\infty} u_n$ we consider the transform

$$(1.1) \qquad A_n = \sum_{\nu=1}^{n} u_\nu \frac{\sin \nu t_n}{\nu t_n}, \qquad \text{where } t_n \downarrow 0 \text{ as } n \longrightarrow \infty .$$

It was shown in a previous paper [5, Section 4, Theorem 3] that the transform (1.1) is regular if and only if

$$(1.2) \qquad n t_n = O(1), \qquad \text{as } n \longrightarrow \infty .$$

We shall now consider the transform (1.1) in relation to Cesàro means. In a forthcoming paper Cornelius Lanczos has found independently that the transform (1.1) is very useful in summing Fourier series and derived series, and gave some very interesting examples; he takes $t_n = \pi/n$. Of our results we quote here the following theorem:

THEOREM 1. *In order that the transform* (1.1) *includes* $(C, 1)$ *summability, it is necessary and sufficient that*

$$(1.3) \qquad n t_n = p\pi + \alpha_n, \quad n\alpha_n = O(1), \qquad p \textit{ a positive integer.}$$

We also discuss other triangular transforms which may be generated by "truncation" of well-known summation processes, such as Riemann summability. The transform A_n and the transform D_n (Section 5) are special cases of the general transform

$$\gamma_n = \sum_{\nu=0}^{n} u_\nu \phi(\nu P_n),$$

Received March 8, 1950. Presented to the American Mathematical Society December 30, 1948. The preparation of this paper was sponsored (in part) by the Office of Naval Research.

Pacific J. Math. 1 (1951), 291-304.

where $\phi(P)$ is a function of the n-dimensional point $P(x_1, x_2, \cdots, x_n)$, and $P_n \longrightarrow 0$. This transform and many special cases of it were discussed by W. Rogosinski [4]; in particular, the special case $\alpha_n = 0$ of our Theorem 4 is included in his result on page 96. The general approach is essentially the same as in the present paper.

2. Proof of Theorem 1. If we write

$$\sum_{\nu=1}^{n} u_\nu = s_n\,, \qquad \sum_{\nu=1}^{n} s_\nu = s_n'\,, \qquad \frac{\sin \nu t_n}{\nu t_n} - \frac{\sin\,(\nu+1)\,t_n}{(\nu+1)\,t_n} = \Delta_\nu\,,$$

$$\frac{\sin \nu t_n}{\nu t_n} - \frac{2\,\sin\,(\nu+1)\,t_n}{(\nu+1)\,t_n} + \frac{\sin\,(\nu+2)\,t_n}{(\nu+2)\,t_n} = \Delta_\nu^2\,,$$

then

$$A_n = \sum_{\nu=1}^{n-1} s_\nu \Delta_\nu + s_n \frac{\sin n t_n}{n t_n}$$

$$= \sum_{\nu=1}^{n-2} s_\nu' \Delta_\nu^2 + s_{n-1}'\,\Delta_{n-1} + (s_n' - s_{n-1}')\,\frac{\sin n t_n}{n t_n}\,,$$

or

$$A_n = \sum_{\nu=1}^{n-2} s_\nu' \Delta_\nu^2 + s_{n-1}' \left[\frac{\sin\,(n-1)\,t_n}{(n-1)\,t_n} - \frac{2\,\sin n t_n}{n t_n}\right] + s_n' \frac{\sin n t_n}{n t_n}\,. \tag{2.1}$$

Now $(C.\,1)$ summability of $\Sigma_{n=1}^{\infty} u_n$ to s means that

$$n^{-1} s_n' \longrightarrow s\,, \qquad \text{as } n \longrightarrow \infty\,. \tag{2.2}$$

If $s_n \equiv 1$, then $A_n = \sin t_n/t_n \longrightarrow 1$.

In order that (2.2) imply $A_n \longrightarrow s$, it is necessary and sufficient [in view of (2.1)] that

$$\text{(2.3)} \qquad \frac{\sin nt_n}{t_n} = O(1), \qquad \frac{\sin (n-1)\, t_n}{t_n} = O(1),$$

$$\text{(2.4)} \qquad \sum_{\nu=1}^{n-2} \nu |\Delta_\nu^2| = O(1), \qquad \text{as } n \longrightarrow \infty .$$

The first condition of (2.3) [in view of (1.2)] is equivalent to

$$\sin nt_n = O(t_n) = O(1/n) ;$$

hence

$$nt_n = p\pi + \alpha_n , \qquad n\alpha_n = O(1) .$$

The second condition of (2.3) now reduces to

$$\cos nt_n \sin t_n = O(t_n) ,$$

or

$$\cos \alpha_n \sin t_n = O(n^{-1}) ,$$

which is satisfied. Finally

$$\frac{\sin \nu t}{\nu} = \int_0^t \cos \nu x \, dx = \Re \int_0^t e^{i\nu x} \, dx ;$$

hence

$$\text{(2.5)} \qquad t_n \Delta_\nu^2 = \Re \int_0^{t_n} \Delta^2 e^{i\nu x} \, dx = \Re \int_0^{t_n} e^{i\nu x} (1 - e^{ix})^2 \, dx ,$$

and

$$\text{(2.6)} \qquad t_n |\Delta_\nu^2| < \int_0^{t_n} |1 - e^{ix}|^2 \, dx = 4 \int_0^{t_n} (\sin x/2)^2 \, dx$$

$$< \int_0^{t_n} x^2 \, dx < t_n^3 .$$

It follows that

$$\sum_{\nu=1}^{n-2} \nu \, |\Delta_\nu^2| < t_n^2 \sum_{\nu=1}^{n} \nu < n^2 t_n^2 = O(1) , \qquad \text{as } n \longrightarrow \infty .$$

This proves Theorem 1.

We can show by an example that the transform A_n may be more powerful than $(C, 1)$. In (1.3) let $p = 1$, $n\alpha_n = -\pi/2$; the series $\Sigma_{\nu=1}^{\infty} (-1)^{n-1} n$ (that is, $u_n = (-1)^n n$) is not summable $(C, 1)$, but summable $(C, 2)$ to $1/4$. Now

$$t_n A_n = \sum_{\nu=1}^{n} (-1)^{\nu-1} \sin \nu t_n$$

$$= \frac{\sin t_n - (-1)^n [\sin nt_n + \sin (n+1) t_n]}{|1 + e^{it}|^2},$$

where $nt_n = \pi - \pi/2n$. Hence, as $n \longrightarrow \infty$,

$$A_n \sim 1/4 + o(1) .$$

An even more striking example is $u_n = (-1)^{n-1} n^2$.

3. Summation by harmonic polynomials. We get a more powerful method if we introduce the harmonic polynomial

$$h_n(\rho, t) = \sum_{\nu=1}^{n} u_\nu \rho^\nu \frac{\sin \nu t}{\nu}, \tag{3.1}$$

and the corresponding transform

$$B_n = \sum_{\nu=1}^{n} u_\nu \rho_n^\nu \frac{\sin \nu t_n}{\nu t_n}, \qquad \rho_n \longrightarrow 1, \quad t_n \downarrow 0, \tag{3.2}$$

or

$$B_n = t_n^{-1} h_n(\rho_n, t_n) .$$

Let

$$s_n^k = \sum_{\nu=0}^{n} s_\nu \gamma_{n-\nu}^{k-1},$$

where

$$\gamma_n^k = \frac{(k+1)\cdots(k+n)}{n!} \sim \frac{n^k}{\Gamma(k+1)} \; ;$$

we also write

$$\Delta^k v_\nu = \sum_{r=1}^{k} (-1)^r \binom{k}{r} v_{\nu+r} \; ,$$

and

$$\sigma_n^k = \frac{s_n^k}{\gamma_n^k} \; .$$

Now (C, k) summability of the sequence $\{s_n\}$ to s is defined by

$$\lim_{n\to\infty} \sigma_n^k = s \; .$$

We quote the following elementary theorem [cf. 6, Theorem 1], which is included in a more general result of Mazur [1, Theorem X]:

LEMMA 1. *Let k be a given positive integer, and let*

$$T_n = \sum_{\nu=0}^{n} a_{n,\nu} s_\nu \; , \qquad n = 0, 1, 2, \cdots .$$

In order that $\lim T_n$ *exist, whenever the sequence $\{s_n\}$ is (C, k) summable to s, it is necessary and sufficient that:*

$$\text{(3.3)} \qquad \sum_{\nu=0}^{n} \gamma_\nu^k \, |\Delta^k a_{n,\nu}| = O(1) \; , \qquad a_{n,\nu} = 0 \text{ for } \nu > n \; ;$$

$$\text{(3.4)} \qquad \lim_{n\to\infty} \gamma_\nu^k \, \Delta a_{n,\nu} = \alpha_\nu \textit{ exists} \; , \qquad \nu = 0, 1, 2, \cdots ;$$

$$\text{(3.5)} \qquad \lim_{n\to\infty} \sum_{\nu=0}^{n} a_{n,\nu} = \beta \textit{ exists} .$$

We then have $\lim T_n = \beta s + \sum_{\nu=0}^{\infty} \alpha_\nu (\sigma_\nu^k - s)$. Since then the transform T_n

is convergence preserving we must have (3.5) and:

$$\lim_{n\to\infty} a_{n,\nu} \text{ exists}, \qquad \nu = 0, 1, 2, \cdots;$$

hence (3.4) and (3.5) hold, so that the conditions of Lemma 1 reduce to (3.3). In the case of the transform B_n, we have

$$a_{n,n} = \rho_n^n \frac{\sin nt_n}{nt_n},$$

$$a_{n,\nu} = \rho_n^\nu \frac{\sin \nu t_n}{\nu t_n} - \rho_n^{\nu+1} \frac{\sin (\nu+1) t_n}{(\nu+1) t_n}, \qquad \nu = 1, 2, \cdots . n-1;$$

hence

$$a_{n,\nu} \longrightarrow 0, \qquad \text{as } n \longrightarrow \infty.$$

To satisfy (3.3) we must have

$$n^k \rho_n^n \frac{\sin nt_n}{nt_n} = O(1), \tag{3.6}$$

$$n^k \rho_n^{n-1} \frac{\sin (n-1) t_n}{(n-1) t_n} = O(1), \tag{3.7}$$

$$\cdots$$

$$n^k \rho_n^{n-k} \frac{\sin (n-k) t_n}{(n-k) t_n} = O(1),$$

and

$$\sum_{\nu=1}^{n-k-1} \nu^k \left| \Delta^{k+1} \rho_n^\nu \frac{\sin \nu t_n}{\nu t_n} \right| = O(1). \tag{3.8}$$

Assume first that $k = 0$; then our conditions become:

$$\rho_n^n \frac{\sin nt_n}{nt_n} = O(1), \tag{3.9}$$

and

$$\sum_{\nu=1}^{n-1} \rho_n^\nu \left| \frac{\sin \nu t_n}{\nu t_n} - \rho_n \frac{\sin (\nu+1) t_n}{(\nu+1) t_n} \right| = O(1). \tag{3.10}$$

We now prove the lemma:

LEMMA 2. *If*

$$\rho_n^n = O(1)\,, \quad \frac{1-\rho_n^n}{1-\rho_n}\, t_n = O(1)\,, \qquad \text{as } t_n \downarrow 0\,, \quad \rho_n \longrightarrow 1\,, \tag{3.11}$$

then B_n is a regular transform.

Clearly (3.9) holds, and we need only to show that (3.10) also holds.

If $\rho_n > 1$, then $\rho_n^\nu < \rho_n^n$, $\nu = 0, 1, \cdots, n-1$; if on the other hand $\rho_n \leq 1$, then $\rho_n^\nu \leq 1$. Hence, in either case,

$$\max_{0 \leq \nu \leq n} \rho_n^\nu = O(1)\,, \qquad \text{as } n \longrightarrow \infty.$$

We have

$$\sum_{\nu=1}^{n} \rho^\nu \left| \frac{\sin \nu t}{\nu} - \rho\, \frac{\sin (\nu+1)\, t}{\nu+1} \right| \leq \sum_{\nu=1}^{n} \rho^\nu \left| \frac{\sin \nu t}{\nu} - \frac{\sin (\nu+1)\, t}{\nu+1} \right| + (1-\rho) \sum_{\nu=1}^{n} \left| \frac{\sin (\nu+1)\, t}{\nu+1} \right| \rho^\nu\,;$$

the second term is $O(t)$, and

$$\frac{\sin \nu t}{\nu} - \frac{\sin (\nu+1)\, t}{\nu+1} = \int_0^t [\cos \nu x - \cos (\nu+1)x]\, dx = O(t^2)\,,$$

so that

$$\sum_{\nu=1}^{n} \rho^\nu \left| \frac{\sin \nu t}{\nu} - \frac{\sin (\nu+1)\, t}{\nu+1} \right| = O\left(t^2\, \frac{1-\rho^n}{1-\rho}\right).$$

Thus (3.10) is satisfied and Lemma 2 holds.

Note that the condition $\rho_n^n = O(1)$ is equivalent to $n(\rho_n - 1) < c$, a positive constant (see [5, p. 73]); furthermore, if $nt_n = O(1)$, then clearly the second condition of (3.11) holds.

Next let $k = 1$; we shall prove the theorem:

THEOREM 2. *If* (3.11) *holds, and if*

$$\rho_n^n \sin nt_n = O(t_n), \qquad n \longrightarrow \infty, \tag{3.12}$$

then B_n *includes* $(C, 1)$.

The conditions (3.6)–(3.8) now become:

$$\rho_n^n \sin nt_n = O(t_n),$$

$$\rho_n^n \sin (n-1) t_n = O(t_n),$$

and

$$\sum_{\nu=1}^{n-2} \nu \left| \Delta^2 \rho_n^\nu \frac{\sin \nu t_n}{\nu} \right| = O(t_n), \qquad \text{as } n \longrightarrow \infty. \tag{3.13}$$

Clearly, we need only to show that (3.13) is satisfied. Now

$$\Delta^2 \rho^\nu \frac{\sin \nu t}{\nu} = \Delta^2 \rho^\nu \int_0^t \cos \nu x \, dx = \Re \Delta^2 \int_0^t \rho^\nu e^{i\nu x} \, dx$$

$$= \Re \int_0^t \rho^\nu e^{i\nu x}(1 - 2\rho e^{ix} + \rho^2 e^{2ix}) \, dx$$

$$= \Re \int_0^t \rho^\nu e^{i\nu x}(1 - \rho e^{ix})^2 \, dx .$$

Hence

$$\left| \Delta^2 \rho^\nu \frac{\sin \nu t}{\nu} \right| < \rho^\nu \int_0^t |1 - \rho e^{ix}|^2 \, dx < \rho^\nu t \{(1-\rho)^2 + \rho t^2\};$$

it follows from (3.11) that

$$\sum_{\nu=1}^{n} \nu \left| \Delta^2 \rho_n^\nu \frac{\sin \nu t_n}{\nu t_n} \right| < \{(1-\rho_n)^2 + \rho_n t_n^2\} \sum_{\nu=1}^{n} \nu \rho_n^\nu = O(1).$$

This proves (3.13) and Theorem 2.

4. Comparison of B_n and (C, k), $k \geq 2$. We wish to prove the following theorem:

THEOREM 3. *Suppose that* (3.11) *holds and that*

$$n^{k-1}\rho_n^n \sin nt_n = O(t_n)\,, \tag{4.1}$$

$$n^{k-1}\rho_n^n \cos nt_n = O(1)\,, \qquad \rho_n \longrightarrow 1\,, \qquad t_n \downarrow 0\,, \tag{4.2}$$

then B_n *includes* (C, k) *summability.*

Now (3.6) holds because of (4.1), and then (3.7) follows from (4.2). It remains to prove (3.8). We have

$$\Delta^{k+1}\rho^\nu \frac{\sin \nu t}{\nu} = \Delta^{k+1}\rho^\nu \int_0^t \cos \nu x\, dx = \Delta^{k+1}\Re \int_0^t \rho^\nu e^{i\nu x}\, dx$$

$$= \Re \int_0^t \rho^\nu e^{i\nu x}(1-\rho e^{ix})^{k+1}\, dx\,;$$

hence

$$\left|\Delta^{k+1}\rho^\nu \frac{\sin \nu t}{\nu}\right| < \rho^\nu \int_0^t |1-\rho e^{ix}|^{k+1}\, dx \tag{4.3}$$

$$< \rho^\nu \int_0^t \{(1-\rho)^2 + \rho t^2\}^{(k+1)/2}\, dx$$

$$= O(\rho^\nu t\{(1-\rho)^{k+1} + t^{k+1}\})\,.$$

It follows that

$$\sum_{\nu=1}^n \nu^k \left|\Delta^{k+1}\rho_n^\nu \frac{\sin \nu t_n}{\nu t_n}\right| = O\left(\sum_{\nu=1}^n \nu^k \rho_n^\nu\{(1-\rho_n)^{k+1} + t_n^{k+1}\}\right) \tag{4.4}$$

$$= O\left((1-\rho_n)^{k+1}\sum_{\nu=1}^n \nu^k\rho_n^\nu\right) + O\left(t_n^{k+1}\sum_{\nu=1}^n \nu^k\rho_n^\nu\right).$$

Here the first term is $O(1)$ by Lemma 2 of [6]; finally

$$t_n^{k+1}\sum_{\nu=1}^n \nu^k\rho_n^\nu = O\left(t_n \sum_{\nu=1}^n \rho_n^\nu\right)^{k+1} = O(1)\,.$$

This proves Theorem 3.

An interesting special case is $t_n = \pi/n$; the conditions now reduce to the single condition

$$n^{k-1}\rho_n^n = O(1) .$$

If, in particular, $n^k \rho_n^n = O(1)$ for *all* k, then B_n includes all (C, k).

Observe that by Lemma 1 of [6] the condition $n^k \rho_n^n = O(1)$ is equivalent to

$$\limsup \{n(\rho_n - 1) + k \log n\} < +\infty .$$

Note also that (4.1) and (4.2) imply:

$$n^{k-1}\rho_n^n = o(1) .$$

5. Truncated Riemann summability. The series $\sum_{\nu=0}^{\infty} u_\nu$ is called (R, k) summable to s if the series

$$u_0 + \sum_{n=1}^{\infty} \left(\frac{\sin nt}{nt}\right)^k u_n = R_k(t) \tag{5.1}$$

converges in some interval $0 < t < t_0$, and if

$$R_k(t) \longrightarrow s , \qquad \text{as } t \longrightarrow 0 .$$

For $k = 1$ it is sometimes called Lebesgue summability. The method (R, k) is regular for $k \geq 2$ and, in fact, it is more powerful than $(C, k-2)$; for $k = 2$, it was employed by Riemann in the theory of trigonometric series. We generate from it by truncation the triangular series to sequence transform $(u_0 = 0)$:

$$D_n = \sum_{\nu=1}^{n} u_\nu \left(\frac{\sin \nu t_n}{\nu t_n}\right)^k = \sum_{\nu=1}^{n-1} s_\nu \Delta \left(\frac{\sin \nu t_n}{\nu t_n}\right)^k + s_n \left(\frac{\sin n t_n}{n t_n}\right)^k ;$$

k is a positive integer. We assume $k \geq 2$; it is then easy to show that D_n is a regular transformation.

From Lemma 1 we find for (C, k) to be included in D_n the conditions:

$$t_n^{-k}(\sin \overline{n-\nu}\, t_n)^k = O(1) , \qquad \text{for } \nu = 0, 1, \cdots, k; \tag{5.2}$$

$$\sum_{\nu=1}^{n-k-1} \nu^k \left| \Delta^{k+1} \left(\frac{\sin \nu t_n}{\nu t_n}\right)^k \right| = O(1) , \qquad n \longrightarrow \infty . \tag{5.3}$$

It follows from (5.2) (see Section 2) that we must have

$$nt_n = p\pi + \alpha_n\ , \qquad n\alpha_n = O(1)\ , \qquad p \text{ a positive integer ;} \tag{5.4}$$

now (5.2) reduces to

$$t_n \sin(\alpha_n - \nu t_n) = O(1)\ , \qquad \nu = 0, 1, \cdots, k\ ,$$

and this is satisfied in view of (5.4).

To show that now (5.3) also holds, we employ a lemma, due to Obreschkoff [2, p. 443]:

LEMMA 3. *We have*

$$\left| \Delta^m \left(\frac{\sin \nu t}{\nu t} \right)^k \right| \leq M \frac{t^{m-k}}{\nu^k}\ ,$$

where M is independent of t and ν.

It now follows that

$$\sum_{\nu=1}^{n} \nu^k \left| \Delta^{k+1} \left(\frac{\sin \nu t_n}{\nu t_n} \right)^k \right| = O(nt_n) = O(1)\ , \qquad n \longrightarrow \infty\ .$$

This yields the following theorem:

THEOREM 4. *If $nt_n = p\pi + \alpha_n$, p a positive integer, $n\alpha_n = O(1)$, then the transform*

$$\sum_{\nu=1}^{n} u_\nu \left(\frac{\sin \nu t_n}{\nu t_n} \right)^k = D_n$$

includes (C, k) summability (k a positive integer).

6. A converse theorem. We shall establish the following result.

THEOREM 5. *If*

$$\liminf \left| \frac{\sin nt_n}{nt_n} \right|^k = \lambda > 1/2\ , \tag{6.1}$$

then the transform D_n is equivalent to convergence.

It follows from (6.1) that $\limsup nt_n < 2^{1/k}$; hence (see Sections 1 and 5) the transform D_n is regular. We now wish to show that $D_n \longrightarrow s$ implies $s_n \longrightarrow s$; we follow a device used by R. Rado [3].

Assume first that $s = 0$, and that $s_n = O(1)$; then

$$0 \leq \limsup_{n\to\infty} |s_n| = \delta < \infty ,$$

and we shall show that $\delta = 0$. To a given $\epsilon > 0$ choose $n = n(\epsilon)$ so that $|s_\nu| < \delta + \epsilon$ for $\nu \geq n$. Next choose $m > n$ and such that $|s_m| > \delta - \epsilon$. We have

$$s_m \left(\frac{\sin mt_m}{mt_m}\right)^k = D_m - \sum_{\nu=1}^{m-1} s_\nu \Delta_\nu ,$$

where

$$\Delta_\nu = \left(\frac{\sin \nu t_m}{\nu t_m}\right)^k - \left(\frac{\sin (\nu+1) t_m}{(\nu+1) t_m}\right)^k ;$$

hence, as $mt_m < \pi$, we have

$$|s_m| \left|\frac{\sin mt_m}{mt_m}\right|^k < |D_m| + \left|\sum_{\nu=1}^{n-1} s_\nu \Delta_\nu\right| + \left|\sum_{\nu=n}^{m-1} s_\nu \Delta_\nu\right|$$

$$< o(1) + (\delta + \epsilon) \left\{\left(\frac{\sin nt_m}{nt_m}\right)^k - \left(\frac{\sin mt_m}{mt_m}\right)^k\right\} .$$

It follows that

$$\delta - \epsilon < |s_m| < o(1) + (\delta + \epsilon) \{1/\lambda - 1 + o(1)\} .$$

But $1/\lambda < 2$, and ϵ is arbitrarily small; hence $\delta = 0$.

We next assume $s = 0$ and $\limsup |s_n| = \infty$; choose $\epsilon > 0$ and ω large. Denote by $m = m(\omega)$ the least m for which $|s_m| > \omega$; then

$$\omega < |s_m| < o(1) + \omega\{1/\lambda - 1 + o(1)\} .$$

But this is impossible for $\lambda > 1/2$, small ϵ, and large m. This proves our theorem for $s = 0$. Finally, applying this result to the sequence $\{s_n - s\}$ and its transform completes the proof of Theorem 5.

7. Application to Fourier series. Suppose that $f(x)$ is a Lebesgue integrable function of period 2π, and let

$$(7.1)\qquad f(x) \sim a_0/2 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \Sigma\, u_n(x)\,;$$

we may assume here $a_0 = 0$. Now (cf. [7, p. 27])

$$F(x) = \int_0^x f(t)\, dt = C + \sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx) \frac{1}{n}\,,$$

where

$$C = \sum_{n=1}^{\infty} \frac{1}{n} b_n\,.$$

It is known [7, p. 55] that at every point x where $F'(x)$ exists and is finite, the series (6.1) is summable (C, r), $r > 1$, to the value $F'(x)$.

It now follows from Theorem 3 for $k = 2$ and $t_n = \pi/n$ that if $n\rho_n^n = O(1)$, then

$$\sum_{\nu=1}^{n} u_\nu(x)\, \rho_n^\nu \frac{\sin \nu\pi/n}{\nu\pi/n} \longrightarrow F'(x)\,.$$

Furthermore, Theorem 4 yields, for $k = 2$, that if

$$nt_n = p\pi + \alpha_n\,, \qquad n\alpha_n = O(1)\,,$$

then

$$\sum_{\nu=1}^{n} u_\nu(x) \left(\frac{\sin \nu t_n}{\nu t_n}\right)^2 \longrightarrow F'(x)\,.$$

An analogous theorem holds for higher derivatives (cf. [7, p. 257]).

REFERENCES

1. St. Mazur, *Über lineare Limitierungsverfahren*, Math. Z. 28 (1928), 599-611.

2. N. Obreschkoff, *Über das Riemannsche Summierungsverfahren*, Math. Z. 48 (1942-43), 441-454.

3. R. Radó, *Some elementary Tauberian theorems (I)*, Quart. J. Math., Oxford Ser. 9 (1938), 274-282.

4. W. Rogosinski, *Abschnittsverhalten bei trigonometrischen und Fourierschen Reihen*, Math. Z. 41 (1936), 75-136.

5. Otto Szász, *Some new summability methods with applications*, Ann. of Math. 43 (1942), 69-83.

6. ———, *On some summability methods with triangular matrix*, Ann. of Math. 46 (1945), 567-577.

7. A. Zygmund, *Trigonometrical series*, Monografje Matematyczne, Warszawa-Lwow, 1935.

NATIONAL BUREAU OF STANDARDS, LOS ANGELES