COMPLETENESS OF SETS OF TRANSLATED COSINES

R. P. BOAS, JR.

1. Introduction. Conditions for the completeness on $(0,\pi)$ of sets $\{\cos \lambda_n x\}$ are well known. Here we shall consider sets $\{\cos(\lambda_n x + q_n)\}$. Such sets seem first to have been considered by Ditkin [3], who proved that $\{\cos(nx + q_n)\}_0^\infty$ is L-complete in $(0,\pi)$ if $0 \le q_n < \pi/2$.

Ditkin's very simple proof uses Fourier series and does not seem capable of extension to the more general sets considered here. Our principal object is to show how the problem may be attacked by complex-variable methods; we shall not attempt an exhaustive discussion.

As a specimen we quote the following case. If $\lambda_n \ge 0$ and $|\lambda_n - n| \le \delta < 1/2$, then the sets $\{\cos(\lambda_n x + q_n)\}_0^\infty$ and $\{\sin(\lambda_n x + q_n)\}_1^\infty$ are L-complete in $(0,\pi)$ if $\pi\delta/2 \le q_n < \pi(1-\delta)/2$. (The statement "$\{f_n(x)\}$ is L^p-complete" means that the only functions of L^p which are orthogonal to all $f_n(x)$ are almost everywhere zero.) A further result, not covered by the present paper, has been given by Bitsadze [1], who showed that every function satisfying a Hölder condition admits a uniformly convergent expansion in terms of the set $\{\cos(nx + \pi/4)\}$; he indicates an application of this result to the Tricomi partial differential equation.

We remark that although Ditkin's set $\{\cos(nx + q_n)\}_0^\infty$ remains complete when all $q_n = \pi/2$, it may fail to be complete if some but not all $q_n = \pi/2$. In fact, the set $\{1, \sin x, \cos 2x, \cos 3x, \cdots\}$ is orthogonal to $\cos x$. However, we shall show that not only is the set $\{\sin(nx + q_n)\}_0^\infty$ complete if $0 \le q_n < \pi/2$, but even the set $\{\sin(nx + q_n)\}_1^\infty$ is complete.

By applying the completeness theorem of Paley and Wiener [5, p. 100] to the equivalent set $\{\cos nx + a_n \sin nx\}$, $0 \le |a_n| < 1$, we can show at once that $\{\cos(nx + q_n)\}_0^\infty$ is L^2-complete if either $0 \le |q_n| \le \delta < \pi/4$ for all n or else $\pi/4 < \delta \le |q_n| \le \pi/2$ for all n. The problem of necessary and sufficient conditions for the completeness of $\{\cos(nx + q_n)\}$ remains open.

2. A general theorem. We shall obtain our results on $\{\cos(\lambda_n x + q_n)\}$ as

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corollaries of a theorem on a related set of more artificial appearance.

THEOREM. *Let* $\{\lambda_n\}_0^\infty$ *be an increasing unbounded sequence of nonnegative numbers; let* $N_1(r)$ *and* $N_2(r)$ *denote respectively the number of* λ_{2n} *and of* λ_{2n+1} *not exceeding* r. *If both*

$$\int_1^r t^{-1} N_1(t)\,dt > \frac{1}{2} r - \gamma \log r - \text{constant}, \tag{2.1}$$

and

$$\int_1^r t^{-1} N_2(t)\,dt > \frac{1}{2} r - \left(\gamma + \frac{1}{2}\right) \log r - \text{constant}, \tag{2.2}$$

where $\gamma = 1/(2p')$ *if* $1 \le p < \infty$, $p' = p/(p-1)$, *and* $\gamma < 1/2$ *if* $p = \infty$, *then the set*

$$\begin{cases} \cos \lambda_{2n} t + a_{2n} \sin \lambda_{2n} t, \\ -a_{2n+1} \cos \lambda_{2n+1} t + \sin \lambda_{2n+1} t \end{cases} \tag{2.3}$$

is L^p*-complete on* $(-\pi/2, \pi/2)$ *if the* a_n *are real numbers all of the same sign.*

COROLLARY 1. *The set* (2.3), *with the* a_n *all of the same sign, is* L^p*-complete on* $(-\pi/2, \pi/2)$ *if* $0 \le \lambda_n \le n + 1 + 1/p'$, $1 \le p < \infty$; *it is* L^∞*-complete if* $0 \le \lambda_n \le n + \delta$, $\delta < 2$.

COROLLARY 2. *If* $\lambda_n \ge 0$ *and*

$$|\lambda_n - n| \le \delta < \frac{1}{2}, \quad \frac{\pi\delta}{2} \le q_n < \frac{\pi(1-\delta)}{2},$$

then the set $\{\cos(\lambda_n x + q_n)\}_0^\infty$ *is L-complete on* $(0, \pi)$.

For $\delta = 0$, Corollary 2 reduces to Ditkin's theorem; for $\delta \ne 0$, the range of q_n is more restricted. If the λ_n are confined to one side of n, a sharper result is true.

COROLLARY 3. *If* $n \le \lambda_n \le n + \delta$, $0 \le \delta < 1$, *and* $0 \le q_n < \pi(1-\delta)/2$, $n \ge 0$; *or if* $n - \delta \le \lambda_n \le n$ *for* $n > 0$, $0 \le \delta < 1$, *and* $\pi(1-\delta)/2 < q_n \le 0$, *then* $\{\cos(\lambda_n x + q_n)\}_0^\infty$ *is L-complete in* $(0, \pi)$.

The following result on sets of sines includes the fact that $\{\sin(nx + q_n)\}_1^\infty$

is L-complete on $(0,\pi)$ if $0 \leq q_n < \pi/2$.

COROLLARY 4. *If* $|n+1-\lambda_n| \leq \delta < 1/2$ *and* $\pi\delta/2 \leq q_n < \pi(1-\delta)/2$, *then the set* $\{\sin(\lambda_n x + q_n)\}_0^\infty$ *is* L-*complete on* $(0,\pi)$.

By demanding only L^p-completeness instead of L-completeness, we can allow the λ_n to be larger than in Corollary 2.

COROLLARY 5. *If* $1 < p < \infty$ *and* $n+2-\delta < \lambda_n < n+2-1/p$, $1/p < \delta < 1$, *then the set* $\{\cos(\lambda_n x + q_n)\}_0^\infty$ *is* L^p-*complete on* $(0,\pi)$ *if* $\pi\delta/2 \leq q_n < \pi/2$.

3. Proof of the general theorem. We now prove the theorem stated above. We must show that if $f(x) \in L^p$ and if

$$\int_{-\pi/2}^{\pi/2} (\cos\lambda_{2n} t + a_{2n} \sin\lambda_{2n} t) f(t)\,dt \tag{3.1}$$

$$= \int_{-\pi/2}^{\pi/2} (-a_{2n+1}\cos\lambda_{2n+1} t + \sin\lambda_{2n+1} t) f(t)\,dt$$

$$= 0 \qquad (n = 0, 1, 2, \cdots),$$

where all a_n satisfy $a_n \geq 0$ or else all a_n satisfy $a_n \leq 0$, then $f(x) = 0$ almost everywhere.

Write

$$F(z) = \int_{-\pi/2}^{\pi/2} f(t)\cos zt\,dt, \qquad G(z) = \int_{-\pi/2}^{\pi/2} f(t)\sin zt\,dt; \tag{3.2}$$

then (3.1) is

$$F(\lambda_{2n}) + a_{2n} G(\lambda_{2n}) = 0, \tag{3.3}$$

$$-a_{2n+1} F(\lambda_{2n+1}) + G(\lambda_{2n+1}) = 0.$$

Let $H(z) = F(z)G(z)$; then $H(0) = 0$; if $\lambda_0 = 0$, then $H'(0) = H''(0) = 0$; and $H(\lambda_{2n})H(\lambda_{2n+1}) \leq 0$. Note that $H(z)$ is an odd function. Let $N(t) = N_1(t) + N_2(t)$, and let $\Lambda(t)$ denote the number of zeros of $H(z)$ in $0 \leq |z| \leq t$.

We prove first that

$$\Lambda(r) \geq 2N(r) + 1. \tag{3.4}$$

To begin with, if $\lambda_0 = 0$, we have, for $0 \leq r < \lambda_1$, the relations $N(t) = 1$, $\Lambda(r) \geq 3$; if $\lambda_0 > 0$, we have $N(t) = 0$ for $0 \leq r < \lambda_0$, $\Lambda(r) = 1$. We proceed by induction. Suppose that (3.4) is true for $r \leq \lambda_k$. Then it remains true for $r < \lambda_{k+1}$, since $N(r)$ does not change in $\lambda_k \leq r < \lambda_{k+1}$. If $H(\lambda_k)H(\lambda_{k+1}) \neq 0$, then $H(\lambda_k)$ and $H(\lambda_{k+1})$ have opposite signs and so $\Lambda(\lambda_{k+1}) \geq \Lambda(\lambda_k) + 2 \geq 2N(\lambda_k) + 3 = 2N(\lambda_{k+1}) + 1$, so that (3.4) is true for $r = \lambda_{k+1}$. If $H(\lambda_{k+1}) = 0$, then (3.4) is true for $r = \lambda_{k+1}$ since $\Lambda(r)$ increases by 2 at $r = \lambda_{k+1}$ while $N(r)$ increases by 1. Finally, suppose $H(\lambda_k) = 0, H(\lambda_{k+1}) \neq 0$. If $H(\lambda_j) = 0$ for $j = 0, 1, 2, \cdots, k$, then $\Lambda(\lambda_{k+1}) \geq \Lambda(\lambda_k) \geq 2k + 3 = 2N(\lambda_{k+1}) + 1$, and (3.4) is verified for $r = \lambda_{k+1}$. Otherwise there is a largest $j < k$ for which $H(\lambda_j) \neq 0$, and $\Lambda(\lambda_j) \geq 2N(\lambda_j) + 1$; there are at least $k - j$ zeros of $H(z)$ in $\lambda_j < x \leq \lambda_{k+1}$; but the number of zeros in this interval is even if $k - j + 1$ is even [since $H(\lambda_{k+1})$ and $H(\lambda_j)$ then have the same sign], odd if $k - j + 1$ is odd; so the number of zeros cannot be $k - j$ and hence must be at least $k - j + 1$. This completes the proof of (3.4).

By combining (3.4) with (2.1) and (2.2), we see that

$$\int_1^r t^{-1}\Lambda(t)\,dt > 2r - 4\gamma \log r - \text{constant}, \tag{3.5}$$

where $4\gamma = 2/p'$ if $1 \leq p < \infty$, $4\gamma < 2$ if $p = \infty$.

We now appeal to a modification of a result of Levinson [4, pp. 7-9] to show that $H(z) \equiv 0$. This is as follows.

LEMMA. *Let $\{x_n\}_{-\infty}^{\infty}$ be a sequence of real numbers arranged in nondecreasing order, and let $H(z)$ be an entire function which is known to vanish at all x_n; if $H(z)$ is known to have a multiple zero at some x_n, that x_n is to be repeated, according to its multiplicity, in the sequence. Let $\nu(r)$ denote the number of x_n such that $|x_n| \leq r$ and suppose that*

$$\int_1^r t^{-1}\nu(t)\,dt \geq 2r - \alpha \log r - \text{constant}.$$

Suppose finally that

$$|H(x+iy)| \leq \left\{\int_0^{\pi/2} h(t)\, e^{t|y|}\,dt\right\}^2,$$

where $h(t) \geq 0$, $h(t) \in L^p(0, \pi/2)$, $1 \leq p < \infty$. Then $H(z) \equiv 0$ if $\alpha \leq 2/p'$, $p' = p/(p-1)$. If $p = \infty$, then $H(z) \equiv 0$ if $\alpha < 2$.

The proof of the lemma is parallel to that given by Boas and Pollard [2] for a similar result, and we omit it.

Since $H(z) \equiv 0$, we have either $F(z) \equiv 0$ or $G(z) \equiv 0$. If $F(z) \equiv 0$, (3.3) shows that $G(\lambda_{2n+1}) = 0$; if $G(z) \equiv 0$, (3.3) shows that $F(\lambda_{2n}) = 0$.

We first consider the case when $F(z) \equiv 0$. Then, in particular, we have

$$\int_{-\pi/2}^{\pi/2} f(t)\,dt = 0,$$

and

$$\int_{-\pi/2}^{\pi/2} f(t) \cos \lambda_{2n+1} t\, dt = 0 \qquad (n = 0, 1, 2, \cdots),$$

$$\int_{-\pi/2}^{\pi/2} f(t) \sin \lambda_{2n+1} t\, dt = 0 \qquad (n = 0, 1, 2, \cdots);$$

hence

$$\int_{-\pi/2}^{\pi/2} f(t)\, e^{i\mu_n t}\, dt = 0 \qquad (n = 0, \pm 1, \pm 2, \cdots), \tag{3.6}$$

where

$$\mu_0 = 0, \quad \mu_n = \lambda_{2n-1}\ (n > 0), \quad \mu_n = -\lambda_{-2n-1}\ (n < 0). \tag{3.7}$$

A result of Levinson [4, p. 6], reduced to the interval $(-\pi/2, \pi/2)$, is that $\{e^{i\mu_n t}\}$ is L^p-complete if $M(t)$, the number of $|\mu_n| \le t$, satisfies

$$\int_1^r t^{-1} M(t)\,dt > r - (1/p') \log r - \text{constant}, \tag{3.8}$$

$1 \le p < \infty$; his proof also shows that L^∞-completeness follows from (3.8) if $1/p'$ is replaced by any number less than 1. Since $M(t) = 2N_2(t) + 1$, (3.8) is true in virtue of (2.2). Thus (2.2) implies $f(t) = 0$ almost everywhere if $F(z) \equiv 0$.

Now suppose that $G(z) \equiv 0$. In the same way we have

$$\int_{-\pi/2}^{\pi/2} f(t)\, e^{i\mu_n t}\, dt = 0,$$

where now

$$\mu_n = \lambda_{2n}\ (n \ge 0), \quad \mu_n = -\lambda_{-2n-2}\ (n < 0). \tag{3.9}$$

In this case $M(t) = 2N_1(t)$ and (3.8) follows from (2.1). The rest of the argument is as before.

4. Proof of Corollary 1. To prove Corollary 1 we have to show that (2.1) and (2.2) follow from $0 \le \lambda_n \le n + \delta$ $(n = 0, 1, 2, \cdots)$, where $\delta = 1 + 1/p'$, $1 \le p < \infty$. In the interval $2k + \delta \le u < 2k + \delta + 2$, where $k = 0, 1, 2, \cdots$, we have $N_1(u) \ge k + 1$. Let $x > 1$ and define n by $2n + \delta \le x < 2n + \delta + 2$. Then

$$\begin{aligned}
\int_\delta^x \frac{N_1(u)}{u}\,du &\ge \int_\delta^{2+\delta} \frac{du}{u} + \int_{2+\delta}^{4+\delta} \frac{2\,du}{u} + \cdots + \int_{2n-2+\delta}^{2n+\delta} \frac{n}{u}\,du \\
&= \sum_{k=1}^{n} k \log\left(1 + \frac{2}{2k+\delta-2}\right) \\
&\ge \sum_{k=1}^{n} k\left\{\frac{2}{2k+\delta-2} - \frac{1}{2}\left(\frac{2}{2k+\delta-2}\right)^2\right\} \\
&\ge \sum_{k=2}^{n} \left\{1 + \frac{2-\delta}{2k} - \frac{1}{2(k-1)}\right\} \\
&= n + \left(1 - \frac{1}{2}\delta - \frac{1}{2}\right) \log n + O(1) \\
&= \frac{1}{2}x + \frac{1-\delta}{2} \log x + O(1) = \frac{1}{2}x - \frac{1}{2p'} \log x + O(1).
\end{aligned}$$

On the other hand, in the interval $2k + 1 + \delta \le u < 2k + 3 + \delta$ $(k = 0, 1, 2, \cdots)$, we have $N_2(u) \ge k + 1$. Thus

$$\begin{aligned}
\int_1^x \frac{N_2(u)}{u}\,du &\ge \int_{1+\delta}^{3+\delta} \frac{du}{u} + \cdots + \int_{2n-1+\delta}^{2n+1+\delta} \frac{n}{u}\,du \\
&= \sum_{k=1}^{n} k \log\left(1 + \frac{2}{2k-1+\delta}\right) \\
&\ge \sum_{k=1}^{n} k\left\{\frac{2}{2k-1+\delta} - \frac{1}{2}\left(\frac{2}{2k-1+\delta}\right)^2\right\}
\end{aligned}$$

$$\geq \sum_{k=1}^{n} \left\{ 1 + \frac{1-\delta}{2k+1} - \frac{1}{2k-1} \right\}$$

$$= x + \frac{1}{2}(1-\delta) - \frac{1}{2} + O(1)$$

$$= x - \frac{1}{2p'} - \frac{1}{2} + O(1) .$$

5. Proof of Corollaries 2–5. In proving Corollaries 2–5, it is convenient to write $-a_n$ instead of a_n, and $t = x - \pi/2$, so that (2.3) becomes

$$\begin{cases} \cos(\lambda_n x - \lambda_n \pi/2) - a_n \sin(\lambda_n x - \lambda_n \pi/2) & (n \text{ even}) ; \\ a_n(\cos \lambda_n x - \lambda_n \pi/2) + \sin(\lambda_n x - \lambda_n \pi/2) & (n \text{ odd}) . \end{cases}$$

Put $a_n(1 + a_n^2)^{-1/2} = \sin b_n$, $(1 + a_n^2)^{-1/2} = \cos b_n$, $0 \leq b_n < \pi/2$ or $-\pi/2 < b_n \leq 0$, according as $a_n \geq 0$ or $a_n \leq 0$. Then the completeness of (2.3) is equivalent to that of

$$\begin{cases} \cos(\lambda_n x - \lambda_n \pi/2) \cos b_n - \sin(\lambda_n x - \lambda_n \pi/2) \sin b_n & (n \text{ even}) ; \\ \sin(\lambda_n x - \lambda_n \pi/2) \cos b_n + \cos(\lambda_n x - \lambda_n \pi/2) \sin b_n & (n \text{ odd}) ; \end{cases}$$

that is, to the completeness of

$$\begin{cases} \cos(\lambda_n x - \lambda_n \pi/2 + b_n) & (n \text{ even}) ; \\ \sin(\lambda_n x - \lambda_n \pi/2 + b_n) & (n \text{ odd}) . \end{cases}$$

Now let $\lambda_n = m - 2\epsilon_n/\pi$, where m is an integer of the same parity as n. Then the completeness of (2.3) is equivalent to that of

$$\cos(\lambda_n x + \epsilon_n + b_n) \qquad (n = 0, 1, 2, \cdots) . \tag{5.1}$$

Thus a set

$$\cos(\lambda_n x + q_n) \tag{5.2}$$

is equivalent to a set of the form (2.3) if for all n either

$$\epsilon_n \leq q_n < \pi/2 + \epsilon_n \tag{5.3}$$

or

$$-\pi/2 + \epsilon_n < q_n \le \epsilon_n . \tag{5.4}$$

We may satisfy (5.3) or (5.4) in various ways. For example, (5.3) is certainly true if $|n - \lambda_n| < \delta$ $(n = 0, 1, 2, \cdots)$, with $\delta < 1/2$ and $\pi\delta/2 \le q_n < \pi(1 - \delta)/2$; this establishes Corollary 2, since the condition of Corollary 1 is certainly satisfied in this case. Corollary 1 requires only that $\lambda_n \le n + 1$ if $p = 1$; if we restrict λ_n to lie always on one side of n we can therefore obtain a stronger result than Corollary 2. In fact, if $n \le \lambda_n \le n + 1$ we have $\epsilon_n \le 0$, and (5.3) is satisfied if $0 \le q_n < \pi/2 + \epsilon_n$, hence certainly if $n \le \lambda_n \le n + \delta$, $\delta < 1$, and $0 \le q_n < \pi(1 - \delta)/2$. On the other hand, if $n - 1 \le \lambda_n \le n$ $(n > 0)$, we have $\epsilon_n \ge 0$ and (5.4) is satisfied if $n - \delta \le \lambda_n < n$ $(n > 0)$, $\delta < 1$, and $-\pi(1 - \delta)/2 < q_n \le 0$.

If we let $\lambda_n = m - 2\epsilon_n/\pi$, where m has opposite parity to n, (2.3) reduces to $\{\sin(\lambda_n x + \epsilon_n + b_n)\}$; by taking $m = n + 1$ we obtain Corollary 4. Finally, Corollary 5 is obtained by taking $m = n + 2$. Further theorems of the same character are readily written down.

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NORTHWESTERN UNIVERSITY