INCIDENCE RELATIONS IN MULTICOHERENT SPACES III

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1. Introduction.

1.1. PRELIMINARIES. The present paper is concerned with relations between systems of sets and their frontiers in a locally connected space S of given degree of multicoherence, $r(S)$. The results are generalizations of those derived in [4] for the unicoherent case $[r(S) = 0]$, and of those in [5] for the case of two sets; the methods are those used in [5] and [6]. First we apply the "analytic" method (cf. [1; 2; 6]) to obtain a general "addition theorem" for arbitrary sets with "nearly disjoint" frontiers (Theorem 1), which is shown to be "best possible" (Theorem 2), and to derive also relations between arbitrary systems of sets and their frontiers (Theorems 3 and 4). Next (§4) we consider a function of sets which measures (roughly speaking) the amount of disconnectedness of the frontiers of the components of the complementary set, and, after deriving some of its properties, use it to extend the Phragmén-Brouwer theorem to arbitrary sets (Theorem 6), and to obtain some related results. A modified "addition theorem" is then established (Theorem 9) which includes both Theorem 1 and Theorem 6 as special cases. Finally, we consider the incidences of sets with disjoint frontiers and subject to further restrictions (for example, that the sets be connected and have connected complements), showing that many problems of this type can be reduced to purely combinatorial problems in graph-theory.

1.2. NOTATIONS. We shall be concerned throughout with subsets of a fixed *nonempty, connected, locally connected, completely normal*[1] T_1 *space*, S. The notations are, in general, the same as in [4; 5; 6]; but the following items are repeated for the convenience of the reader.

The number of components, less one, of a set E, is denoted by $b_0(E)$; thus $b_0(0) = -1$. If the number of components of E is infinite, we write $b_0(E) = \infty$, without distinction as to cardinality. The *degree of multicoherence* of S is defined by $r(S) = \sup b_0(A \cap B)$, where A and B are closed connected sets such that $A \cup B = S$. It is known [5] that "closed" can be replaced by "open" here.

If $A_1, A_2, \cdots, A_n$ are any n sets (that is, subsets of S), and J is any nonempty collection of distinct suffixes

[1] As was remarked in [5, §6.6(3)], there would be no difficulty in reformulating the theorems so as to apply if complete normality were weakened to normality.

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$$j_1, j_2, \cdots, j_k \qquad (1 \le j_1 < j_2 < \cdots < j_k \le n),$$

we write A_J as an abbreviation for $A_{j_1} \cap A_{j_2} \cap \cdots \cap A_{j_k}$, and write

$$\bigcup \{ A_J \mid |J| = k \} \text{ as } X_k (A_1, A_2, \cdots, A_n),$$

or simply as X_k. Thus

$$\bigcup A_j = X_1 \supset X_2 \supset \cdots \supset X_n = \bigcap A_j .$$

For convenience, we introduce the conventions $X_0 = S$ and $X_k = 0$ if $k > n$. We write

$$(1) \qquad h(A_1, A_2, \cdots, A_n) = \sum b_0 (X_k) - \sum b_0 (A_k) \qquad (1 \le k \le n),$$

with the convention that in interpreting an equality or inequality involving $h(A_1, \cdots, A_n)$ in which $\sum b_0 (A_k) = \infty$, we first transpose all negative terms. If the sets A_j are all closed, or all open, or more generally have separated differences[2], it is known [4, Th. 6b] that $h(A_1, \cdots, A_n) \ge 0$.

Again, following Eilenberg [1], we consider (continuous) mappings f of subsets of S into the circle S^1 of complex numbers of unit modulus, and write "$f \sim 1$ on X" to mean that there exists a real (continuous) function ϕ on X such that $f(x) = \exp [i\phi (x)]$ when $x \in X$. Mappings $f_1, f_2, \cdots, f_m$ of X in S^1 are *independent* on X if the only (positive or negative) integers $p_1, p_2, \cdots, p_m$, for which the product (in the sense of complex numbers)

$$f_1^{p_1} f_2^{p_2} \cdots f_m^{p_m} \sim 1 \text{ on } X,$$

are $p_1 = p_2 = \cdots = p_m = 0$. If $A_1, A_2, \cdots, A_n$ are closed sets whose union is X, the greatest number of mappings f of X in S^1 which are independent on X and such that $f \sim 1$ on each A_j (or ∞ if there is no such greatest number) is denoted by $p(A_1, A_2, \cdots, A_n)$. For fixed X and n, we write

$$(2) \qquad r_n (X) = \sup p (A_1, \cdots, A_n),$$

the supremum being taken over all systems of n closed sets $A_1, \cdots, A_n$ whose union is X. Clearly $0 = r_1(X) \le r_2(X) \le \cdots$; it is known [1] that

$$\sup_n r_n (X) = b_1 (X)$$

and [1; 6] that $r_2 (S) = r(S)$.

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[2] That is, $A_j - A_k$ and $A_k - A_j$ are separated ($1 \le j < k \le n$). (Two sets are "separated" if neither meets the closure of the other.)

1.3. SOME LEMMAS. We shall require the following lemmas, some of which are known; the proofs of the rest are easy.

(1) If $A_1, A_2, \cdots, A_n$ have separated differences, then

(i) $\bigcup \mathrm{Fr}(A_j) = \bigcup \mathrm{Fr}(X_j)$;

(ii) $A_j \cap A_k$ and $\mathrm{Co}(A_j \cup A_k)$ are separated $(1 \leq j < k \leq n)$ if and only if $\mathrm{Cl}(X_k) \subset X_j$ and $X_k \subset \mathrm{Int}(X_j)$;

(iii) $\mathrm{Fr}(A_j \cap A_k) \cap \mathrm{Fr}(A_j \cup A_k) = 0$ $(1 \leq j < k \leq n)$ if and only if $X_1, X_2, \cdots, X_n$ have disjoint[3] frontiers; that is, $\mathrm{Cl}(X_k) \subset \mathrm{Int}(X_j)$;

(iv) $A_1, A_2, \cdots, A_n$ are of finite incidence[4] if and only if

$$\sum b_0(X_j) < \infty.$$

(2) If A_1 and A_2 are both open, or both closed, then $A_1 - A_2$ and $A_2 - A_1$ are separated; and further, $A_1 \cap A_2$ and $\mathrm{Co}(A_1 \cap A_2)$ are separated if and only if $\mathrm{Fr}(A_1 \cap A_2) \cap \mathrm{Fr}(A_1 \cup A_2) = 0$. If A_1 and A_2 are open, this condition is equivalent to $\mathrm{Fr}(A_1) \cap \mathrm{Fr}(A_2) \cap \mathrm{Fr}(A_1 \cap A_2) = 0$.

(3) "Approximation lemma." If $A_j - A_k$ and $A_k - A_j$ are separated, and also $A_j \cap A_k$ and $\mathrm{Co}(A_j \cup A_k)$ are separated $(1 \leq j < k \leq n)$, then, given any open sets $W(J) \supset A_J$ (where J runs over all nonempty sets of suffixes between 1 and n), there exist open sets $A_j^* \supset A_j$ such that, for any open sets B_j satisfying $A_j \subset B_j \subset A_j^*$, we have $B_J \subset W(J)$ and

$$\mathrm{Fr}(B_j) \cap \mathrm{Fr}(B_k) \cap \mathrm{Fr}(B_j \cap B_k) = 0 \quad (1 \leq j < k \leq n).$$

If further $\mathrm{Fr}(A_j) \cap \mathrm{Fr}(A_k) \cap \mathrm{Fr}(A_j \cup A_k) = 0$ $(j \neq k)$, the sets A_j^* can be chosen so that the sets B_j have disjoint frontiers.

(If $n = 2$, this reduces to [5, Ths. 7 and 7a]; the general case follows by a straightforward induction over n.)

(4) If $A_1, A_2, \cdots, A_n$ are closed sets of finite incidence, then

$$p(A_1, A_2, \cdots, A_n) \leq h(A_1, A_2, \cdots, A_n);$$

if further no three of the sets A_j have a common point (for example, if $n = 2$), then $p = h$. (Cf. [6, §2.6].)

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[3] Throughout this paper, "disjoint" means "pairwise disjoint".

[4] That is, the sets A_j and all their intersections A_J have only finitely many components.

(5) If f maps X in S^1, and X is a finite union of disjoint closed sets on each of which $f \sim 1$, then $f \sim 1$ on X. (Trivial.)

(6) If f maps S in S^1, and $f \sim 1$ on a closed set $A \subset S$, then there exists an open set $V \supset A$ such that $f \sim 1$ on V. (Cf. [1, p. 157; 6, §2.2 (2)].)

(7) If f maps A in S^1, and $f \sim 1$ on $\mathrm{Fr}(A)$, then f may be extended to a mapping f^* of S in S^1 such that $f^* \sim 1$ on $\mathrm{Cl}(S - A)$.

For $f = \exp(i\phi)$ on $\mathrm{Fr}(A)$; since $\mathrm{Cl}(S - A)$ is normal, ϕ can be extended to a continuous real function ϕ^* on $\mathrm{Cl}(S - A)$; define $f^* = \exp(i\phi^*)$ on $\mathrm{Cl}(S - A)$, and $f^* = f$ elsewhere.

(8) *If $A_1, A_2, \cdots, A_n$ are n closed sets, and $1 \leq m \leq n$, then*

$$p(A_1, A_2, \cdots, A_n) \leq r_m(X_1) + r_{n+1-m}(X_m) \leq r_m(X_1) + b_1(X_m).$$

For consider N mappings $f_1, \cdots, f_N$ of X_1 ($= \cup A_j$) in S^1 which are independent on X_1 and satisfy $f_k \sim 1$ on A_j ($1 \leq k \leq N$, $1 \leq j \leq n$). We must prove

$$N \leq r_m(X_1) + r_{n+1-m}(X_m).$$

Let s be the greatest number of mappings f_k which are independent on X_m; since $X_m \subset A_1 \cup A_2 \cup \cdots \cup A_{n+1-m}$, clearly $s \leq r_{n+1-m}(X_m)$. We may suppose that the mappings f_k are independent on X_m for $N - s < k \leq N$, and then have, for each $k \leq N - s$, a relation of the form

$$g_k \equiv f_k^{p_k} \prod_{t > N - s} f_t^{q_{kt}} \sim 1$$

on X_m, where the exponents p_t, q_{kt} are integers not all zero, so that clearly $p_k \neq 0$. It readily follows that the mappings g_k ($1 \leq k \leq N - s$) of X_1 in S^1 are independent on X_1, and they clearly satisfy $g_k \sim 1$ on each A_j. Further, from (6) above, there exists an open set $V_m \supset X_m$ such that each $g_k \sim 1$ on $\mathrm{Cl}(V_m)$. Now $X_{m-1} - V_m$ is a finite union of disjoint closed sets of the form $A_J - V_m$ (where $|J| = k - 1$), on each of which each $g_k \sim 1$: hence, by (5), $g_k \sim 1$ on $X_{m-1} - V_m$, so that there exists an open set $V_{m-1} \supset X_{m-1} - V_m$ such that each $g_k \sim 1$ on $\mathrm{Cl}(V_{m-1})$. Proceeding in this way, we obtain open sets

$$V_\lambda \supset X_\lambda - (V_{\lambda+1} \cup V_{\lambda+2} \cup \cdots \cup V_m) \quad (1 \leq \lambda \leq m)$$

such that each $g_k \sim 1$ on $\mathrm{Cl}(V_\lambda)$. Since $\cup \mathrm{Cl}(V_\lambda) \supset X_1$, the number $N - s$ of mappings g_k is at most $p(\overline{V}_1, \overline{V}_2, \cdots, \overline{V}_m) \leq r_m(X_1)$, and the result follows.

As corollaries, we have:

(9) If, in the proof of (8), each of the mappings $f_k \sim 1$ on X_m, then $N \leq r_m(X_1)$.

(10) *If no* $m + 1$ *of the sets* A_j *in* (8) *can have a common point* (for example, if $m = n$), *then* $p(A_1, A_2, \cdots, A_n) \leq r_m(X_1)$.

For in this case, X_m falls into disjoint closed sets A_J, each contained in a single A_j; hence, from (5), each $f_k \sim 1$ on X_m.

2. An additional theorem.

2.1. INTRODUCTION. The last result, 1.3 (10), combined with 1.3 (4), gives another proof of the fact [6, Ths. 3 and 4a] that if $A_1, A_2, \cdots, A_n$ are closed sets which cover S, and no three of them have a common point, then

$$h(A_1, \cdots, A_n) \leq r(S).$$

In the present section we shall obtain a considerable extension of this property (Theorem 1), and show that it is the "best possible" of its kind, incidentally obtaining a new characterization of $r(S)$ (Theorem 2).

2.2. THEOREM 1. *Let* $A_1, A_2, \cdots, A_n$ *be any subsets of* S *having separated differences and such that* $A_j \cap A_k$ *and* $\mathrm{Co}(A_j \cup A_k)$ *are separated whenever* $j \neq k$.[5] *Suppose that no point belongs to* A_j *for more than m distinct values of j, where* $2 \leq m \leq n$.[6] *Then*

$$0 \leq h(A_1, A_2, \cdots, A_n) \leq (m-1)r(S).$$

Proof. Clearly we may assume that $r(S)$ and $b_0(A_j)$ are finite ($1 \leq j \leq n$); from [5, Th. 9], the sets A_j are then of finite incidence. Further, it will suffice to prove the theorem under the additional assumptions that the sets A_j are *closed* and have *disjoint frontiers*. For if the theorem is known in this case, the method of "approximation" extends it first [applying the second part of 1.3 (3) to the sets $\mathrm{Co}(A_j)$] to the case in which the sets A_j are open and satisfy

[5]These hypotheses are implied by: (a) the sets $\mathrm{Fr}(A_j)$ are disjoint, or (b) $A_1, \cdots, A_n$ are all open, or all closed, and $\mathrm{Fr}(A_j \cap A_k) \cap \mathrm{Fr}(A_j \cup A_k) = 0$ whenever $j \neq k$, or (c) $A_1, \cdots, A_n$ are all closed and $\mathrm{Fr}(A_j) \cap \mathrm{Fr}(A_k) \cap \mathrm{Fr}(A_j \cup A_k) = 0$ ($j \neq k$), or dually, and thus also by: (d) $A_1, \cdots, A_n$ are closed and cover S, and no three of them have a common point. A slight relaxation of the hypotheses on the sets A_j is possible; see 2.3 (3) below.

[6]The case $m = 1$ is trivial. If equality holds in the conclusion of Theorem 1, and both sides are finite, then the sets A_j must in fact satisfy stronger frontier conditions; see 5.6 below.

$$\mathrm{Fr}(A_j) \cap \mathrm{Fr}(A_k) \cap \mathrm{Fr}(A_j \cap A_k) = 0 \qquad (j \neq k),$$

and thence [by the first part of 1.3 (3)] to the general case; we omit the details, since the argument is a straightforward generalization of that in [5, §§7.4 and 7.5] (cf. also [6, §4.4]).

We write $X_s(A_1, A_2, \cdots, A_t)$ as X_s^t $(1 \leq s \leq t \leq n)$, and introduce the conventions $X_s^t = S$ if $1 \leq s \leq n < t$, or if $0 = s < t$, and $X_s^t = 0$ if $s > t$. Now (all the numbers involved being finite here) one readily verifies that

(1) $$h(A_1, A_2, \cdots, A_n) = h(A_1, A_2, \cdots, A_{n-1}) + \sum h(A_n \cap X_{s-1}^{n-1}, X_s^{n-1}) \quad (1 \leq s \leq n-1),$$

and repeated application of this identity gives

(2) $$h(A_1, A_2, \cdots, A_n) = \sum\nolimits_1 + \sum\nolimits_2 + \cdots + \sum\nolimits_{n-1},$$

where

$$\sum\nolimits_s = \sum\nolimits_t h(A_{t+1} \cap X_{s-1}^t, X_s^t) \qquad (s \leq t \leq n-1).$$

We first show that

(3) $$\sum\nolimits_s \leq r(S) \qquad (1 \leq s \leq n-1).$$

For, from 1.3 (4), we have

$$\sum\nolimits_s = \sum\nolimits_t p(A_{t+1} \cap X_{s-1}^t, X_s^t) \qquad (s \leq t \leq n-1).$$

Let f_{tj}, where $j = 1, 2, \cdots, n_t$, be mappings of

$$X_s^{t+1} = (A_{t+1} \cap X_{s-1}^t) \cup X_s^t$$

in the unit circle such that

(i) $f_{tj} \sim 1$ on $A_{t+1} \cap X_{s-1}^t$,

(ii) $f_{tj} \sim 1$ on X_s^t,

(iii) for fixed t, these mappings are independent on X_s^{t+1}.

To prove (3), it suffices to show that the total number $\sum n_t$

($s \leq t \leq n-1$) of these mappings is at most $r(S)$.

We have

$$\mathrm{Fr}[\mathrm{Cl}(S - X_s^{t+1})] \subset \mathrm{Fr}(X_s^{t+1}) \subset \mathrm{Fr}(A_{t+1} \cap X_{s-1}^t) \cup \mathrm{Fr}(X_s^t),$$

a union of two closed sets which are easily seen [from 1.3(1)] to be disjoint. Hence [from 1.3(5) and 1.3(7)] $f_{tj} \sim 1$ on $\mathrm{Fr}[\mathrm{Cl}(S - X_s^{t+1})]$, and so f_{tj} can be extended to a mapping, which we still denote by f_{tj}, of S in the unit circle, in such a way that

(iv) $f_{tj} \sim 1$ on $\mathrm{Cl}(S - X_s^{t+1})$.

We assert that the extended mappings f_{tj} are *all* independent on S. For suppose not; then, for each t, there exists a mapping of the form

$$g_t = \Pi_j f_{tj}^{p_{tj}} \qquad (1 \leq j \leq n_t).$$

where the exponents p_{tj} are positive or negative integers, not all zero for all t, such that

(4) $g_s g_{s+1} \cdots g_{n-1} \sim 1$ on S.

From (ii), we have $g_t \sim 1$ on X_s^t; and so, if $t > s$, we have $g_t \sim 1$ on X_s^{s+1}. Thus (4) gives $g_s \sim 1$ on X_s^{s+1}; hence, from (iii), it follows that $g_s = 1$, and all the exponents p_{sj} are zero. A similar argument, with s replaced by $s+1$, then proves $g_{s+1} = 1$, and so on; finally all the exponents p_{tj} must be zero, giving the desired contradiction.

Now write

$$E_k = \mathrm{Cl}(X_s^{s+k-1} - X_s^{s+k-2}), \qquad k = 1, 2, \cdots, n+2-s;$$

thus the sets E_k are closed and cover S, and it is easy to see that no three of them have a common point. We shall show:

(5) $f_{tj} \sim 1$ on E_k.

In fact, if $k \leq t+1-s$, then $E_k \subset X_s^{s+k-1} \subset X_s^t$; if $k = t+2-s$, then $E_k \subset A_{t+1} \cap X_{s-1}^t$; and if $k \geq t+3-s$, then $E_k \subset \mathrm{Cl}(S - X_s^{t+1})$; thus in each case (5) follows from (ii), (i), or (iv).

Thus the total number of mappings f_{tj} is at most

$$p(E_1, E_2, \cdots, E_{n+2-s});$$

but, by 1.3(10), this number is at most $r_2(\bigcup E_k) = r(S)$; thus (3) is established.

Now we further have $\sum_s = 0$ if $s \geq m$, since the sets $A_{t+1} \cap X^t_{s-1}$ and X^t_s are then disjoint (for $X_{s+1} = 0$). Thus the theorem follows from (2) and (3).

2.3. Corollaries and Remarks. We make the following observations.

(1) *For any two sets A, B, satisfying the hypotheses of Theorem* 1, *we have*

$$b_0(A) + b_0(B) \leq b_0(A \cup B) + b_0(A \cap B) \leq b_0(A) + b_0(B) + r(S).$$

(this generalizes [5, Th. 9].)

(2) *For any set E, we have*

$$b_0(\mathrm{Fr}(E)) \leq b_0(\overline{E}) + b_0(\mathrm{Cl}(\mathrm{Co}(E)) + r(S).$$

(this generalizes [4, §6.5].)

(3) *In Theorem* 1, *the hypothesis that $A_j \cap A_k$ and* $\mathrm{Co}(A_j \cup A_k)$ *be separated* $(j \neq k)$ *may be omitted for each pair j, k for which $A_j \subset A_k$*; that is, it may be replaced by: For each j, k $(1 \leq j, k \leq n)$, either $A_j \subset A_k$, or $A_j \supset A_k$, or $A_j \cap A_k$ and $\mathrm{Co}(A_j \cup A_k)$ are separated. This is proved by noting that a more careful application of the approximation argument will still lead to closed sets with disjoint frontiers.

(4) Other results may be derived by observing that, under suitable conditions on the sets $A_1, \cdots, A_n$, further sums $\sum_s$ in 2.2(1) above will vanish. For example, Theorem 1 can be slightly sharpened as follows:

If $A_1, \cdots, A_n$ satisfy the hypotheses of Theorem 1 *(as weakened in* (3) *above), and if they can be renumbered so that*

$$A_{\lambda+1} \subset A_{\lambda+2} \subset \cdots \subset A_n,$$

then

$$h(A_1, \cdots, A_n) \leq \min(\lambda, m-1)\, r(S).$$

For the approximation argument enables us to assume, as before, that the sets $A_1, \cdots, A_n$ are closed and have disjoint frontiers. In 2.2(2) we easily verify that now $X^t_s \subset A_{t+1} \cap X^t_{s-1}$ whenever

$$\lambda + 1 \leq s \leq t \leq n - 1;$$

hence $\sum_s = 0$ whenever $s > \lambda$.

(5) A further slight sharpening of Theorem 1 is implied by the following result.

If the sets $A_1, \cdots, A_n$ *have separated differences, and if* A_n *(say) is either disjoint from, or contains, or is contained in, each other set, then*

$$h(A_1, \cdots, A_{n-1}, A_n) = h(A_1, \cdots, A_{n-1}).$$

We may assume that A_n is disjoint from $A_1, \cdots, A_k$, contains

$$A_{k+1}, \cdots, A_l,$$

and is contained in $A_{l+1}, \cdots, A_{n-1}$ (where $0 \leq k \leq l \leq n-1$). It is easy to see that we may take $A_1, \cdots, A_n$ to be of finite incidence, and then, by 2.2(1), have only to prove that

$$h(A_n \cap X_{s-1}^{n-1}, X_s^{n-1}) = 0 \qquad (1 \leq s \leq n-1).$$

If $s < n - l$, then $A_n \subset X_s^{n-1}$, and the result is trivial. If $s \geq n - l$, write

$$Y_p = X_p(A_1, \cdots, A_k, A_{l+1}, \cdots, A_{n-1})$$

and

$$Z_q = Z_q(A_{k+1}, \cdots, A_l, A_{l+1}, \cdots, A_{n-1});$$

it is easily verified that $X_s^{n-1} \doteqdot Y_s \cup Z_s$ and that $Y_s \subset \mathrm{Co}(A_n)$ and $Z_s \subset A_n$, from which again the result follows.

(6) Finally, as a corollary from (4), we have the following extension of (1):

If $B_1, \cdots, B_p, C_1, \cdots, C_q$ *are arbitrary sets such that* $B_j - C_k$ *and* $C_k - B_j$ *are separated, and* $B_j \cap C_k$ *and* $\mathrm{Co}(B_j \cup C_k)$ *are separated, whenever* $1 \leq j \leq p$, $1 \leq k \leq q$, *then*

$$h(B_1, \cdots, B_p) + h(C_1, \cdots, C_q) \leq h(B_1, \cdots, B_p, C_1, \cdots, C_q)$$

$$\leq h(B_1, \cdots, B_p) + h(C_1, \cdots, C_q) + \min(p, q, m-1)\, r(S),$$

where m *is the greatest number of the* $p + q$ *sets* $B_1, \cdots, C_q$ *which have a common point.*

This follows on application of (4) and Theorem 1 to the $p + q$ sets $X_j(B_1, \cdots, B_p)$, $X_k(C_1, \cdots, C_q)$.

2.4. CONVERSE. The converse of Theorem 1 holds in the following rather strong form, which represents an extension to any number of sets of the defining property of $r(S)$.

THEOREM 2. *Let integers m, n be given, where $2 \leq m \leq n$. Let $A_1, \cdots, A_n$ be any n closed connected sets, no $m + 1$ of which have a common point, such that $\mathrm{Fr}(A_j) \cap \mathrm{Fr}(A_k) = 0$ whenever $j \neq k$, and such that $A_j \cup A_k = S$ whenever $1 \leq j < k \leq m$.*[7] *Then*

$$\sup b_0(X_m) = (m - 1)\, r(S) + n - m.$$

In this statement, the word "closed" may be replaced by "open".

To show that

(1) $b_0(X_m) \leq (m - 1)\, r(S) + n - m,$

we clearly may assume $X_m \neq 0$; then $b_0(X_s) \geq 0$ if $s \leq m$, and

$$b_0(X_s) = -1$$

for $m < s \leq n$, so that (1) is a trivial consequence of Theorem 1.

To complete the proof, let N be any integer such that

$$0 \leq N \leq (m - 1)\, r(S).$$

We first construct m closed connected sets $B_1, B_2, \cdots, B_m$, such that

(2) $B_j \cup B_k = S \ (1 \leq j < k \leq m)$ and $b_0(\bigcap B_j) \geq N$.

If $r(S) = \infty$, this is trivial (take all but two of the sets B_j to be S), so we may assume $r(S) < \infty$. From [6, §4.1], there exists a finite covering of S by closed connected sets $E_1, E_2, \cdots, E_M$, no three of which have a common point, whose nerve G satisfies $r(G) = r(S) = r$, say, and such that G is arbitrarily often "dispersed"; this implies [6, §3.4(7)] that G is obtainable from a graph H by subdividing each arc l_λ of H which belongs to a simple closed curve in H, into at least $2m + 2$ subarcs by extra vertices of order 2. We can select[8] r such (disjoint, open)

[7] Note that we do not require *every* two sets A_j, A_k to cover S. In fact, if n nonempty closed sets are such that every two of them cover S, then trivially all of them have a common point.

[8] See, for example, the argument proving [6, §4.1(3)].

arcs l_λ in H, say $l_1, l_2, \cdots, l_r$, whose removal does not disconnect H; let l_λ (where $1 \leq \lambda \leq r$) contain the consecutive vertices $p_{\lambda,0}, p_{\lambda,1}, p_{\lambda,2}, \cdots, p_{\lambda,2m}$ of order 2 in G. Denote by $E_{\lambda,j}$ the set E_k which corresponds to $p_{\lambda,j}$; thus, if $1 \leq \lambda \leq r$ and $1 \leq j \leq 2m-1$, each $E_{\lambda,j}$ meets two and only two other sets E_k, namely $E_{\lambda,j-1}$ and $E_{\lambda,j+1}$. Define B_q, where $1 \leq q \leq m$, to be the union of all the sets E_k *except*

$$E_{1,2q-1}, E_{2,2q-1}, \cdots, E_{r,2q-1}.$$

Then B_q is closed, and is easily seen to be connected (cf. [6, Th. 1]). Further, since $\mathrm{Co}(B_q) \subset \bigcup_\lambda E_{\lambda,2q-1}$, we have $\mathrm{Co}(B_q) \cap \mathrm{Co}(B_s) = 0$ if $q \neq s$, so that $B_q \cup B_s = S$. On the other hand, let D be the union of those sets E_k which are not of the form $E_{\lambda,j}$ $(1 \leq \lambda \leq r, \quad 1 \leq j \leq 2m-1)$; then

$$\bigcap B_q \subset D \cup \bigcup E_{\lambda,2h} \qquad (1 \leq \lambda \leq r,\ 1 \leq h \leq m-1),$$

a union of $1 + (m-1)r$ disjoint closed sets, each of which it meets; thus $b_0(\bigcap B_q) \geq (m-1)r \geq N$.

There exist (cf. 1.3(3) and [6, §6.1]) connected open sets $C_q \supset B_q$ whose closures A_q have the same incidences as the sets B_q; then

$$\mathrm{Fr}(A_j) \cap \mathrm{Fr}(A_k) \subset \mathrm{Fr}(C_j) \cap \mathrm{Fr}(C_k) \subset \mathrm{Co}(C_j \cup C_k) = 0$$

whenever $j \neq k$, and moreover we have $A_j \cup A_k = S$ $(1 \leq j < k \leq m)$ and $b_0(\bigcap A_j) \geq N$.

If $n = m$, the theorem is thus established. If $n > m$, we note that the open set $\mathrm{Int}\,[X_{m-1}(A_1, \cdots, A_m)] - X_m(A_1, \cdots, A_m)$ is nonempty, from 1.3(1), and take $A_{m+1}, \cdots, A_n$ to be $n - m$ distinct points in it; clearly

$$b_0[X_m(A_1, \cdots, A_m, \cdots, A_n)] \geq N + n - m,$$

and the proof is complete.

The modifications required to produce *open* sets A_j with similar properties are obvious.

3. Index inequalities for arbitrary sets.

3.1. AN INEQUALITY. Let $E_1, E_2, \cdots, E_n$ be arbitrary subsets of S. As in [4, §7], we write

$$A_j = \mathrm{Cl}(E_j),\ B_j = \mathrm{Cl}(S - E_j),\ P_j = X_j(A_1, \cdots, A_n),\ Q_j = X_j(B_1, \cdots, B_n).$$

An argument entirely analogous to that in [4, §7], based on 2.3 (1) and (2), gives:

THEOREM 3. *We have*

$$\begin{aligned} h[\mathrm{Fr}(E_1), \cdots, \mathrm{Fr}(E_n)] - nr(S) &\leq h(A_1, \cdots, A_n) + h(B_1, \cdots, B_n) \\ &\quad + h(P_1 \cap Q_n, P_2 \cap Q_{n-1}, \cdots, P_n \cap Q_1) \\ &\leq h[\mathrm{Fr}(E_1), \cdots, \mathrm{Fr}(E_n)] + nr(S). \end{aligned}$$

COROLLARY. We have

$$h(\overline{E}_1, \overline{E}_2, \cdots, \overline{E}_n) \leq h[\mathrm{Fr}(E_1), \mathrm{Fr}(E_2), \cdots, \mathrm{Fr}(E_n)] + nr(S).$$

3.2. THE CASE $m = 2$. It is easy to see that the inequalities in Theorem 3 are "best possible"; however, Theorem 1 suggests that in the Corollary the term $nr(S)$ could be replaced by $(n-1)\,r(S)$, or more generally by $(m-1)\,r(S)$, where no $m+1$ of the sets $\mathrm{Cl}(E_j)$ have a common point. I have been able to prove this only in the case $m = 2$:

THEOREM 4. *If $E_1, E_2, \cdots, E_n$ are arbitrary subsets of S, no three of whose closures have a common point, then*

$$h(\overline{E}_1, \overline{E}_2, \cdots, \overline{E}_n) \leq h[\mathrm{Fr}(E_1), \mathrm{Fr}(E_2), \cdots, \mathrm{Fr}(E_n)] + r(S).$$

Proof. We can assume that $r(S)$ is finite, and that the systems of sets $[\mathrm{Cl}(E_1), \cdots, \mathrm{Cl}(E_n)]$ and $[\mathrm{Fr}(E_1), \cdots, \mathrm{Fr}(E_n)]$ are both of finite incidence, since otherwise (in view of the convention regarding infinite terms in the h-function; see 1.2) Theorem 4 asserts no more than Theorem 3, Corollary. Hence, in view of 1.3(4), Theorem 4 will follow [if we take $A_j = \mathrm{Cl}(E_j)$ and $F_j = \mathrm{Fr}(E_j)$] from:

THEOREM 4a. *Let $A_1, A_2, \cdots, A_n, F_1, F_2, \cdots, F_n$ be any closed sets such that $A_j \supset F_j$ and $\bigcup F_j \supset \bigcup \mathrm{Fr}(A_j)$. Then*

$$\mathrm{p}(A_1, A_2, \cdots, A_n) \leq \mathrm{p}(F_1, F_2, \cdots, F_n) + r(S).$$

3.3. PROOF OF THEOREM 4a. Let $f_1, f_2, \cdots, f_N$ be N independent mappings of $\bigcup A_j$ in the unit circle such that each $f_k \sim 1$ on each A_j; we must prove that $N \leq \mathrm{p}(F_1, \cdots, F_n) + r(S)$. Let s be the greatest number of mappings f_t which are independent on $\bigcup F_j$: clearly $s \leq \mathrm{p}(F_1, \cdots, F_n)$. We may suppose that the mappings f_k are independent on $\bigcup F_j$ for $N - s + 1 \leq k \leq N$, and then have, for each $t \leq N - s$, a relation (say)

$$g_t \equiv f_t^{p_t} \prod_k f_k^{q_{kt}} \sim 1$$

on $\bigcup F_j$, where $N - s + 1 \leq k \leq N$. Thus g_t is a mapping of $\bigcup A_j$ in S^1 which ~ 1 on each A_j ; and, since clearly $p_t \neq 0$, the mappings g_t $(1 \leq t \leq N - s)$ are independent on $\bigcup A_j$.

Write $C_0 = \mathrm{Cl}(S - \bigcup A_j)$; then $\mathrm{Fr}(C_0) \subset \bigcup F_j$, so that, from 1.3(7), each g_t may be extended to a mapping (still denoted by g_t) of S in S^1 such that $g_t \sim 1$ on C_0. Now define $C_1 = A_1$, $C_j = \mathrm{Cl}[A_j - (A_1 \cup A_2 \cup \cdots \cup A_{j-1})]$ $(2 \leq j \leq n)$; then the sets $C_0, C_1, \cdots, C_n$ are closed and cover S, and each $g_t \sim 1$ on each C_j. Let $Z = \bigcup(C_j \cap C_k)$, where $0 \leq j < k \leq n$; then $Z \subset \bigcup \mathrm{Fr}(A_j) \subset \bigcup F_j$, so that each $g_t \sim 1$ on Z. From 1.3(9), the number $N - s$ of mappings g_t is at most $r(S)$, and the theorem follows.

3.4. REMARK. We remark that no inequality similar to Theorem 4, but in the reverse direction, can hold in general. For example, take S to be the plane, and let A be a circular disc and B an inscribed convex polygon plus its interior; then A, B are closed and connected, and $h(A, B) = 0$, but $h[\mathrm{Fr}(A), \mathrm{Fr}(B)]$ can be arbitrarily large.

4. Frontiers of complementary components.

4.1. DEFINITION. For any $A \subset S$, let $\{C_\lambda\}$ be the components of the *complement* of A, and write

(1) $c(A) = \sum b_0(\mathrm{Fr}(C_\lambda))$,

with the usual convention that a vacuous sum is zero. [Thus $c(S) = 0$, $c(0) = -1$.] From [5, Th. 4] we have

(2) $c(A) + b_0[\mathrm{Cl}(S - A)] \geq b_0[\mathrm{Fr}(A)]$,

and (a weaker statement unless $b_0[\mathrm{Cl}(S - A)]$ is infinite)

(3) $c(A) \geq b_0(\overline{A})$.

If A is open, we evidently have equality in (2). (Note that (3) contains the well-known fact that, if $\overline{A}$ is not connected, at least one component of $\mathrm{Co}(A)$ has a disconnected frontier.)

4.2 LEMMA. *Let C be a component of $S - A$, and let U be an open set containing* $\mathrm{Fr}(C)$. *Then there exists an open set $V \supset \overline{A}$ such that $\overline{V} \cap \overline{C} \subset U$.*

This follows from [6, §6.1] applied to the sets $\bar{A}$, $\bar{C}$; a direct proof is also easy.

4.3 THEOREM 5. *If $c(A) \geq n$, then there exists an open set $A^* \supset \bar{A}$ such that, for each set B satisfying $A \subset B \subset A^*$, we have $c(B) \geq n$.*

For if $c(A) \geq n$, then there exist finitely many components, say C_1, C_2, $\cdots$, C_m, of $\mathrm{Co}(A)$, such that $b_0[\mathrm{Fr}(C_j)] \geq n_j$ where $\sum n_j \geq n$ $(1 \leq j \leq m)$. Thus, for each j, $\mathrm{Fr}(C_j)$ is a union of $n_j + 1$ disjoint closed nonempty sets F_{jk} $(1 \leq k \leq n_j + 1)$, and there exist open sets $U_{jk} \supset F_{jk}$ such that $\mathrm{Cl}(U_{jk}) \cap \mathrm{Cl}(U_{jl}) = 0$ $(j \neq l)$. Let $U_j = \bigcup_k U_{jk}$, an open set containing $\mathrm{Fr}(C_j)$; from the lemma in 4.2, there exists an open set $V_j \supset \bar{A}$ such that $\mathrm{Cl}(V_j) \cap \mathrm{Cl}(C_j) \subset U_j$. Take $A^* = \bigcap_j V_j$, and suppose that B is any set satisfying $A \subset B \subset A^*$. Then, since $\bigcup_k F_{jk} \subset \bar{B} \cap \bar{C}_j \subset \bigcup_k U_{jk}$, we have $b_0(\bar{B} \cap \bar{C}_j) \geq n_j$. Now let $\{D_{j\mu}\}$ be those components of $\mathrm{Co}(B)$ which are contained in C_j, and write $E_j = \bigcup_\mu D_{j\mu}$. One readily verifies that $\mathrm{Fr}(E_j) \subset \bar{B} \cap \bar{C}_j \subset \mathrm{Cl}(S - E_j)$, and that $E_j \cup (\bar{B} \cap \bar{C}_j) = \bar{C}_j$; hence, from [5, Th. 4], $\sum_\mu b_0(\mathrm{Fr}(D_{j\mu})) \geq b_0(\bar{B} \cap \bar{C}_j) \geq n_j$, so that $c(B) \geq \sum_{j,\mu} b_0(\mathrm{Fr}(D_{j\mu})) \geq \sum n_j \geq n$.

COROLLARY. *We have $c(A) \leq c(\bar{A})$.*

4.4. EXTENSION OF THE PHRAGMÉN-BROUWER THEOREM. This theorem, as extended in [5, Th. 5], can now be extended still further.

THEOREM 6. *For any set A, we have $c(A) \leq b_0(\bar{A}) + r(S)$.*

The proof is almost identical with that for the case in which $\bar{A}$ is connected, in [5, §4.2]; the difference arises from the fact that the sets L, M there constructed need not here be connected. But we may assume without loss that $b_0(\bar{A}) < \infty$, and have $b_0(L) \leq b_0(\bar{A})$ and $b_0(M) \leq b_0(\bar{A})$; hence, from 2.3 (1), we have $b_0(L \cap M) < 2b_0(\bar{A}) + r(S)$. Since $b_0(\bar{A}) + 1$ of the components of $L \cap M$ now arise from $\bar{A}$, the argument can be concluded in the same way as before.

COROLLARY 1. *If $r(S)$ is finite, and A is any subset of S such that $\bar{A}$ has only a finite number of components, then all but at most a finite number of the components of $S - A$ have connected frontiers.*

COROLLARY 2. *If S is unicoherent, then $c(A) = b_0(\bar{A})$; and, conversely, this equality is characteristic of unicoherence.*

(This follows from 4.1 (2) and [5, Th. 5].)

4.5. ANOTHER EXTENSION. It has been shown in [5, Th. 5] that, conversely, Theorem 6 serves to characterize $r(S)$, even when restricted to the case in which A is closed (or open) and connected. However, Theorem 6 can be restated in a slightly different though equally natural way, in which the converse question is more difficult.

THEOREM 6a. *For any set A, we have*

$$\text{(i)}\quad b_0(\mathrm{Fr}(A)) \le c(A) + b_0(\mathrm{Cl}(S-A)) \le c(\bar{A}) + b_0(\mathrm{Cl}(S-A)) \le b_0(\mathrm{Fr}(A)) + r(S).$$

Conversely, if for some fixed (finite) n we have

$$\text{(ii)}\quad c(A) \le b_0(\mathrm{Fr}(A)) + n$$

whenever A is nowhere dense, and if further

(iii) *S is metrizable, or $r(S)$ is finite,*

then $r(S) \le n$.

The first inequality in (i) is a restatement of 4.1 (2), the second follows from Theorem 5, Corollary, and the third from Theorem 6 applied to $\bar{A}$, in view of the fact [4, §6.2] that $b_0(\bar{A}) + b_0(\mathrm{Cl}(S-A)) \le b_0(\mathrm{Fr}(A))$. For the converse, suppose that (ii) holds, but that $r(S) > n$. From [5, Th. 5a), there exists a closed connected set A' such that $S - A'$ has only a finite number of (open) components $C_1, C_2, \cdots, C_m$, and $b_0(\mathrm{Fr}(A')) > m + n - 1$; thus from [5, Th. 4], we have $\sum b_0(\mathrm{Fr}(C_j)) > n$. Suppose now that $r(S)$ is finite, and write $A = \mathrm{Fr}(A')$; thus A is nowhere dense, and, from 2.3 (2), $b_0(A) < \infty$. Let $\{D_\lambda\}$ be the components of $\mathrm{Int}(A')$; then [5, Th. 4] we have $\sum b_0(\mathrm{Fr}(D_\lambda)] \ge b_0(A) = b_0[\mathrm{Fr}(A)]$. But the components of $\mathrm{Co}(A)$ are precisely the sets C_j, D_λ; hence $c(A) > b_0[\mathrm{Fr}(A)] + n$, contradicting (ii).

If $r(S) = \infty$, the above argument still applies provided that $b_0[\mathrm{Fr}(A')] < \infty$. Hence we may assume $b_0[\mathrm{Fr}(A')] = \infty$, so that there must exist some C_j, say C, for which $b_0[\mathrm{Fr}(C)] = \infty$. Now, the complement (say) F of C is closed and connected. If it is assumed that S is metrizable, then there exists a sequence of open sets G_n such that $G_n \supset \mathrm{Cl}(G_{n+1})$ $(n = 1, 2, \cdots)$, and $\bigcap G_n = F$. Let $X = C - \bigcup \mathrm{Fr}(G_n)$; from a theorem of Hewitt [3], there exist disjoint sets Y, Z such that $Y \cup Z = X$ and $\bar{Y} = \bar{Z} = \bar{X} = \bar{C}$. We take $A = C - Y$. Thus $\mathrm{Cl}(S-A) = S$; and $\mathrm{Fr}(A) = \bar{C}$, which is connected. But $\mathrm{Co}(A)$ can be separated, by one of the sets $\mathrm{Fr}(G_n)$, between F and any given point of Y; thus one of the components of $\mathrm{Co}(A)$ is F itself, and again (ii) is contradicted.

COROLLARY. *If S is unicoherent, and $\{C_\lambda\}$ are the components of an* *arbitrary set E, then*

$$b_0[\mathrm{Fr}(C_\lambda)] + b_0(\bar{E}) = b_0[\mathrm{Fr}(E)];$$

and this property characterizes unicoherence among metrizable (locally connected and connected) spaces.

It would be interesting to know whether the extra hypotheses on S imposed in (iii) are needed. It would be easy to replace them by others (for example, local compactness plus perfect normality).

5. Modified addition theorems.

5.1. A MODIFICATION. As in the case of two connected sets [5, Ths. 11 and 11a], special cases of Theorem 1 can be obtained under alternative hypotheses. As an example, we state:

THEOREM 7. *If A and B are any sets satisfying*

$$\mathrm{Fr}(A) \cap \mathrm{Fr}(B) \cap \mathrm{Fr}(A \cap B) = 0,$$

then

$$b_0(A \cup B) + b_0(A \cap B) \leq b_0(A) + b_0(B) + r(S);$$

and if there is finite equality here, then $A - B$ and $B - A$ are separated (so that Theorem 1 then in fact applies).[9]

The proof is a fairly straightforward generalization of that of [5, Th. 11], with 2.3(1) replacing [5, §7.4]. The extension of Theorem 7 to n sets, however, appears to present some difficulty.

5.2. ANOTHER MODIFICATION. A more interesting modification of Theorem 1 is the following, in which $r(S)$ does not enter explicitly; in some cases (in view of Theorem 6) it gives more information than does Theorem 1.

THEOREM 8. *If A and B are arbitrary sets such that*

$$\mathrm{Fr}(A) \cap \mathrm{Fr}(B) \cap \mathrm{Fr}(A \cup B) = 0,$$

then

$$h(\bar{A}, \bar{B}) + b_0(\bar{A}) \leq c(A).$$

[9] It follows (see 5.6 below) that, in the case of finite equality, we have for each component E of $A \cup B$ that $\mathrm{Fr}(A \cap E) \cap \mathrm{Fr}(B \cap E) = 0$. It is false, in general, that $\mathrm{Fr}(A) \cap \mathrm{Fr}(B) = 0$.

Proof. Write $C = \mathrm{Cl}[\mathrm{Co}(A)]$, and apply [4, Th. 6b] to the closed sets $\bar{A} \cup \bar{B}$, $\bar{A} \cap \bar{B}$, C. We obtain

(1) $b_0(\bar{A} \cup \bar{B}) + b_0(\bar{A} \cap \bar{B}) + b_0(C) \leq b_0[\bar{B} \cup \mathrm{Fr}(A)] + b_0[\bar{B} \cap \mathrm{Fr}(A)]$.

From the frontier relation satisfied by the sets A and B, it readily follows that $\mathrm{Fr}(A) \cap \mathrm{Co}(\bar{B})$ is closed, and thence that each component of $\mathrm{Fr}(A)$ which meets $\bar{B}$ is contained in $\bar{B}$. Hence we see that

$$b_0[\bar{B} \cup \mathrm{Fr}(A)] + b_0[\bar{B} \cap \mathrm{Fr}(A)] = b_0(\bar{B}) + b_0[\mathrm{Fr}(A)],$$

and consequently

(2) $b_0(\bar{A} \cup \bar{B}) + b_0(\bar{A} \cap \bar{B}) + b_0(C) \leq b_0(\bar{B}) + b_0[\mathrm{Fr}(A)]$.

But by 4.1 (2), we have $b_0[\mathrm{Fr}(A)] \leq b_0(C) + c(A)$. Thus, *provided that $b_0(C)$ is finite*, we have proved

(3) $b_0(\bar{A} \cup \bar{B}) + b_0(\bar{A} \cap \bar{B}) \leq b_0(\bar{B}) + c(A)$,

from which the theorem follows immediately.

To complete the proof, we deduce that (3) continues to hold even when $b_0(C) = \infty$; and in doing so, we may assume that $b_0(\bar{B}) + c(A) < \infty$. Define B^* to be the union of those components of $\bar{B}$ which meet $\bar{A}$, and A^* to be the union of A with all components of $\mathrm{Co}(A)$ which have connected frontiers. It is easy to verify that

$$\mathrm{Fr}(A^*) \cap \mathrm{Fr}(B^*) \cap \mathrm{Fr}(A^* \cap B^*) = 0,$$

and that, since $c(A) < \infty$, $b_0[\mathrm{Co}(A^*)]$ is finite. Hence (3) holds for the sets A^*, B^*; and it is a routine matter to deduce that (3) also holds for A and B^*, and thence finally for A and B.

There is no difficulty in extending this theorem to any number of sets; for example, (2) can be extended to the following property, valid in an arbitrary topological space S (and generalizing [4, §7.4 (1)]):

(4) If $A_1, \cdots, A_m, B_1, \cdots, B_n$ are arbitrary sets such that

$$\mathrm{Fr}(A_j) \cap \mathrm{Fr}(B_k) \cap \mathrm{Fr}(A_j \cup B_k) = 0 \qquad (1 \leq j \leq m, 1 \leq k \leq n),$$

and $C_j = \mathrm{Cl}[\mathrm{Co}(A_j)]$, then

$$\sum b_0[X_h(\overline{A}_1, \cdots, \overline{A}_m, \overline{B}_1, \cdots, \overline{B}_n)] + \sum b_0[X_j(C_1, \cdots, C_m)]$$
$$\leq \sum b_0[X_j[\mathrm{Fr}(A_1), \cdots, \mathrm{Fr}(A_m)]\}$$
$$+ \sum b_0[X_k(\overline{B}_1, \cdots, \overline{B}_n)] + mb_0(S),$$

the ranges of summation being $1 \leq h \leq m+n$, $1 \leq j \leq m$, $1 \leq k \leq n$; and (3) can be extended similarly.

5.3. AN INCLUSIVE RESULT. The next theorem includes both Theorem 1 and the extended Phragmén-Brouwer theorem (Theorem 6) as special cases. We shall need the following lemma.

LEMMA. *If G is a set with only finitely many components, then there exists a finite set of points $x_1, x_2, \cdots, x_q \in \mathrm{Fr}(G)$ such that*

$$b_0[G \cup (x_1) \cup \cdots \cup (x_q)] = b_0(\overline{G}).$$

For if G has components $G_1, G_2, \cdots, G_s$, we have only to take at least one point x_j in every nonempty set $\overline{G}_\lambda \cap \overline{G}_\mu$ $(\lambda \neq \mu)$.

5.4. THEOREM 9. *Let $A_1, A_2, \cdots, A_n$ be any subsets of S having separated differences and such that $A_j \cap A_k$ and $\mathrm{Co}(A_j \cup A_k)$ are separated whenever $j \neq k$; and suppose that no point belongs to A_j for more than m values of j, where $2 \leq m \leq n$. Then*

(1) $h(A_1, \cdots, A_n) + c(\overline{X}_1) + c(\overline{X}_2) + \cdots + c(\overline{X}_{m-1})$

$$\leq b_0(\overline{X}_1) + b_0(\overline{X}_2) + \cdots + b_0(\overline{X}_{m-1}) + (m-1)\, r(S),$$

where $X_j = X_j(A_1, \cdots, A_n)$. Further, if there is finite equality in (1), *then, for each $q \leq n-1$, for each set J of $q+1$ distinct suffixes $j_1, j_2, \cdots, j_{q+1}$ between 1 and n, and for each component E of X_q, we have*

(2) $\bigcap \{\mathrm{Fr}(A_j \cap E) \mid j \in J\} \subset E.$

To prove (1), we may assume throughout that $r(S)$ and $\sum b_0(A_j)$ are finite; it then follows from Theorems 1 and 6 that the numbers $b_0(X_j)$, $b_0(\overline{X}_j)$, and $c(X_j)$ are also finite. Further, we may obviously suppose that $X_{m-1} \neq 0$ (otherwise (1) would be derived with a smaller value of m). Again, by using the method of approximation, we may assume in addition that the sets A_j are all open and, by 1.3 (2), satisfy

(3) $\mathrm{Fr}(A_j) \cap \mathrm{Fr}(A_k) \cap \mathrm{Fr}(A_j \cap A_k) = 0 \qquad (j \neq k).$

For, in the general case, we apply 1.3(3) to replace the sets A_j by slightly larger relatively connected sets $A_j{}^*$ having the same incidences and satisfying (3); and, in view of Theorem 5, the truth of (1) for the sets $A_j{}^*$ will imply (1) for the sets A_j.

From (3) and 1.3(1), the open sets X_j satisfy

(4) $X_1 \supset \overline{X}_2 \supset X_2 \supset \overline{X}_3 \supset \cdots \supset X_{m+1} = 0.$

We shall define inductively, for $j = 1, 2, \cdots, m-1$, open sets G_j consisting of a finite number of components C_{jk} of $\mathrm{Co}(\overline{X}_j)$, and open sets $V_j \supset \mathrm{Fr}(G_j)$, such that[11]

(5) $G_j \cup V_j \subset G_k$ whenever $j < k$;

$\overline{V}_j \subset X_{j-1}$; $\overline{V}_j \cap \overline{X}_{j+1} = 0$; $\overline{V}_j \cap \overline{V}_k = 0$ if $j \neq k$;

$\mathrm{Fr}(V_j) \cap \mathrm{Fr}(A_k) = 0$ (for all j, k); and $\mathrm{Fr}(V_j) \cap \mathrm{Fr}(X_j) = 0$.

Further,

(6) $b_0(V_j) < \infty$, $b_0(X_j \cup V_j) \leq b_0(\overline{X}_j)$, and

$$b_0(V_j \cap G_j) \geq c(\overline{X}_j) + b_0(G_j).$$

For suppose this done for all $j < p$, where $1 < p < m$. Define G_p to be the union of all those components of $\mathrm{Co}(\overline{X}_p)$ which either (a) have disconnected frontiers, or (b) meet $G_{p-1} \cup V_{p-1}$. Since $G_{p-1} \cup V_{p-1} \subset \mathrm{Co}(\overline{X}_p)$, this gives $G_{p-1} \cup V_{p-1} \subset G_p$; and since further

$$b_0(G_{p-1} \cup V_{p-1}) < \infty,$$

Theorem 6, Corollary 1, shows that $b_0(G_p) < \infty$. Let G_p consist of the components C_{pk} of $\mathrm{Co}(\overline{X}_p)$ ($k = 1, 2, \cdots, n_p$); thus

$$\textstyle\sum_k b_0[\mathrm{Fr}(C_{pk})] = c(\overline{X}_p).$$

Hence, if the components of $\mathrm{Fr}(C_{pk})$ are denoted by F_{pkl} ($l = 1, 2, \cdots, n_{pk}$), we have $\sum_k (n_{pk} - 1) = c(\overline{X}_p)$. For fixed p and k, there exist open sets $N_{pkl} \supset F_{pkl}$ with disjoint closures (for varying l); and it follows from the lemma in 4.2 that an open set U_p exists such that $U_p \supset \mathrm{Fr}(X_p)$ and $U_p \cap \mathrm{Cl}(C_{pk}) \subset \bigcup_l N_{pkl}$. We may further suppose, from (4), that $\overline{U}_p \subset X_{p-1}$, $\overline{U}_p \cap \overline{X}_{p+1} = 0$, and $\overline{U}_p \cap \overline{V}_{p-1} = 0$. It readily follows that $\mathrm{Fr}(U_p) \cap \mathrm{Fr}(A_k) = 0$ for each k ($1 \leq k \leq n$). Again, from

[11] By convention, $X_0 = S$ and $X_{n+1} = 0$.

the lemma in 5.3, there exists a finite set $Q_p \subset \operatorname{Fr}(X_p)$ such that

$$b_0(X_p \cup Q_p) = b_0(\overline{X}_p).$$

Let V_p = union of those components of U_p which meet $Q_p \cup \bigcup F_{pkl}$; clearly $b_0(V_p)$ is finite; and, since $\operatorname{Fr}(V_p) \subset \operatorname{Fr}(U_p)$, it is easily seen that (5) continues to hold when $j = p$. Also the sets $V_p \cap C_{pk} \cap N_{pkl}$ are (for varying k and l) all pairwise separated and nonempty; hence the number of components of $V_p \cap G_p$ is at least $\sum_k n_{pk} = c(\overline{X}_p) + n_p$, so that (6) holds.

To start the induction, we take G_1 to consist of the components of $\operatorname{Co}(\overline{X}_1)$ with disconnected frontiers; the rest of the construction is exactly as in the general case. Thus (5) and (6) hold for $j = 1, 2, \cdots, m-1$. We remark that it follows trivially from (5) that

(7) $\operatorname{Fr}(V_j) \cap \operatorname{Fr}(G_k) = 0$ whenever $j \neq k$, and $V_j \cap G_k = 0$ if $j > k$.

Now consider the "elementary symmetric sets"

$$Y_j = X_j(G_1, G_2, \cdots, G_{m-1}, V_1, V_2, \cdots, V_{m-1}) \qquad (1 \leq j \leq 2m-2)$$

and

$$Z_k = X_k(A_1, A_2, \cdots, A_n, G_1, \cdots, G_{m-1}, V_1, \cdots, V_{m-1})$$
$$(1 \leq k \leq 2m+n-2).$$

Using (5) and (7), we obtain

(8) $Y_1 = G_{m-1} \cup V_{m-1}$, $Y_j = G_{m-j} \cup V_{m-j} \cup (G_{m-j+1} \cap V_{m-j+1})$ if

$2 \leq j \leq m-1$; $Y_m = G_1 \cap V_1$; and $Y_j = 0$ if $j > m$.

Thus, since $Z_k = X_k \cup \bigcup(X_{k-p} \cap Y_p) \cup Y_k$, we find:

(9) $Z_k \neq 0$ if $1 \leq k \leq m$; $Z_m = X_m \cup \bigcup(X_p \cap V_p) \cup \bigcup(G_q \cap V_q)$

$(p, q = 1, 2, \cdots, m-1)$; and $Z_k = 0$ if $k > m$.

Now the open sets $A_1, A_2, \cdots, A_n$; $G_1, \cdots, G_{m-1}$; $V_1, \cdots, V_{m-1}$ satisfy the hypotheses of Theorem 1, since this is true of $A_1, \cdots, A_n$ from (3), while we readily verify that

$$\operatorname{Fr}(A_j) \cap \operatorname{Fr}(G_k) \cap \operatorname{Fr}(A_j \cap G_k) = 0,$$

and that in all the remaining cases the sets have disjoint frontiers. Hence

$$(10)\quad \sum b_0(Z_p) \leq \sum b_0(A_j) + \sum b_0(G_k) + \sum b_0(V_k) + (m-1)\,r(S)$$

$$(1 \leq p \leq n + 2m - 2,\ 1 \leq j \leq n,\ 1 \leq k \leq m-1).$$

But (9) shows that

$$\sum b_0(Z_p) \geq b_0(X_m) + \sum b_0(X_k \cap V_k) + \sum b_0(V_k \cap G_k) + m - n.$$

Also

$$b_0(X_k \cap V_k) \geq b_0(X_k) + b_0(V_k) - b_0(X_k \cap V_k) \quad (\text{cf. } [4, \S 6.2])$$

$$\geq b_0(X_k) + b_0(V_k) - b_0(\overline{X}_k),$$

and, from (6).

$$b_0(V_k \cap G_k) \geq c(\overline{X}_k) + b_0(G).$$

Thus finally, since all the numbers involved here are finite, (1) follows from (10).

5.5. THE CASE OF FINITE EQUALITY. Suppose now that there is finite equality in (1) above, and that a point y exists in (say)

$$\mathrm{Fr}(A_1 \cap E) \cap \mathrm{Fr}(A_2 \cap E) \cap \cdots \cap \mathrm{Fr}(A_{p+1} \cap E) - E,$$

where E is a component of X_p; thus $y \not\subset X_p$. It is easy to see that we may assume without loss of generality that $p \leq m-1$ and that the sets A_j are all open. Clearly $y \subset \mathrm{Fr}(X_p)$; thus we may carry out the preceding construction in such a way that $y \subset Q_p \subset V_p$. But, from the way in which (1) was derived from (10), we must now have $h(X_p, V_p) = 0$, so that the component W of V_p which contains y must meet E in a connected set; consequently, since $W \cap \mathrm{Cl}(X_{p+1}) = 0$, it follows that $W \cap E$ meets one and only one of the sets

$$A_J\ (= \bigcap \{A_j,\ j \subset J\})$$

with $|J| = p$. Since W meets $A_1 \cap E$, we have $1 \subset J$; similarly $2 \subset J, \cdots$, and $(p+1) \subset J$, giving a contradiction.

5.6. REMARKS. We observe that the preceding results contain those concerning modified addition theorems in [5, Ths. 11 and 11a]. For, in the first place, 1.3 (1) together with an "approximation" argument shows that the relation

(2) above is equivalent to the apparently stronger relation

(2a) $\qquad \bigcap \{\mathrm{Fr}(A_j \cap E)\} \subseteq \mathrm{Cl}(X_{p+1}) \qquad\qquad (j \in J,\ |J| = p+1,\ p < n).$

(In fact, the left side here is contained in

$$\mathrm{Fr}(X_{p+1}) \cup \mathrm{Fr}(X_{p+2}) \cup \cdots \cup \mathrm{Fr}(X_m).\,)$$

Hence if $A_1, \cdots, A_n$ also satisfy the condition (slightly stronger than in Theorem 9) that $\mathrm{Fr}(A_j \cap A_k) \cap \mathrm{Fr}(A_j \cup A_k) = 0$ $(j \neq k)$, finite equality in Theorem 9 will imply, again from 1.3(1), that

(2b) $\qquad \bigcap \{\mathrm{Fr}(A_j \cap E)\} \subset \mathrm{Int}\,(E) \qquad\qquad (j \in J,\ |J| = p+1,\ p < n),$

a relation which is slightly stronger than (2). And if the sets A_j satisfy the even stronger condition

$$\mathrm{Fr}(A_j) \cap \mathrm{Fr}(A_k) \cap \mathrm{Fr}(A_j \cap A_k) = 0 \qquad\qquad (j \neq k),$$

it can be deduced from (2b) that

(2c) $\qquad \bigcap \{\mathrm{Fr}(A_j \cap E)\} = 0 \ \text{ if } 2p \geq n \qquad\qquad (j \in J,\ |J| = p+1,\ p < n).$

Finally, if there is finite equality in Theorem 1, then there will be finite equality in Theorem 9, for $c(\overline{X}_j) \geq b_0(\overline{X}_j)$, by 4.1(3); and thus the above considerations will apply.

5.7. Other Inequalities. Many other inequalities can be derived from Theorem 9; for example:

Theorem 9a. *Under the hypotheses of Theorem* 9, *we have*

$$\text{(i)}\quad c(\overline{X}_1) + c(\overline{X}_2) + \cdots + c(\overline{X}_{m-1}) + b_0(X_m) + b_0(X_{m+1}) + \cdots + b_0(X_n) \leq \sum b_0(A_j) + (m-1)\, r(S).$$

Further, if there is finite equality in (i), *we have*

$$\text{(ii)}\quad X_p(\overline{A}_1, \overline{A}_2, \cdots, \overline{A}_n) = \overline{X}_p \qquad\qquad (1 \leq p \leq m).[12]$$

Proof. Relation (i) is a trivial consequence of Theorem 9, (1), since

$$b_0(\overline{X}_j) \leq b_0(X_j)\,.$$

[12] Condition (ii) need not hold for $p > m$.

Suppose there is finite equality in (i); as before it will suffice, by an approximation argument, to prove (ii) assuming that the sets A_j are open. Now, finite equality in (i) implies that Theorem 9, (2), holds, and also that $b_0(\overline{X}_j) = b_0(X_j)$ for all $j < m$. If J, K denote sets of p distinct suffixes between 1 and n, and $J \neq K$, we find (writing $A_J = \bigcap\{A_j \mid j \in J\}$) that

$$\overline{A}_J \cap \overline{A}_K \subset \overline{X}_{p+1} \qquad \text{if } p < m.$$

This includes (ii) when $p = 2$; and (ii) follows in general by an easy induction over p.

5.8. Geometrical Considerations. To illustrate the geometrical content of these theorems, we consider the case of two sets in more detail.

Theorem 10. *Let A and B be sets, neither of which contains the other, having separated differences and connected complements, and suppose that $A \cap B$ and $\mathrm{Co}(A \cup B)$ are separated. Then*

$$b_0(A \cap B) + b_0(\mathrm{Co}(A \cup B)) \leq b_0(A) + b_0(B) + r(S) - 1.$$

If there is finite equality here, and further $\mathrm{Fr}(A) \cap \mathrm{Fr}(B) \cap \mathrm{Fr}(A \cup B) = 0$, then each component of $\mathrm{Co}(A \cup B)$ has a frontier consisting of exactly two components.

Proof. We can assume that $r(S)$ is finite. Write $P = \mathrm{Co}(A)$, $Q = \mathrm{Co}(B)$; then P and Q are connected, so that, from Theorem 1, $b_0(P \cap Q)$ is finite. Let $P \cap Q$ $(= \mathrm{Co}(A \cup B))$ have components $H_1, H_2, \cdots, H_n$. Then

(1) A and H_j are not separated,

since otherwise $Q = \{(Q \cap A) \cup (P \cap Q - H_j)\} \cup H_j$, a union of two nonempty separated sets. Similarly B and H_j are not separated. Hence

(2) $\mathrm{Fr}(H_j)$ meets both $\mathrm{Fr}(A)$ and $\mathrm{Fr}(B)$.

Let $A^* = A \cup (P \cap Q)$, $B^* = B \cup (P \cap Q)$; from (1), A^* is connected relative to A, so that $b_0(A^*) \leq b_0(A)$, and similarly $b_0(B^*) \leq b_0(B)$. It is easy to see that $A^* - B^*$ and $B^* - A^*$ are separated, and that $A^* \cap B^*$ and $\mathrm{Co}(A^* \cup B^*)$ are separated; hence 2.3 (1) gives

$$b_0(A^* \cap B^*) \leq b_0(A) + b_0(B) + r(S).$$

But $A^* \cap B^* = (A \cap B) \cup \mathrm{Co}(A \cup B)$, a union of two separated sets; thus the first part of the theorem follows.

From Theorem 9a and Theorem 5, Corollary, we also have

$$b_0(A \cap B) + \sum b_0(\mathrm{Fr}(H_j)) \leq b_0(A) + b_0(B) + r(S).$$

If further $\mathrm{Fr}(A) \cap \mathrm{Fr}(B) \cap \mathrm{Fr}(A \cup B) = 0$, (2) shows that $b_0[\mathrm{Fr}(H_j)] \geq 1$ for each j. Hence finite equality in Theorem 10 requires $b_0[\mathrm{Fr}(H_j)] = 1$ for each j, and the proof is complete.

COROLLARY. *If A and B are simple sets*[13] *with disjoint frontiers, and neither A nor B contains the other, then $b_0(A \cap B) + b_0[\mathrm{Co}(A \cup B)] \leq r(S) - 1$; and if there is finite equality here, then each component of $A \cap B$ or of $\mathrm{Co}(A \cup B)$ has a frontier with exactly two components.*

This follows on applying Theorem 10 first to A, B and then to $\mathrm{Co}(A)$, $\mathrm{Co}(B)$. If S is unicoherent, the first part of this corollary reduces to [4, §4.5].

6. Simple sets with disjoint frontiers.

6.1. FINITELY MULTICOHERENT SPACES. Throughout this section, we shall assume that *$r(S)$ is finite.*

THEOREM 11. *Let $A_1, A_2, \cdots, A_n$ be simple*[14] *subsets of S, every two of which meet, and which have disjoint frontiers. Then there exist N or fewer of the sets A_j whose union is $\bigcup_1^n A_j$, where*

$N = 2r(S)$ if $r(S) > 1$, or if $r(S) = 1$ and $\bigcap A_j \neq 0$.

$N = 3$ if $r(S) = 1$ and $\bigcap A_j = 0$, and

$N = 2$ if $r(S) = 0$.

These values of N are the smallest possible.

It is easy to see by examples (it suffices to take S to be a linear graph) that no smaller values of N are possible in general. To prove the rest of the theorem, we need two graph-theoretic lemmas.

6.2. LEMMA 1. *Let G be a connected linear graph having no end-points, and let $E_1, E_2, \cdots, E_n$ be closed connected subsets of G, every two of which meet. If $r(G) > 1$, or if $r(G) = 1$ and $\bigcap E_j \neq 0$, then $\bigcup E_j$ is the union of $2r(G)$ or fewer of the sets E_j; if $r(G) = 1$ and $\bigcap E_j = 0$, then $\bigcup E_j$ is the union of at most 3 sets E_j.*

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[13] A set E is "simple" if E and $S - E$ are both connected.

[14] It would suffice to require only that $\mathrm{Cl}(A_j)$ and $\mathrm{Cl}[\mathrm{Co}(A_j)]$ be connected ($1 \leq j \leq n$).

The proof is by induction over $r(G)$. If $r(G) \leq 2$, the lemma can be verified by inspection of the possible graphs G. Suppose, then, that $r(G) \geq 3$, and that the lemma is true for all graphs of smaller degree of multicoherence but not for G, and that n is the smallest number of sets for which the lemma fails for G. Thus no E_j is contained in the union of the others.

From G we derive a homeomorphic graph G^* by suppressing all vertices of order 2; the (open) 1-cells of G^* will thus be the components of G minus its vertices of orders other than 2; we call them the "maximal 1-cells" of G. (A maximal 1-cell may have coincident end-points.) We consider three cases:

(1) If G has a cut-point R which is not a vertex of G, let PQ be the maximal 1-cell of G which contains R; thus here $P \neq Q$, and $G - PQ$ is a union of two disjoint, closed connected nonempty subgraphs H, K, neither of which has an end-point. From [6, §3.2(1)], we see that $r(H) + r(K) = r(G)$, while, since G has no end-points, $r(H) \geq 1$ and $r(K) \geq 1$. For the moment we assume that neither $r(H)$ nor $r(K)$ is 1. Write $E_j' = E_j \cap H$, $E_j'' = E_j \cap K$; it is easy to see that these sets are closed and connected, though possibly empty. Further, every two nonempty sets E_j' must meet, since both must contain P unless one of the corresponding sets E_j is contained in H. Hence the hypothesis of induction applies to H and the nonempty sets E_j', and $\bigcup E_j'$ must be contained in the union of at most $2r(H)$ sets E_j. Similarly $\bigcup E_j''$ is contained in the union of at most $2r(K)$ sets E_j. Thus we obtain at most $2r(G)$ sets E_j in all, which together contain $\bigcup E_j' \cup \bigcup E_j''$; further, their union is connected and so contains PQ and thus $\bigcup E_j$, unless $\bigcup E_j'$ or $\bigcup E_j''$ is empty.

If $\bigcup E_j''$, say, is empty but $\bigcup E_j' \neq 0$, it is easy to see that at most $2r(H) + 1 < 2r(G)$ sets E_j will suffice, namely those selected to contain $\bigcup E_j'$, together with the set E_j which contains the largest subarc of PQ. If $\bigcup E_j' = \bigcup E_j'' = 0$, all the sets E_j are contained in PQ, and two of them will suffice.

If $r(H)$, say, is 1 (so that H is a circle), the above argument needs modification only if one of the given sets is contained in $H - (P)$; we leave the details to the reader.

(2) If G has a cut-vertex R, but no cut-point other than a vertex, the argument is essentially the same as before, with PQ degenerating to R.

(3) Finally, if G has no cut-points, pick $x \in E_1 - (E_2 \cup \cdots \cup E_n)$; replacing x by a sufficiently nearby point if necessary, we can suppose that x is not a vertex and so belongs to a unique maximal 1-cell PQ of G. Here $P \neq Q$, since G has no cut-points, and the subgraph $H = G - PQ$ is connected and has no end-lines. We easily find $r(H) = r(G) - 1$. Write $E_j' = E_j \cap H$; as before, at most $2r(H)$ of the sets $E_2, \cdots, E_n$, say $E_2, \cdots, E_m$ ($m \leq 2r(H) + 1$), must contain

$\bigcup E_j{}'$ ($j \geq 2$). The connected set $E_1 \cup E_2$ joins x to H (for we may clearly assume $\bigcup E_j{}' \neq 0$), and so contains one of the arcs Px, Qx, say Px. If none of $E_{m+1}, \cdots, E_n$ meets Qx, the m sets $E_1, E_2, \cdots, E_m$ contain $\bigcup E_j$. If $Qx \cap (E_{m+1} \cup \cdots \cup E_n) \neq 0$, let y be its point on Qx closest to x, and let $y \in E_k$; then the connected set $E_2 \cup E_k$ joins y to H without containing x, and so contains QX; thus the $m+1$ sets $E_1, E_2, \cdots, E_m, E_k$ contain $\bigcup E_j$. Since $m + 1 \leq 2r(G)$, the proof is complete.

6.3. LEMMA 2. *Let $B_1, B_2, \cdots, B_n$ be n simple closed subsets of a connected linear graph G, every two of which meet. If $r(G) > 1$, or if $r(G) = 1$ and $\bigcap B_j \neq 0$, then $\bigcup B_j$ is the union of $2r(G)$ or fewer of the sets B_j; if $r(G) = 1$ and $\bigcap B_j = 0$, then $\bigcup B_j$ is the union of at most 3 sets B_j.*

As before, we may assume that the lemma is false, and that n is the smallest number of sets for which it fails; thus no B_j is contained in the union of the others. Define a "maximal end-line" PQ of G to be a maximal 1-cell PQ of G in which Q is an end-point of G; thus $P \neq Q$. If B_1, say, meets a maximal end-line PQ which it does not contain, then (being closed and simple) B_1 must be either a closed arc xQ, where $x \in PQ$, or the closure of the complement in G of such an arc. In the latter case, it is clear that B_1 together with one other set B_j will contain the rest; in the former case, we see similarly that either $B_1 \cup B_2 = G$, or $B_2 \supset B_1$, or $B_1 \supset B_2$—all of which are excluded. This proves, then, that each B_j contains all maximal end-lines of G which it meets. Let H be the graph obtained from G by removing all end-points and maximal end-lines, and write $E_i = B_i \cap H$. On applying Lemma 1 to the sets $E_1, \cdots, E_n$ in the graph H, we see that $\bigcup E_j$ is the union of the desired number of sets E_j; the analogous conclusion for the sets B_j follows.

6.4. *Proof of Theorem* 11. We shall consider only the case $r(S) > 1$ explicitly; the modifications needed when $r(S) = 1$ will be obvious, and the case $r(S) = 0$ is covered by [4, §4.5]. It will thus suffice to prove that, if $n > 2r(S) \geq 4$, one of the sets A_j is contained in the union of the others. Consider the 2^n intersections $Y_k = \overline{D}_1 \cap \overline{D}_2 \cap \cdots \cap \overline{D}_n$ ($1 \leq k \leq 2^n$), where each D_j takes the two values A_j, $\mathrm{Co}(A_j)$, in all possible combinations. The sets Y_k are closed and cover S; and, since the sets $\mathrm{Fr}(A_j)$ are disjoint, no three of them have a common point. Further, from Theorem 1, $b_0(Y_k)$ is finite, and so the sets Y_k are of finite incidence. Let G denote the modified nerve (cf. [6]) of the sets Y_k; as in [6, §6.4], G is connected and $r(G) \leq r(S)$. Let B_p denote the subgraph of G consisting of (i) all vertices which correspond to intersections Y_k in which the pth "factor" D_p is A_p, and (ii) all edges of G both of whose end-points have been assigned to B_p. Let C_p be defined similarly, but with $\mathrm{Co}(A_p)$ replacing A_p. Thus, for each p $(1 \le p \le n)$, B_p and C_p are disjoint subgraphs which together contain all the vertices of G; and it is easy to see that B_p and C_p are connected, since A_p and $\mathrm{Cl}[\mathrm{Co}(A_p)]$ are. Hence $B_1, B_2, \cdots, B_n$ are simple closed subsets of G. Further, if $p \neq q$, B_p and B_q have at least a common vertex. Thus, by Lemma 2, one of the sets B_p is contained in the union of the others; say $B_1 \subset B_2 \cup \cdots \cup B_n$. It readily follows that $A_1 \subset A_2 \cup \cdots \cup A_n$, whence the proof is completed.

6.5. COROLLARY. *For any collection of more than $2r(S)$ simple subsets of S with disjoint frontiers, the union of some two of the sets contains the intersection of the rest.*

6.6. FURTHER RESULTS. Evidently the method which was employed to prove Theorem 11 is of more general applicability; it shows, roughly speaking, that the incidences of a system of sets with disjoint frontiers are no worse than if S were a linear graph of the same degree of multicoherence. In the same way we may prove:

THEOREM 11a. *Let $A_1, A_2, \cdots, A_n$ be n simple subsets of S, every two of which meet, and which have disjoint frontiers. If n is large enough compared with $r(S)$ (assumed finite), then some A_j is contained in the union of two others.*

(Note that no A_j need be contained in *one* other, irrespective of how large n is.) Here the determination of the "best" bound for n seems to be difficult: it can be shown, however, that, disregarding the trivial case $r(S) = 0$, it lies between $\exp\{\exp[c_1 r(S)]\}$ and $\exp\{\exp[c_2 r(S)]\}$, where c_1, c_2 are positive constants.

Another related theorem, proved in a similar way, is:

THEOREM 11b. *Let $A_1, A_2, \cdots, A_n$ be connected subsets of S such that $b_0[\mathrm{Co}(A_j)] \le q$ $(j = 1, 2, \cdots, n)$. Suppose that every two of the sets A_j meet, and that they have disjoint frontiers. Then there exists a function N of q and $r(S)$ (independent of n) such that $\bigcup A_j$ is contained in the union of N or fewer of the sets A_j.*

It is easy to show by examples that, with $q \ge 1$, we have

$$N \ge (q+1)(q+2)\,r(S) \quad \text{if } r(S) \ge 1,$$

and

$$N \geq q^2 + q + 2 \quad \text{if} \quad r(S) = 0;$$

but the author does not know if these values are in fact the best.

For theorems of this type, the conditions that the sets A_j (or, more generally, their closures) be connected, and that the numbers $b_0\{\mathrm{Cl}[\mathrm{Co}(A_j)]\}$ be bounded, cannot be omitted. In [4, §8] a theorem in a similar order of ideas was obtained for *arbitrary* connected sets in a unicoherent space; it can indeed be extended to the multicoherent case, but at the cost of requiring not only that certain intersections of the sets A_j be nonempty, but that they have sufficiently many components. For example, the theorem for three sets becomes:

(1) *If A_1, A_2, A_3, are connected subsets of S such that $A_1 \cap A_2 \cap A_3 = 0$, and $b_0(A_j \cap A_k) \geq r(S)$ whenever $j \neq k$, then every two of the sets* $\mathrm{Fr}(A_j)$ *meet.*

The proof of (1) is an easy consequence of [5, §7.2].

We finally remark that the present technique can be used to give a direct "elementary" proof of Theorem 1, without using mappings in S^1. However, though the basic idea (showing that the sets have the same incidences as if S were a linear graph) is simple, a quite lengthy and tedious argument is needed to reduce the general theorem to the case in which the complements of the sets are of finite incidence; and the proof given in 2.2 above is considerably shorter.

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