A NOTE ON THE BANACH SPACES OF CALKIN AND MORREY

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1. Introduction. Let G be a bounded domain in an n-dimensional real Euclidean space, and for $\alpha > 1$, let L_α be the space of real-valued functions f such that f^α is summable over G. The class $\mathfrak{P}_\alpha$ as defined by Calkin and Morrey [2; 6; 7] is then the class of functions which together with their first "generalized derivatives" [2, Def. 3.4; 7, p. 4] are in L_α. With a suitable norm, $\mathfrak{P}_\alpha$ becomes a Banach space [2, p. 185]. Morrey proved [7, p. 8] that in this Banach space the solid sphere V of radius K and with the origin as center is "weakly compact"[1]). Using this fact together with lower semicontinuity theorems, he obtained very general existence theorems for minima of multiple integrals[2]).

The object of the present note is to point out that some of the results in this direction may be obtained by the use of general Banach space theory: the starting point is the simple remark that the Banach space $\mathfrak{P}_\alpha$ is reflexive (§ 2). The weak compactness of the solid sphere V is, by Alaoglu's theorem [1], a corollary to this remark [§ 3]. It now follows almost immediately that a real-valued function $I(x)$, which is "weakly" lower semicontinuous, takes a minimum in V (Theorem 3.1). In § 4 some sufficient conditions for weak lower semicontinuity are given. Finally, as an example of the applicability of these considerations to calculus of variation problems, a theorem on the existence of minima of multiple integrals is given which is related to, but not identical with, the results of Morrey referred to at the end of the previous paragraph (§ 5).

2. The uniform convexity and reflexivity of the space $\mathfrak{P}_\alpha$. Let t denote the point with coordinates $t_1, t_2, \cdots, t_n$ of the domain G of § 1. Let $f(t) = f^{(0)}(t)$ be an element of $\mathfrak{P}_\alpha$, and $f^{(i)}(t)$ $(i = 1, 2, \cdots, n)$ its first generalized derivative with respect to t_i. Let $||f||$ be defined by the equation

$$(2.1) \qquad ||f||^\alpha = \int_G \sum_{i=0}^{n} |f^{(i)}(t)|^\alpha dt.$$

[1] See [7] for Morrey's definition of weak compactness. The weak topology used in the present paper is defined in § 3.

[2] See [7, Chap. III], where also the relation to the results of Tonelli is discussed.

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We then have:

LEMMA 2.1. *Let $\alpha > 1$. With the norm defined by (2.1) the space of classes of functions of $\mathfrak{P}_\alpha$ equivalent under this norm is a Banach space*[3].

From now on $\mathfrak{P}_\alpha$ will always denote the Banach space of Lemma 2.1, and it will always be supposed that $\alpha > 1$.

THEOREM 2.1. *The space $\mathfrak{P}_\alpha$ is uniformly convex*[4].

Proof. Let L_α be the Banach space of classes of equivalent functions f which are defined in G and for which

$$\left\{ \int_G |f|^\alpha dt \right\}^{1/\alpha} \tag{2.2}$$

exists. L_α is uniformly convex [3, p.403, Corollary]. Since a finite "uniformly convex" direct product of uniformly convex spaces is uniformly convex [3, p. 397-398] it follows that the direct product of L_α taken $(n+1)$ times by itself, that is, the space $\tilde{\mathfrak{P}}_\alpha$ of $(n+1)$-tuples $f_0, f_1, \cdots, f_n$ $(f_\nu \in L_\alpha;\ \nu = 0, 1, \cdots n)$ with the norm

$$\left\{ \int_G \sum_{\nu=0}^{n} |f_\nu|^\alpha dt \right\}^{1/\alpha}$$

is likewise uniformly convex. This proves the theorem since $\mathfrak{P}_\alpha$ is obviously a linear subspace of $\tilde{\mathfrak{P}}_\alpha$.

Since a uniformly convex space is reflexive [5; 8], we have the following corollary to Theorem 2.1.

COROLLARY. *For $\alpha > 1$, $\mathfrak{P}_\alpha$ is reflexive.*

3. **The compactness of the sphere V.** We recall first a few well-known definitions and facts. Let E be an arbitrary Banach space in the strong topology, that is in the topology induced by the norm of the space. Let K be a positive number, and V be the solid sphere $\|x\| \leqq K$ of E. By V_K we denote then the topological space whose elements are those of V and whose topology is induced by the following neighborhood definition: A neighborhood of the point x_0 of V_K is determined by a positive number ϵ and a finite number of linear continuous functionals $l_1(x), \cdots, l_n(x)$, and consists of all points x of V_K for which

[3] See [2, p. 185]. The definition of the norm given by Calkin is slightly different from the one used in the present paper. However, the proof of Lemma 2.1 is essentially unaltered.

[4] For the definition of the term "uniformly convex" see [3].

$$|l_i(x) - l_i(x_0)| < \epsilon \qquad (i = 1, 2, \cdots, n).$$

If E is the conjugate space of another Banach space F, $E = F^*$, we denote by V_K^* the topological space whose elements are again those of V, but whose topology is induced by the following neighborhood definition: A neighborhood of a point x_0 of V_K^* is determined by a positive number ϵ and a finite number of elements $f_1, \cdots, f_n$ of F, and consists of all points x of V_K for which

$$|x(f_i) - x_0(f_i)| < \epsilon \qquad (i = 1, 2, \cdots, n).$$

A well-known theorem of Alaoglu [1, Theorem 1.3] states that V_K^* is compact. Since for a reflexive space we have $V_K = V_K^*$, we obtain:

LEMMA 3.1. *If E is reflexive then V_K is compact.*

Since a strongly closed convex subset of V is also closed in the weak topology (that is, in the topology of V_K) we have as a consequence of Lemma 3.1 the following:

LEMMA 3.2. *Let C be a convex subset of V which is closed in the strong topology, and C_K the same set in the topology of V_K. Then C_K is compact.*

An easy consequence of Lemma 3.2 is:

LEMMA 3.3. *Let C and C_K have the same meaning as in Lemma 3.2, and let $I(x)$ be a real-valued function which is lower semicontinuous in C_K. Then $I(x)$ reaches a minimum in some point of C.* [5]

The preceding lemmas, together with the corollary to Theorem 2.1, now yield the main result of the present section:

THEOREM 3.1. *Let C be a bounded closed convex subset of $\mathfrak{P}_\alpha$. Let the norms of the elements of C be bounded by the positive constant K. Let V and V_K have the same meaning as in the first paragraph of this section, with E replaced by $\mathfrak{P}_\alpha$, and denote the set C in the topology of V_K by C_K. Then C_K is compact, and a real-valued function $I(x)$, which is lower semicontinuous in C_K, reaches a minimum in C.*

4. Sufficient conditions for lower semicontinuity. We prove now:

THEOREM 4.1. *Let C and C_K have the same meaning as in Theorem 3.1,*

[5] For a proof that Lemma 3.2 implies Lemma 3.3 see [9, p. 423-424].

and let $I(x)$ be a real-valued function defined on C. Then the following condition is sufficient for the lower semicontinuity of $I(x)$ on C_K (and therefore, by Theorem 3.1, for the existence of a minimum of $I(x)$ on C): to each $x_0 \in C$ there exists a bounded linear functional $l(x)$ such that

$$I(x) - I(x_0) \geqq l(x - x_0) \tag{4.1}$$

for all $x \in C$.

Proof. By definition of the lower semicontinuity we have to prove: to any given $\epsilon > 0$ there exists a neighborhood $N(x_0)$ of x_0 in V_K such that

$$I(x) - I(x_0) \geqq -\epsilon \tag{4.2}$$

for all x in the intersection, $N(x_0) \cap C$. But by (4.1) the inequality (4.2) will certainly be satisfied if we choose

$$N(x_0) = \{x \,|\, |l(x) - l(x_0)| < \epsilon,\ x \in V_K\}.$$

THEOREM 4.2. *With the same notations as in Theorem 4.1 let $I(x)$ have first and second order Fréchet differentials $D(x, h)$ and $D^2(x, h, k)$ at every point x of C. Moreover, let*

$$D^2(x, h, h) \geqq 0 \quad \text{for} \quad x \in C. \tag{4.3}$$

Then $I(x)$ is lower semicontinuous in C_K.

Proof. From the Taylor expansion [4, Theorem 5],

$$I(x_0 + h) - I(x_0) = D(x_0, h) + \frac{1}{2}\int_0^1 D^2(x_0 + th, h, h)\, dt,$$

together with (4.3), we obtain

$$I(x_0 + h) - I(x_0) \geqq D(x_0, h).$$

This inequality shows that the assumptions of Theorem 4.1 are satisfied with

$$l(x - x_0) = D(x_0, x - x_0).$$

5. An application to a multiple integral variational problem. Let G be the domain of § 1 with points $t = (t_1, \cdots, t_n)$. For each $\mu = 1, \cdots, m$ let $z_\mu(t) \in \mathfrak{P}_\alpha$ and let Π_α be the space of classes of equivalent m-tuples $z = (z_1(t), \cdots, z_m(t))$ with the norm

$$(5.1) \qquad \|z\| = \left[\int_G \sum_{\mu=1}^{m} \left\{ |z|^\alpha + \sum_{\nu=1}^{n} \left| \frac{\partial z_\mu}{\partial t_\nu} \right|^\alpha \right\} dt \right]^{1/\alpha}.$$

LEMMA 5.1. *Theorems* 3.1 *and* 4.1 *still hold if* $\mathfrak{P}_\alpha$ *is replaced by* Π_α.

This lemma is obvious from the proofs of the theorems in question.

THEOREM 5.1. *Let*

$$f(t, z, p) = f(t_1, \cdots, t_n, z_1, \cdots, z_m, p_{11}, \cdots, p_{mn})$$

be a real-valued function of the indicated variables with the following properties:

(1) *f is defined for* $t = (t_1, \cdots, t_n) \in G$ *and for all values of the real variables* z_μ, $p_{\mu\nu}$ $(\mu = 1, \cdots, m;\ \nu = 1, \cdots, n)$, *and for the same domain of the varibles* $\partial f/\partial t_\nu$, $\partial f/\partial z_\mu$, *and* $\partial f/\partial p_{\mu\nu}$ *are supposed to exist*;

(2) *if* $z_\mu(t) \in \mathfrak{P}_\alpha$ *then the functions of t obtained by replacing z by* $z_\mu(t)$ *and* $p_{\mu\nu}$ *by* $\partial z_\mu/\partial t_\nu$ *in f*, $\partial f/\partial z_\mu$, *and* $\partial f/\partial p_{\mu\nu}$ *are in* L_β, *where* β *is defined by*

$$1/\beta + 1/\alpha = 1;$$

(3) $e(t, z, z^0, p, p^0) \geqq 0$,

where by definition

$$e(t, z, z^0, p, p^0) = f(t, z, p) - f(t, z^0, p^0)$$

$$- \sum_{\mu=1}^{m} \left\{ f_{z_\mu}(t, z^0, p^0)(z_\mu - z_\mu^0) + \sum_{\nu=1}^{n} f_{p_{\mu\nu}}(t, z^0, p^0)(p_{\mu\nu} - p_{\mu\nu}^0) \right\}.$$

Under these assumptions, if

$$I(z) = \int_G f\left[t_1, \cdots, t_n, z_1(t), \cdots, z_m(t), \frac{\partial z_1}{\partial t_1}, \cdots, \frac{\partial z_m}{\partial t_n}\right] dt,$$

then there exists a

$$z^{(1)} = z^{(1)}(t) = [z_1^{(1)}(t), \cdots, z_m^{(1)}(t)]$$

in the sphere

$$\|z\| \leqq K \tag{5.2}$$

such that

$$I(z) \geqq I(z^{(1)})$$

for all z in the sphere (5.2).

Proof. By Lemma 5.1 and Theorem 3.1, it will be sufficient to prove that $I(z)$ is lower semicontinuous at each point z^0 of the sphere (5.2). To such z^0 we define the linear functional $l(\zeta)$ of

$$\zeta = [\zeta_1(t), \cdots, \zeta_m(t)]$$

by setting

$$l(\zeta) = \int_G \sum_{\mu=1}^{m} \left\{ (f_{z_\mu})_0 \, \zeta_\mu + \sum_{\nu=1}^{n} (f_{p_{\mu\nu}})_0 \frac{\partial \zeta_\mu}{\partial t_\nu} \right\} dt,$$

where the symbol $(\ )_0$ indicates that the arguments are

$$t_1, \cdots, t_n, z_1^0(t), \cdots, z_m^0(t), \partial z_1^0/\partial t_1, \cdots, \partial z_m^0/\partial t_n,$$

and where

$$\zeta = [\zeta_1(t), \cdots, \zeta_m(t)] \in \Pi_\alpha.$$

The assumption (2) assures us that the linear functional $l(\zeta)$ is bounded. From the definition of $l(\zeta)$ and the assumption (3) we obtain

$$I(z) - I(z^0) = l(z - z^0) + e \geqq l(z - z^0).$$

Thus the assumption (4.1) of Theorem 4.1 is satisfied, and the theorem to be proved follows from Theorem 4.1 in conjunction with Lemma 5.1.

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