ON A THEOREM OF BEURLING AND KAPLANSKY

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1. Introduction. The object of this paper is to remark that a natural and simple proof of the theorem of Beurling and Kaplansky (Theorem 1 below) can be obtained by adapting to general groups a classical proof already given in the books of Wiener [8] and Zygmund [9]. In fact, Theorem 1 is an immediate consequence of a lemma (Lemma 1 below) which was proved by these authors in the case when the group is the integers or the real numbers. An easy generalization of Lemma 1 (Lemma 2 below) yields immediately the generalization of the Beurling and Kaplansky theorem stated as Theorem 2 below. For the history of the development of this theorem, see [3, p. 149] and [5]; the book [3] did not appear until the present paper had been submitted, but it seemed wise to add the reference.

2. Statement of results. Let $A = \{a, b, \cdots\}$ be a locally compact abelian group and $X = \{x, y, \cdots\}$ the dual group (the group operations will be written multiplicatively). Let

$$L^1(A) = \{f, g, h, p, \cdots\}$$

denote the set of all integrable functions with respect to the Haar measure of A,

$$\|f\| = \|f\|_1$$

the L^1-norm of f, $\hat{f}(x)$ the Fourier transform of $f(a)$,

$$f_1 * f_2$$

the product of convolution (that is, the product in the group algebra),

$$f_1 f_2 = f_1(a) f_2(a)$$

the ordinary product of functions, and

$$(x, a) = x(a) = a(x)$$

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the value of the character $x \in X$ at the point $a \in A$. Subsets of A will be denoted by $C, D, \cdots$, subsets of X by $P, Q, S, \cdots$, and subsets of $L^1(A)$ by $I, J, \cdots$.

The spectrum $S(f)$ of a function $f \in L^1(A)$ is the set of the points $x \in X$ such that $\hat{f}(x) = 0$, and the spectrum $S(I)$ of a set $I \subset L^1(A)$ is the set of the points $x \in X$ such that $\hat{f}(x) = 0$ for all $f \in I$.

We suppose known the following Tauberian theorem of Segal and Godement (see [**1**] or [**4**]).

THEOREM A. *If I is a closed ideal of $L^1(A)$, and $f \in L^1(A)$ is such that $S(I)$ is interior to $S(f)$, then $f \in I$.*

Theorem A is a consequence of the regularity (in the sense of Silov) of the algebra $L^1(A)$, and the following Lemma A (see [**7**], [**1**], or [**4**]).

LEMMA A. *Given $f \in L^1(A)$ and $\epsilon > 0$, there is a function $g \in L^1(A)$ with the following properties:*

(i) $\hat{f}(x) = 0$ *implies* $\hat{g}(x) = 0$; *that is,* $S(f) \subset S(g)$.

(ii) *If $h = f - g$, then $\hat{h}(x)$ vanishes in a neighborhood of the point ∞ (that is outside of a compact set $P \subset X$).*

(iii) $||g|| \leq \epsilon$.

It is known [**6**] that Theorem A is not true if $S(f)$ is merely contained in but not interior to $S(f)$; however, if $S(I)$ consists of a single point, the following theorem is true:

THEOREM 1 (Beurling and Kaplansky). *If I is a closed ideal such that $S(I)$ consists of a single point x_0, then $S(f) \supset S(I)$ implies $f \in I$.*

This is a special case of the following:

THEOREM 2. *Let I be a closed ideal such that the boundary P of $S(I)$ is a reducible set (or that the intersection of P with the boundary of $S(f)$ is a reducible set). Then $S(f) \supset S(I)$ implies $f \in I$.*

A set is said to be reducible if it contains no nonvoid perfect subsets.

Theorem 1 was proved by Beurling in the case when A consists of the real numbers, using complex-variable methods. Kaplansky proved the theorem in the general case using the structure theory of groups. A direct and simple proof of Theorem 1 is given in a recent paper of Helson [**2**], and in the same paper is given a complete proof of Theorem 2.

We want to show that a still more natural and simple proof of Theorems 1 and 2 can be obtained as follows.

2. Proofs. We first reduce Theorem 1 to the following Lemma 1 (observe that Lemma A is obtained from Lemma 1 by replacing the point x_0 by ∞).

LEMMA 1. *Given a point $x_0 \in S(f)$, $f \in L^1(A)$, and $\epsilon > 0$, there is a function $g \in L^1(A)$ with the following properties:*

(i) $S(f) \subset S(g)$:

(ii) *if $h = f - g$, then $\hat{h}(x)$ vanishes in a neighborhood $U(x_0)$ of the point x_0;*

(iii) $||g|| \leq \epsilon$.

It is easy to see that Theorem 1 is an immediate consequence of Lemma 1 and Theorem A. In fact, if $S(I)$ consists of a single point $x_0 \in S(f)$, then by Lemma 1 there is a function h such that $||f - h|| < \epsilon$, and x_0 is interior to $S(h)$; hence, by Theorem A, $h \in I$. Since ϵ is arbitrary and $||f - h|| \leq \epsilon$, it follows that $f \in I$, and this proves Theorem 1.

Similarly it is easy to see that Theorem 2 is an immediate consequence of Theorem A, Lemma A, and the following Lemma 2.

LEMMA 2. *Given a compact reducible set $Q \subset S(f)$, $f \in L^1(A)$, and $\epsilon > 0$, there is a function $g \in L^1(A)$ with the following properties:*

(i) $S(f) \subset S(g)$;

(ii) *if $h = f - g$, then $\hat{h}(x)$ vanishes in a neighborhood $U(Q)$* of the set Q;

(iii) $||g|| \leq \epsilon$.

Hence Theorems 1 and 2 will be proved if we prove Lemmas 1 and 2.

3. Proof of Lemma 1. Without loss of generality we may suppose $x_0 = 1 =$ unit of X. Then by hypothesis

$$\hat{f}(x_0) = \int_A f(a)\, da = 0.$$

Given $\epsilon > 0$, there is a compact set $C \subset A$ such that

$$\int_{A-C} |f(a)|\, da < \epsilon/4, \tag{1}$$

hence also

$$(2) \qquad \left| \int_C f(a)\, da \right| = \left| \int_{A-C} f(a)\, da \right| < \epsilon/4.$$

If $p(a)$ is any function from $L^1(A)$, and $g = p * f$, we have

$$g(a) = \int_A f(b)\, p(ab^{-1})\, db = \int_C + \int_{A-C} f(b)\, p(ab^{-1})\, db,$$

$$(3) \qquad ||g|| \leq \int_A \left| \int_C f(b)\, p(ab^{-1})\, db \right| da$$

$$+ \int_A \left| \int_{A-C} f(b)\, p(ab^{-1})\, db \right| da = M + N.$$

Using (1) and (2), and denoting the characteristic function of the set $C' = A - C$ by $\phi_{C'}$, we have

$$(3a) \qquad N = \int_A \left| \int_A f(b)\, \phi_{C'}(b)\, p(ab^{-1})\, db \right| da$$

$$= ||(f\phi_{C'}) * p|| \leq ||f\phi_{C'}|| \cdot ||p||$$

$$= ||p|| \cdot \int_{C'} |f(a)|\, da \leq \epsilon/4 \cdot ||p||,$$

$$(3b) \qquad M \leq \int_A \left| \int_C f(b)\, [p(ab^{-1}) - p(a)]\, db \right| da$$

$$+ \int_A \left| \int_C f(b)\, db \right| |p(a)|\, da$$

$$\leqq \{ \sup_{b \in C} \int_A |p(ab^{-1}) - p(a)|\, da \} ||f|| + \epsilon/4\, ||p||.$$

Let us denote $p(ab^{-1})$ by $p^b(a)$; then

$$(4) \qquad ||g|| \leq \epsilon/2\, ||p|| + ||f|| \sup_{b \in C} ||p^b - p||.$$

Since

$$\hat{g}(x) = \hat{f}(x)\hat{p}(x),$$

$\hat{f}(x) = 0$ implies $\hat{g}(x) = 0$, and inequality (4) shows that Lemma 1 will be proved if we prove the following proposition.

PROPOSITION A. *Given $\epsilon > 0$ and a compact set $C \subset A$, there is a function $p(a)$ such that:*

a) $p \in L^1(A)$ *and* $||p|| \leq 2$;

b) *there is a neighborhood $U(1)$ of the point $1 \in X$ such that $\hat{p}(x) = 1$ for $x \in U(1)$;*

e) $||p^b - p|| < \epsilon$ *for b in the compact set C.*

Proof of Proposition A. Take two compact neighborhoods V and V' of the $1 \in X$, of measures η and η', and such that

$$\overline{V} \subset V'; \ \eta' \leq 4\eta, \tag{5}$$

and define

$$\hat{p}(x) = 1/\eta \{ \hat{\phi}_V * \hat{\phi}_{V'} \} = 1/\eta \{ \hat{\phi} * \hat{\phi}' \}, \tag{6}$$

where $\hat{\phi} = \hat{\phi}_V$ ($\hat{\phi}' = \hat{\phi}_{V'}$) is the characteristic function of the set $V(V')$. Since $\hat{\phi}$, $\hat{\phi}' \in L^2(X)$, by Plancherel's theorem $\hat{p}(x)$ is the Fourier transform of a function $p(a) \in L^1(A)$. Since $\overline{V} \subset V'$, there is a neighborhood $U = U(1)$ such that $V \cdot U \subset V'$, and from (6) it is clear that $\hat{p}(x) = 1$ for $x \in U$. Using the Plancherel theorem it is easy to see that $p(a)$ satisfies also the conditions a) and c), provided V' is taken small enough (cfr. [5]). For instance, let us prove condition c). Since the Fourier transform of $\phi^b - \phi$ is $\hat{\phi}(x)[(x, b) - 1]$, and since $\hat{\phi}(x) = 0$ outside of $V' \cdot V'$, it follows that if $b \in C$, and V' is small enough, then

$$||\phi^b - \phi||_2 = ||[(x, b) - 1]\hat{\phi}||_2 \leq \epsilon_1 ||\hat{\phi}||_2 = \epsilon_1 \eta^{1/2},$$

for every $b \in C$, where $\epsilon_1 > 0$ is arbitrarily small. Since

$$p(a) = \phi(a)\phi'(a)/\eta,$$

by Plancherel's theorem,

$$||p^b - p||_1 = 1/\eta\, ||\phi\phi' - \phi^b\phi'^b|| \leq 1/\eta [||\phi'(\phi - \phi^b)|| + ||\phi^b(\phi' - \phi'^b)||]$$

$$\leq 1/\eta[\|\phi'\|_2\, \epsilon_1\, \|\phi\|_2 + \|\phi\|_2\, \epsilon_1\, \|\phi'\|_2] \leq 2\epsilon_1 (\eta\eta')^{1/2}/\eta \leq 4\epsilon_1,$$

and this proves condition c).

REMARK. As we already mentioned, the foregoing proof of Lemma 1 is an adaptation of a proof given in Zygmund's book. Zygmund considers the particular case when A consists of the integers and X is the unit circle, so that the functions $\hat{f}(x)$ are periodic functions with absolutely convergent Fourier series, and he takes for $\hat{p}(x)$ the function

$$\hat{p}(x) = 1 \text{ if } |x| \leq \eta,$$

$$\hat{p}(x) = 0 \text{ if } |x| \geq 2\eta,$$

$$\hat{p}(x) \text{ linear if } \eta \leq |x| \leq 2\eta.$$

Then he proves that the total variation of the derivative of the function is bounded by a fixed number, and from this he deduces properties a), b), c) of the function $p(a)$. This is the only point in Zygmund's proof which does not apply to general groups; however, it is easy to see that the function $\hat{p}$ used by Zygmund is exactly what formula (6) reduces to when V is taken to be an interval, and thus the proof can be adapted to the general case.

4. Proof of Lemma 2. Let $Q \subset S(f)$ be a compact reducible set, and let $Q^{(1)} = Q'$ be the set of the points x such that any neighborhood of x contains an infinite subset of Q. Define

$$Q^{(2)} = (Q^{(1)})',$$

and form in the usual way the sequence of derivative sets:

$$Q \supset Q^{(1)} \supset Q^{(2)} \supset \cdots \supset Q^{(\alpha)} \supset \cdots.$$

Let w be such that

$$Q^{(w)} = Q^{(w+1)};$$

then $Q^{(w)}$ is a perfect set; and since Q is reducible, $Q^{(w)} = 0$. If $w = 1$, then Q is a finite set and n successive applications of Lemma 1 yields Lemma 2 in this case. We will now prove Lemma 2 by induction on w.

Suppose that Lemma 2 is true if $Q^{(w)} = 0$ for $w < w_0$; we shall prove that it is also true if $Q^{(w)} = 0$ for $w = w_0$. Consider first the case when $w_0 = w' + 1$. Then $Q^{(w')}$ is a finite set, and hence there is a function $h \in L^1(A)$ such that

$$\|f - h\| \leq \epsilon/2, \; S(f) \subset S(h),$$

and $\hat{h}(x)$ vanishes on an open set $U \supset Q^{(w')}$. Since $Q - U$ has the property

$$(Q - U)^{(w')} = 0,$$

and $w' < w_0$, by the inductive assumption there is a function h' such that

$$S(f) \subset S(h) \subset S(h'), \; \|h - h'\| \leq \epsilon/2,$$

and $\hat{h}'(x)$ vanishes on an open set $U' \supset Q - U$. Hence $\hat{h}'(x)$ vanishes on $U \cup U' \supset Q$, and

$$\|f - h'\| \leq \|f - h\| + \|h - h'\| \leq 2\epsilon/2 = \epsilon.$$

If w_0 is not of the form $w' + 1$, then by definition

$$Q^{(w_0)} = \bigcap_{w < w_0} Q^{(w)};$$

hence for some $w' < w_0$ we must have $Q^{(w')} = 0$, and by the inductive assumption Lemma 2 is true in this case.

This proves Lemma 2.

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