COMMENTS ON THE PRECEDING PAPER BY HERZOG AND PIRANIAN

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1. Our main purpose here is to extract and formulate explicitly the general principle underlying the construction of Herzog and Piranian. The results in this note are implicitly contained in the computations on pp. 535 and 537 of their paper, and the full credit belongs to them.

2. We use the notation $M(r, f) = \max |f(z)|$ on $|z| = r$.

THEOREM 1. *Let f_n be analytic in $|z| \le 1$, let r_n be increasing, $0 < r_n \longrightarrow 1$ as $n \longrightarrow \infty$, let $a_n > 0$,*

$$A = \sum_{n=1}^{\infty} a_n < +\infty,$$

let $R(t) = \sum a_k$ over all k such that $r_k \ge t$, and let $g = \sum_{n=1}^{\infty} f_n$. If

(a) $M(r_n, f_{n+1}) \le a_n$,

and

(b) $M(1, f_n') \le a_n (1 - r_n)^{-1}$

for all n, then g is analytic in $|z| < 1$, and for $|z| \le 1$, $r_{n-1} \le r \le r_n$, we have

$$\left| g(rz) - \sum_{1}^{n-1} f_k(z) - f_n(rz) \right| \le A(1-r)^{1/2} + R(1 - (1-r)^{1/2}), \tag{1}$$

$$|g(r_n z) - g(r_{n-1} z) - f_n(z)| \le 2A(1 - r_{n-1})^{1/2} + 2R(1 - (1 - r_{n-1})^{1/2}) + R(r_n). \tag{2}$$

Proof. We have

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$$|f_k(rz) - f_k(z)| \le a_k(1-r)/(1-r_k)$$

$$\le \begin{cases} a_k(1-r)^{1/2} & \text{if } r_k \le 1-(1-r)^{1/2} \\ \\ a_k & \text{if } k \le n-1, \end{cases}$$

and $|f_k(rz)| \le a_{k-1}$ for $k > n$. Inequality (1) now follows from

$$g(rz) - \sum_{k=1}^{n-1} f_k(z) - f_n(rz) = \sum_{k=1}^{n-1} (f_k(rz) - f_k(z)) + \sum_{k=n+1}^{\infty} f_k(rz).$$

We now apply (1) with $r = r_n$ and $r = r_{n-1}$ to estimate

$$h(z) = g(r_n z) - g(r_{n-1} z) - f_n(r_n z) + f_n(r_{n-1} z),$$

and obtain (2) from

$$g(r_n z) - g(r_{n-1} z) - f_n(z) = h(z) - f_n(r_{n-1} z) + (f_n(r_n z) - f_n(z)).$$

3. **We denote by** $E(g)$ the set of radial continuity of g.

COROLLARY 1a. *If* $|z_0| = 1$, $\limsup_{n\to\infty} |f_n(z_0)| > 0$, *then* $z_0 \notin E(g)$.

COROLLARY 1b. *If* $|z_0| = 1$, *and* $\lim f_n(rz_0)$ *exists as* $r \longrightarrow 1$ *and* $n \longrightarrow \infty$ *simultaneously,*[1] *then*

$$\lim_{r\to\infty} g(rz_0) \text{ and } \sum_{n=1}^{\infty} f_n(z_0) = g(z_0)$$

either both exist or both do not exist. If $\lim f_n(rz_0) = 0$, *then*

$$\lim_{r\to 1} g(rz_0) = g(z_0)$$

if either exists. Hence if $M(1, f_n) \longrightarrow 0$ *as* $n \longrightarrow \infty$, *then* $E(g)$ *is the set of convergence of* $\sum_{n=1}^{\infty} f_n(z)$ *on* $|z| = 1$.

4. **We now establish:**

[1] The weaker condition that $f_n(rz_0)$ has a limit as $n \longrightarrow +\infty$ and $r \longrightarrow 1$ in such a way that $r_{n-1} \le r \le r_n$ for all n is sufficient for this corollary.

THEOREM 2. *If F_n is analytic in $|z| \leq 1$, $M(1, F_n) \leq M_n$, $M(1, F_n') \leq M_n$ for all n, and $a_n > 0$ (all n), $\sum_{n=1}^{\infty} a_n < +\infty$, then there exist sequences r_n and k_n such that $f_n(z) = z^{k_n} F_n(z)$ satisfies* (a) *and* (b) *of Theorem* 1.

Proof. Let $k_1 = 0$ and suppose that $k_2, \cdots, k_n, r_1, \cdots, r_{n-1}$ are defined. Then (b) is satisfied if

$$r_n \geq 1 - \frac{a_n}{M_n(k_n+1)} .$$

Choose any r_n such that

$$1 > r_n > \max \left[r_{n-1}, 1 - \frac{a_n}{M_n(k_n+1)} \right].$$

Then (a) is satisfied if

$$k_{n+1} \geq \frac{\log (a_n/M_{n+1})}{\log r_n} .$$

5. As a consequence, we have:

COROLLARY 2a. *If*

$$\limsup_{n \to \infty} |\alpha_n| > 0, \quad \limsup_{n \to \infty} k_n^{-1} \log |\alpha_n| = 0,$$

$$a_n > 0, \quad \sum a_n < +\infty, \quad \text{and} \quad \frac{k_{n+1}}{k_n} \geq \frac{|\alpha_n|}{a_n} \log \frac{|\alpha_{n+1}|}{a_n}$$

for all n, then $E(g) = 0$, where $g(z) = \sum \alpha_n z^{k_n}$.

If $\alpha_n = O(1)$, $\limsup_{n \to \infty} |\alpha_n| > 0$, *$k_n$ increasing, and*

$$\sum \frac{k_n}{k_{n+1}} \log \frac{k_{n+1}}{k_n} < +\infty,$$

then $E(g) = 0$.

COROLLARY 2b. *Suppose that f is analytic in the circle $|z| \leq 1$, $f(1) = 1$, $M(1, f') \leq 1$, and that $a_n > 0$ (all n),*

$$\sum_{n=1}^{\infty} a_n < +\infty.$$

Let

$$g(z) = \sum_{n=1}^{\infty} z^{k_n} f(ze^{-i\theta_n}).$$

If

$$\liminf_{n\to\infty} \left[\frac{k_{n+1}}{k_n} + 3\,\frac{\log a_n}{a_n}\right] > 0,$$

then $z = e^{i\theta} \notin E(g)$ *if* $|\theta - \theta_n| \leq (\pi/3) - h$, $0 < h < \pi/3$, *for infinitely many* n. *In particular,* $E(g) = 0$ *if the set* $\{\theta_n\}$ *is dense in the interval* $[0, 2\pi]$.

6. The discussion of $C_n(z)$ on pp. 534, 535 of the preceding paper shows that they are constructed essentially in accordance with Theorem 2 above. The gap theorem in Corollary 2a is very crude, and can certainly be improved. The high-indices theorem of Hardy and Littlewood and Tauberian methods (see [2] and [3]) yield much sharper results.

7. The construction on p. 537 of Herzog and Piranian can also be carried out as follows.

LEMMA. *If A and B are disjoint closed sets in the plane and B is bounded and has a simply connected compliment, and* $\epsilon > 0$, *then there is a polynomial* $P(z)$ *such that* $|P(z)| \leq \epsilon$ *on B and* $|P(z)| \geq 1$ *on A.*

Proof. Let $T_n(z)$ be the Chebyshev polynomial of degree n for B; that is, T_n is the polynomial of degree n with highest coefficient 1 whose maximum modulus on B is the least possible. Then $T_n(z)^{1/n} \longrightarrow \phi(z)$ in the exterior of B, where $\phi(z)$ is the function which maps the exterior of B onto the exterior of a circle $|w| > c$ and whose Taylor expansion at ∞ begins thus: $\phi(z) = z + \cdots$. Let $c < C < R$ be such that $|\phi(z)| \geq R + \epsilon$ on A. Then there is an n such that

$$|T_n(z)|^{1/n} \geq R \text{ on } A \quad \text{and} \quad |T_n(z)|^{1/n} \leq C \text{ on } B.$$

If n is chosen such that $\epsilon(R/C)^n \geq 1$, then $R^{-n}\,T_n$ is a polynomial with the desired properties.

There are, of course, many other ways of constructing such a polynomial.

Now in the construction on p. 537, take a convergent double series $\sum a_{kh}$ with $a_{kh} > C$. Choose $A = I_{kh}$ and let B be the sector $z = re^{i\theta}$ with $0 \leq r \leq 1$ and θ in the closed interval complimentary to I_{kh} and its two adjacent intervals in G_k. Let P_{kh} be a polynomial such that $|P_{kh}(z)| \geq 1$ on I_{kh} and $|P_{kh}(z)| \leq a_{kh}$ on B. Arrange the pairs (k, h) in a sequence by the diagonal process, and apply Theorem 2, then Theorem 1.

8. The polynomials C_n used by Herzog and Piranian are of the desired type for the sets A and B considered in the preceding paragraph. They provide a simple explicit construction and enjoy other interesting properties which seem to be useful in a number of problems. The fact that they are small on the whole set B above follows from the following remark which is surely known:

If

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad s_n(z) = \sum_{k=0}^{n} a_k z^k,$$

and $0 \leq r \leq 1$, $|z| \leq 1$, then $|f(rz)| \leq \sup_n |s_n(z)|$.

This is a trivial consequence of the identity $f(rz) = O(1-r) \sum_0^\infty r^n s_n(z)$.

References

1. F. Herzog and G. Piranian, *Sets of radial continuity of analytic functions*, Pacific J. Math. 4 (1954), 533-538.

2. N. Levinson, *Gap and density theorems*, Amer. Math. Soc. Colloquium Publications, New York, 1940.

3. N. Wiener, *A Tauberian gap theorem of Hardy and Littlewood*, J. Chinese Math. Soc. 1 (1936), 15.

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