CERTAIN GENERALIZED HYPERGEOMETRIC IDENTITIES OF THE ROGERS-RAMANUJAN TYPE

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1. Introduction. In a recent paper H. L. Alder [1] has obtained a generalization of the well-known Rogers-Ramanujan identities. In this paper I have deduced the above generalizations as simple limiting cases of a general transformation in the theory of hypergeometric series given by Sears [5]. This method, besides being much simpler than that of Alder, also gives a simple form for the polynomials $G_{k,\mu}(x)$ given by him. In Alder's proof the polynomials $G_{k,\mu}(x)$ had to be calculated for every fixed k with the help of certain difference equations but in the present case we get directly the general form of these polynomials.

2. Notation. I have used the following notation throughout the paper. Assuming $|x|<1$, let

$$(a)_s \equiv (a;\, s) = (1-a)(1-ax)\cdots(1-ax^{s-1})\,, \qquad (a)_0 = 1$$

$$\prod^{s}(a_1, a_2, \cdots, a_r;\, b_1, b_2, \cdots, b_t) = \frac{(a_1;\, s)(a_2;\, s)\cdots(a_r;\, s)}{(b_1;\, s)(b_2;\, s)\cdots(b_t;\, s)}$$

$$\prod(a) = \prod_{n=0}^{\infty}(1-ax^n)$$

$$\prod(a_1, a_2, \cdots, a_r;\, b_1, b_2, \cdots, b_t) = \frac{\prod(a_1)\prod(a_2)\cdots\prod(a_r)}{\prod(b_1)\prod(b_2)\cdots\prod(b_t)}$$

$$K_s = \frac{(k;\, s)(x\sqrt{k}\,;\, s)(-x\sqrt{k}\,;\, s)}{(x;\, s)(\sqrt{k}\,;\, s)(-\sqrt{k}\,;\, s)}$$

$$K_{s,r} = K_s \frac{(x^{-r};\, s)}{(kx^{r+1};\, s)}\, x^{rs}$$

$$S_{n,n-1} = \sum_{r_n=0}^{r_{n-1}} \frac{k^{r_n} x^{r_n^2} (x^{r_{n-1}-r_n+1};\, r_n)}{(x;\, r_n)}\,, \qquad S_{1,0} = \sum_{r_1=0}^{r} \frac{k^{r_1} x^{r_1^2} (x^{r-r_1+1};\, r_1)}{(x;\, r_1)}$$

$$T_{n,M} = \sum_{t_n=0}^{\left[\frac{M-n-1}{M-n} - t_{n-1}\right]} \frac{(x^{t_{n-1}-2t_n+1};\, 2t_n)\, x^{-2t_n(t_{n-1}-t_n)}}{(x;\, t_n)(x^{t_{n-2}-2t_{n-1}+1};\, t_n)}\,, \qquad (M=3, 4, 5, \cdots)$$

where $[a]$ denotes the integral part of a.

The numbers $s, r, r_1, r_2, \cdots, t, t_1, t_2, \cdots$ are either zero or positive

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integers. r_0 and t_0, wherever they occur, have been replaced simply by r and t respectively. Empty products are to mean unity.

3. **Sears** [5, § 4] has proved the following theorem:

$$(3.1)\quad \sum_{s=0}^{\infty} x^{\frac{1}{2}s(s-1)}(kx/a_1a_2)^s \prod^{s} (a_1, a_2; x, kx/a_1, kx/a_2)\theta_s$$
$$=\prod (kx, kx/a_1a_2; kx/a_1, kx/a_2) \sum_{r=0}^{\infty} (kx/a_1a_2)^r \prod^{r} (a_1, a_2; x, kx)$$
$$\times \sum_{t=0}^{r} \frac{(x^{-r}; t)(-1)^t x^{rt}}{(kx^{r+1}; t)(x; t)} \theta_t ,$$

wrere $|kx/a_1a_2|<1$, $|x|<1$ and θ_s is any sequence. The theorem holds provided only that the series on the left converges.

Take

$$\theta_s=\prod^{s}\left[\begin{matrix} k, x\sqrt{k}, -x\sqrt{k}, a_3, a_4, \cdots, a_{2M+1}; \\ \sqrt{k}, -\sqrt{k}, kx/a_3, kx/a_4, \cdots, kx/a_{2M+1} \end{matrix}\right]$$
$$\times \frac{(k^{M-1}x^{M-1})^s}{(a_3a_4\cdots a_{2M+1})^s} x^{\frac{1}{2}s(1-s)}, \qquad (M=1, 2, 3, \cdots)$$

Then

$$(3.2)\quad \sum_{s=0}^{\infty} K_s \frac{(a_1; s)(a_2; s)\cdots(a_{2M+1}; s)}{(kx/a_1; s)(kx/a_2; s)\cdots(kx/a_{2M+1}; s)} \frac{(k^M x^M)^s}{(a_1a_2\cdots a_{2M+1})^s}$$
$$=\prod (kx, kx/a_1a_2; kx/a_1, kx/a_2) \sum_{r=0}^{\infty} (kx/a_1a_2)^r \prod^{r} (a_1, a_2,; x, kx)$$
$$\times \sum_{t=0}^{r} K_{t,r} \frac{(a_3; t)(a_4; t)\cdots(a_{2M+1}; t)(-1)^t x^{\frac{1}{2}t(1-t)} (k^{M-1}x^{M-1})^t}{(kx/a_3; t)(kx/a_4; t)\cdots(kx/a_{2M+1}; t)(a_3a_4\cdots a_{2M+1})^t} .$$

Now let $a_1, a_2 \cdots, a_{2Mn1} \to \infty$ in (3.2). Then we get

$$(3.3)\quad \sum_{s=0}^{\infty} K_s(-1)^s k^{Ms} x^{\frac{1}{2}s\{(2M+1)s-1\}}$$
$$=\prod (kx) \sum_{r=0}^{\infty} \frac{k^r x^{r^2}}{(x; r)(kx; r)} \sum_{t=0}^{r} K_{t,r} k^{(M-1)t} x^{(M-1)t^2} .$$

And in (3.2) if we take $(M-1)$ for M, $a_1=x^{-r}$ and let $a_2, a_3, \cdots, a_{2M-1}$ tend to ∞, we have

$$(3.4)\quad \sum_{t=0}^{r} K_{t,r} k^{(M-1)t} x^{(M-1)t^2}$$
$$=(kx; r) \sum_{t=0}^{r} \frac{k^t x^{t^2}(x^{r-t+1}; t)}{(x; t)} \sum_{s=0}^{t} K_{s,t} k^{(M-2)s} x^{(M-2)s^2} .$$

On repeated application of (3.4) on the right-hand side of (3.3) it follows that

$$\{\prod(kx)\}^{-1}\sum_{s=0}^{\infty}K_s(-1)^s k^{Ms}x^{\frac{1}{2}s\{(2M+1)s-1\}}=\sum_{r=0}^{\infty}\frac{k^r x^{r^2}}{(x;\,r)}\prod_{n=1}^{M-2}S_{n,n-1}\,,$$

there being $(M-2)$ terminating series on the right since

$$\sum_{s=0}^{t}K_{s,t}=0 \tag{3.5}$$

by Watson's transformation [(2); § 8.5 (2)] of a terminating ${}_8\phi_7$ into a Saalschützian ${}_4\phi_3$.

Now it is easily verified that

$$\prod_{n=1}^{M-2}S_{n,n-1}$$

can, by suitable rearrangements, be simplified to

$$\sum_{t_1=0}^{(M-2)r}\frac{k^{t_1}x^{t_1^2}(x^{r-t_1+1};\,t_1)}{(x;\,t_1)}\sum_{t_2=0}^{\left[\frac{M-3}{M-2}t_1\right]}\frac{(x^{t_1-2t_2+1};\,2t_2)x^{-2t_2(t_1-t_2)}}{(x;\,t_2)(x^{r-t_1+1};\,t_2)}\prod_{n=3}^{M-2}T_{n,M}\,,$$

where $t_h=r_h+r_{h+1}+\cdots+r_{M-2}$, $(h=1,2,\cdots,M-2)$.

Thus on putting $r+t_1=t$, we finally have

$$\begin{aligned}\{\prod(kx)\}^{-1}&\sum_{s=0}^{\infty}K_s(-1)^s k^{Ms}x^{\frac{1}{2}s\{(2M+1)s-1\}}\\&=\sum_{t=0}^{\infty}\frac{k^t x^{t^2}}{(x;\,t)}\sum_{t_1=0}^{\left[\frac{M-2}{M-1}t\right]}\frac{(x^{t-2t_1+1};\,2t_1)x^{-2t_1(t-t_1)}}{(x;\,t_1)}\prod_{n=2}^{M-2}T_{n,M}\,.\end{aligned}\tag{3.6}$$

This is a k-cum-M generalization of the Rogers-Ramanujan identities. For any assigned values of M and t, the repeated terminating series can, by dividing out by the denominator factors, be evaluated as polynomials in x.

Let us now write

$$G_{M,t}(x)=x^{t^2}\sum_{t_1=0}^{\left[\frac{M-2}{M-1}t\right]}\frac{(x^{t-2t_1+1};\,2t_1)x^{-2t_1(t-t_1)}}{(x;\,t_1)}\prod_{n=2}^{M-2}T_{n,M}\,. \tag{3.7}$$

Then, as usual, for $k=1$ and $k=x$ respectively, the left-hand side of (3.6) can be expressed as a product by means of Jacobi's classical identity

$$\sum_{n=-\infty}^{\infty}(-1)^n x^{n^2}z^n=\prod_{n=1}^{\infty}(1-x^{2n-1}z)(1-x^{2n-1}/z)(1-x^{2n}) \tag{3.8}$$

and we get Alder's generalization of the first and second Rogers-

Ramanujan identities in the form

$$(3.9)\qquad \prod_{n=0}^{\infty} \frac{(1-x^{(2M+1)n+M})(1-x^{(2M+1)n+M+1})}{(1-x^{(2M+1)n+1})(1-x^{(2M+1)n+2})\cdots(1-x^{(2M+1)n+2M})} = \sum_{t=0}^{\infty} \frac{G_{M,t}(x)}{(x;\,t)}$$

and

$$(3.10)\qquad \prod_{n=0}^{\infty} \frac{1}{(1-x^{(2M+1)n+2})(1-x^{(2M+1)n+3})\cdots(1-x^{(2M+1)n+2M-1})} = \sum_{t=0}^{\infty} \frac{x^t G_{M,t}(x)}{(x;\,t)}$$

where $G_{M,t}(x)$ is given by (3.7). The polynomials $G_{M,t}(x)$ can be seen by easy verification to be identical with $G_{k,\mu}(x)$ of Alder.

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Added in Proof. If in (3.2) we take $a_1 = -\sqrt{kx}$, make $a_2, a_3, \cdots, a_{2M+1}$ tend to ∞, and proceed as in § 3, we get for $k=1$ and $k=x$ the respective identities

$$\prod_{n=1}^{\infty} \frac{(1-x^{2Mn-(M-\frac{1}{2})})(1-x^{2Mn-(M+\frac{1}{2})})(1-x^{2Mn})}{(1-x^n)}$$

$$= \{\Pi(-x^{\frac{1}{2}})\}^{-1} \sum_{t=0}^{\infty} \frac{x^{\frac{1}{2}t^2}(-x^{\frac{1}{2}})_t}{(x)_t} \sum_{t_1=0}^{\left[\frac{M-2}{M-1}t\right]} \frac{x^{-t_1(t-\frac{3}{2}t_1)}}{(x)_{t_1}}$$

$$\times \frac{(x^{t-2t_1+1})_{2t_1}}{(-x^{\frac{1}{2}+t-t_1})_{t_1}} \prod_{n=2}^{M-2} T_{n,M}$$

and

$$\prod_{n=1}^{\infty} \frac{(1-x^{2Mn-1})(1-x^{2Mn-(2M-1)})(1-x^{2Mn})}{(1-x^n)}$$

$$= \{\pi(-x)\}^{-1} \sum_{t=0}^{\infty} \frac{x^{\frac{1}{2}t(t+1)}(-x)_t}{(x)_t} \sum_{t_1=0}^{\left[\frac{M-2}{M-1}\right]} \frac{x^{\frac{1}{2}t_1} x^{-t_1(t-\frac{3}{2}t_1)}}{(x)_{t_1}}$$

$$\times \frac{(x^{t-2t_1+1})_{2t_1}}{(-x^{1+t-t_1})_{t_1}} \prod_{n=2}^{M-2} T_{n,M}\,.$$

References

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