UNIQUENESS THEORY FOR ASYMPTOTIC EXPANSIONS IN GENERAL REGIONS

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1. Introduction. Let D be a simply connected region with an analytic boundary C. Assume that $z=0$ is an interior point while $z=1$ lies on the boundary. We assume further that the tangent to C at $z=1$ is not parallel to the real axis. In this case, we shall be able to fit into D small angles Γ placed symmetrically about the real axis and with vertex at $z=1$. These angles will be of the form $-\delta \leqq \theta \leqq \delta$ or $\pi-\delta \leqq \theta \leqq \pi+\delta$, $\delta > 0$, depending upon the location of $z=1$. For a given $f(z)$ regular in D, we consider the following limits defined recursively

$$
\begin{aligned}
a_0 &= \lim_{z\to 1} f(z) \\
a_1 &= \lim_{z\to 1} (z-1)^{-1}[f(z)-a_0] \\
a_2 &= \lim_{z\to 1} (z-1)^{-2}[f(z)-a_0-a_1(z-1)] \\
&\cdots\;.
\end{aligned} \tag{1}
$$

If each limit in (1) exists and is independent of the manner in which $z\to 1$ through values in some angle Γ, then $f(z)$ is said to possess an asymptotic expansion at $z=1$ in the sense of Poincaré, and this is indicated by writing

$$f(z) \sim \sum_{n=0}^{\infty} a_n(z-1)^n\,. \tag{2}$$

We shall designate by $A(=A(D))$ the linear class of functions which are regular in D and which possess asymptotic expansions at $z=1$ in the sense of Poincaré. The angle Γ in which (1) is valid may depend upon the particular $f \in A$ selected.

Uniqueness theory is concerned with distinguishing nontrivial subclasses of A within which the expansion $\sum_{n=0}^{\infty} a_n(z-1)^n$ determines the corresponding function uniquely. Write for the remainder

$$R_n(z)=f(z)-a_0-a_1(z-1)-\cdots-a_{n-1}(z-1)^{n-1}\,, \tag{3}$$

and consider the ratios

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$$(4) \qquad f_n(z)=(z-1)^{-n}R_n(z) \qquad (n=1, 2, \cdots),\ f_0=f.$$

For $f \in A$, the functions $f_n(z)$ are regular in D and are bounded as $z \to 1$ in Γ. For a given sequence of positive quantities $\{m_n\}$, we consider the subset $A(m_n)$ of A consisting of those functions which satisfy in addition

$$(5) \qquad \|f_n\|^2 < Mk^n m_n^2 \qquad (n=0, 1, 2, \cdots)$$

for some $M>0$, $k>0$. Here $\|\ \ \|$ designates some conveniently chosen norm. The constants M and k may vary from function to function within the class. With the selection

$$(6) \qquad \|f\| = \max_{z \in D} |f(z)|,$$

it has been shown by Watson [1] and F. Nevanlinna [5] that when D is a sector, we may produce uniqueness classes by restricting the growth of the sequence $\{m_n\}$ sufficiently. When D is the unit circle, T. Carleman [2] has given necessary and sufficient conditions on $\{m_n\}$ in order that the resulting subclass $A(m_n)$ be a uniqueness class. At the same time Carleman raises the problem of giving necessary and sufficient conditions in the case of a more general region D. This problem (with the norm (6)) has been known in the literature at the generalized problem of Watson. It has been treated by Mandelbrojt and MacLane [3] using the theory of distortion in conformal mapping. See also Meili [4]. In the present paper, we adopt the norm

$$(7) \qquad \|f\|^2 = \int_C |f(z)|^2\, ds,$$

and show how it is possible to combine Carleman's idea of introducing an appropriate minimum problem with the techniques afforded by the theory of conformal kernel functions to arrive at a solution to this general problem. The class $A(m_n)$ will henceforth refer to the norm (7). Thus the question which we are treating may be worded as follows: *What are necessary and sufficient conditions on the sequence of constants $\{m_n\}$ in order that*

$$(8) \qquad \|f_n\|^2 = \int_C |f_n(z)|^2\, ds = \int_C \left| \frac{f(z)-a_0-a_1(z-1)-\cdots-a_{n-1}(z-1)^{n-1}}{(z-1)^n} \right|^2 ds < Mk^n m_n^2$$

determine $f(z)$ uniquely from the asymptotic coefficients a_n.

2. Preliminary observations. We must first explain the sense in which we shall understand the expression

$$\int_C |f(z)|^2 \, ds$$

when $f(z)$ is regular in D but not necessarily in its closure. Let $w = m(z)$ map D conformally onto the unit circle with $m(0)=0$ and $m(1)=1$. The images of $|w|=r$ will be designated by C_r, $0<r<1$. It is well known that the set of functions

$$\phi_n(z)=\frac{1}{\sqrt{2\pi}}\frac{[m'(z)]^{1/2}}{r^{n+1/2}}[m(z)]^n \qquad (n=0,\ 1,\ 2,\ \cdots) \tag{9}$$

is complete and orthonormal over each C_r, $0<r<1$, relative to the inner product

$$(f,\ g)=\int_{C_r} f\bar{g} \, ds \, .$$

Suppose then that we are given a function $f(z)$ which is regular in D. Then for any fixed $0<r<1$, $f(z)$ is continuous on C_r. Hence we can write

$$f(z)=\sum_{n=0}^{\infty} a_n \phi_n(z) \tag{10}$$

holding uniformly and absolutely in the interior of C_r. The coefficients a_n are given by

$$a_n=\int_{C_r} f(z)\,\overline{\phi_n(z)} ds \qquad (n=0,\ 1,\ \cdots) \, . \tag{11}$$

Hence, for $r^*<r$, we have from (9) and (10),

$$\int_{C_{r^*}} |f(z)|^2 \, ds=\sum_{n=0}^{\infty} |a_n|^2 \frac{r^{*2n+1}}{r^{2n+1}} \, . \tag{12}$$

This equation tells us that

$$\int_{C_{r^*}} |f(z)|^2 \, ds$$

is an increasing function of r^* and hence

$$\lim_{r^*\to 1^-} \int_{C_{r^*}} |f(z)|^2 \, ds$$

exists (or equals $+\infty$). For $f(z)$ regular in D we shall therefore agree that

$$\int_C |f(z)|^2\, ds = \lim_{r\to 1-} \int_{C_r} |f(z)|^2\, ds\ .$$

LEMMA. *Given an arbitrary sequence of positive constants* $\{m_n\}$; *the class* $A(m_n)$ *is not a uniqueness class for asymptotic expansions at* $z=1$ *if and only if there exists an* $f \not\equiv 0$ *regular in* D *and constants* $M>0$, $k>0$, *for which*

$$\left\| \frac{f(z)}{(z-1)^n} \right\|^2 < Mk^n\, m_n^2 \qquad (n=0,\ 1,\ 2,\ \cdots). \tag{13}$$

Proof. If $A(m_n)$ is not a uniqueness class, there will exist two functions $g(z)$, $y(z) \in A(m_n)$, $g \not\equiv h$, possessing the same asymptotic expansion, say $\sum_{n=0}^{\infty} a_n(z-1)^n$, and satisfying

$$\int_C \left| \frac{g(z) - \sum_{k=0}^{n-1} a_k(z-1)^k}{(z-1)^n} \right|^2 ds < M_1 k_1^n m_n^2 \qquad (n=0,\ 1,\ \cdots) \tag{14}$$

$$\int_C \left| \frac{h(z) - \sum_{k=0}^{n-1} a_k(z-1)^k}{(z-1)^n} \right|^2 ds < M_2 k_2^n m_n^2$$

with $k_1 \leqq k_2$. Therefore, by Minkowski's inequality,

$$\begin{aligned} \int_C \left| \frac{g(z)-h(z)}{(z-1)^n} \right|^2 ds &< (M_1^{1/2} k_1^{n/2} + M_2^{1/2} k_2^{n/2})^2 m_n^2 \\ &= (M_1^{1/2}(k_1/k_2)^{n/2} + M_2^{1/2})^2 k_2^n m_n^2 \\ &< (M_1^{1/2} + M_2^{1/2})^2 k_2^n m_n^2 \end{aligned} \tag{15}$$

so that $g-h$ does not vanish identically and satisfies (13) with $M=(M_1^{1/2} + M_2^{1/2})^2$ and $k=k_2$.

Conversely, let $f \not\equiv 0$ satisfy (13). We shall show that (13) implies

$$\lim_{z\to 1} \frac{f(z)}{(z-1)^n} = 0 \qquad (n=0,\ 1,\ 2,\ \cdots) \tag{16}$$

as $z \to 1$ through values in some angle Γ. Assuming, for the moment, that this is so, (16) and (1) imply that

$$f(z) \sim 0 + 0\cdot(z-1) + 0\cdot(z-1)^2 + \cdots\ . \tag{17}$$

That is, $f(z)$ possesses an identically zero asymptotic expansion at $z=1$. Furthermore $f_n = f(z)(z-1)^{-n}$, so that (13) implies that $f \in A(m_n)$. Thus, $A(m_n)$ is not a uniqueness class for asymptotic expansions at $z=1$.

We show now that (13) implies (16). Given any $g(z)$ regular in D. Select any $0<r<1$. We have from (9), (10), (11), and the Schwarz inequality

$$|g(z)|^2 < K_{C_r}(z, \bar{z}) \int_{C_r} |g(z)|^2\, ds, \tag{18}$$

for all z interior to C_r. K_{C_r} is the so-called Szegö kernel function for C_r whose explicit expression is (Szegö [6], Bergman [1])

$$K_{C_r}(z, \bar{z}) = \sum_{n=0}^{\infty} \phi_n(z)\overline{\phi_n(z)} = \frac{1}{2\pi}\frac{r|m'(z)|}{r^2-|m(z)|^2}. \tag{19}$$

Writing $f(z)/(z-1)^n$ in place of $g(z)$ in (18), and using (13) and the monotonicity with r of

$$\int_{C_r} |f(z)|^2\, ds,$$

we find for $j \leqq n$ and z interior to C_r,

$$\left|\frac{f(z)}{(z-1)^j}\right|^2 \leqq \frac{|(z-1)^{n-j}|^2 r|m'(z)|}{(2\pi)(r^2-|m(z)|^2)} Mk^n m_n^2 \qquad (n=0, 1, 2, \cdots). \tag{20}$$

For each z in D we select an $r=r(z)=|m(z)|+\varepsilon(z)<1$ where $\varepsilon(z)$ is defined by

$$\varepsilon(z) = \frac{1}{2}(1-|m(z)|). \tag{21}$$

Thus,

$$\lim_{z\to 1} \varepsilon(z) = 0. \tag{22}$$

Here, $z\to 1$ through values in D. From (20), (21), and $r<1$,

$$\left|\frac{f(z)}{(z-1)^j}\right|^2 \leqq \frac{|(z-1)^{n-j}|^2}{2\pi}\cdot\frac{|m'(z)|Mk^n m_n^2}{2|m(z)|\varepsilon(z)+\varepsilon^2(z)} \tag{23}$$

$$< \frac{|(z-1)^{a-j}|^2|m'(z)|Mk^n m_n^2}{4\pi|m(z)|\varepsilon(z)}.$$

We are now ready to consider the limit of (23) as $z\to 1$. First consider

$$\frac{\varepsilon(z)}{|z-1|} = \frac{1-|m(z)|}{2|z-1|} = \frac{1}{2}(1+|m(z)|)^{-1}\frac{(1-|m(z)|^2)}{|z-1|}. \tag{24}$$

Since $m(z)$ is by assumption analytic at $z=1$, we have in a neighborhood of $z=1$,

$$m(z)=1+(z-1)R(z)\,, \tag{25}$$

where $R(z)$ is analytic there. Note that $R(1)=m'(1)\neq 0$, and write $R(z)=\sigma(z)e^{i\alpha(z)}$, $\sigma(z)>0$. We have $\sigma(1)\neq 0$ and $\alpha(1)\neq \pi/2$, $3\pi/2$, inasmuch as the tangent to C at $z=1$ is assumed not parallel to the real axis. Furthermore, write $z=1+\rho e^{i\theta}$. Then, from (25),

$$\begin{aligned}\frac{1-|m(z)|^2}{|z-1|}&=\frac{-2\mathscr{R}\{(z-1)R(z)\}}{|z-1|}-\frac{|z-1|^2|R(z)|^2}{|z-1|}\\&=-2\mathscr{R}\{e^{i\theta}\sigma(z)^{i\alpha(z)}\}-|z-1||R(z)|^2\\&=-2\sigma(z)\cos(\theta+\alpha(z))-|z-1||R(z)|^2\,.\end{aligned} \tag{26}$$

If $z\to 1$ through some angle Γ: $-\delta\leqq\theta\leqq\delta$ or $\pi-\delta\leqq\theta\leqq\pi+\delta$, then, since $\alpha(1)\neq\pi/2$, $3\pi/2$, it follows from the above that for δ sufficiently small, the expression (26) will be bounded away from 0. In view of (24) we will have

$$\frac{\varepsilon(z)}{|z-1|}\geqq\tau>0\ ;\ z\to 1 \tag{27}$$

for z in some Γ. From (23), we have,

$$\left|\frac{f(z)}{(z-1)^j}\right|^2<|z-1|^{2n-2j-1}|m'(z)|Mk^n m_n^2\Big/\frac{4\pi|m(z)|\cdot\varepsilon(z)}{|z-1|}\,. \tag{28}$$

Thus, for $2n-2j-1>1$ it is now clear from (28) and (27) that

$$\lim_{z\to 1}\frac{f(z)}{(z-1)^j}=0\,.$$

For each j considered we need only use an $n>j+1$. This completes the proof of the lemma.

3. The uniqueness theorem.

THEOREM. *Given an arbitrary sequence of positive constants m_n. The class $A(m_n)$ is a uniqueness class for asymptotic expansions at $z=1$ if and only if for all $t>0$,*

$$\limsup_{n\to\infty}\int_C\log\left\{\sum_{k=0}^{n}\frac{t^k}{m_k^2}|(z-1)^{n-k}|^2\right\}\frac{\partial}{\partial n}\log|m(z)|ds=\infty\,. \tag{20}$$

Here $\partial/\partial n$ designates normal differentiation in the positive sense.

The above statement is equivalent to saying that $A(m_n)$ is not a uniqueness class if and only if there exists a $t>0$ and a $K>0$ such that

$$\int_C \log\left\{\sum_{k=0}^{n} \frac{t^k}{m_k^2} |(z-1)^{n-k}|^2\right\}\frac{\partial}{\partial n} \log|m(z)|\,ds < K, \qquad n=0, 1, 2, \cdots \tag{30}$$

K may depend upon t, but is independent of n.

In view of the lemma of the preceding section, we shall prove that (30) is a necessary and sufficient condition for the existence of an $f(z) \neq 0$, and M, and a k which satisfy (13).

Consider the following sequence of integrals

$$I_n(f)=\sum_{k=0}^{n} \frac{t^k}{m_k^2} \int_C \left|\frac{f(z)}{(z-1)^k}\right|^2 ds; \tag{31}$$

$$= \sum_{k=0}^{n} \frac{t^k}{m_k^2} \|f\|_k^2 \qquad n=0, 1, \cdots,$$

where we have written

$$\|f\|_k^2 = \int_C \left|\frac{f(z)}{(z-1)^k}\right|^2 ds; \qquad k=0, 1, \cdots. \tag{32}$$

We can also write (31) in the form

$$I_n(f)=\left\|\frac{\rho_n(z)f(z)}{(z-1)^n}\right\|^2 \tag{33}$$

where $\rho_n(z)$ is an analytic function which is regular in D, continuous on C and is such that

$$|\rho_n(z)|=\left\{\sum_{k=0}^{n} \frac{t^k}{m_k^2} |(z-1)^{n-k}|^2\right\}^{1/2}, \quad \text{for } z \text{ on } C. \tag{34}$$

We shall show below how a $\rho_n(z)$ may be constructed which has these properties and has, in addition, the property that

$$\rho_n(z) \neq 0 \quad \text{for} \quad z \text{ in } D. \tag{35}$$

Let n be fixed, and consider the following minimum problem P_n. *Determine that function $f(z)$ regular in D with $f(0)=1$ and such that*

$$I_n(f)=\textit{minimum}. \tag{36}$$

This problem can be solved by passing to a related problem P_n'. *Determine that function $g(z)$ regular in D with $g(0)=1$ and such that*

$$\|g\|^2=\textit{minimum} \tag{37}$$

The solution of the problem P_n' is given by the function (see, for example Szegö [6], Bergman [1])

$$g^*(z)=K_D(z, 0)/K_D(0, 0) \tag{38}$$

where $K_D(z, \overline{w})$ is the Szegö kernel function of the region D. The minimum value of the integral (37) is $1/K_D(0, 0)$. If we write

$$I_n(f)=|\rho_n(0)|^2\left\|\frac{\rho_n(z)f(z)}{\rho_n(0)(1-z)^n}\right\|^2, \tag{39}$$

we see, in view of (35) that the function $\rho_n(z)f(z)/\rho_n(0)(1-z)^n$ can play the role of $g(z)$ in the problem P_n'. The minimizing function f_n^* of the problem P_n is therefore

$$f_n^*(z)=\frac{K_D(z, 0)(1-z)^n\rho_n(0)}{\rho_n(z)K_D(0, 0)}, \tag{40}$$

and the minimum value of the integral is

$$I_n(f_n^*)=\frac{|\rho_n(0)|^2}{K_D(0, 0)}. \tag{41}$$

We now assert: a necessary and sufficient condition in order that there exist an $f(z)\not\equiv 0$ and constants $M>0$, $k>0$ such that

$$\|f\|_n^2=\left\|\frac{f(z)}{(z-1)^n}\right\|^2<Mk^n m_n^2 \qquad (n=0, 1, \cdots) \tag{42}$$

is that there exists a $t>0$ and a $K>0$ such that

$$I_n(f_n^*)\leqq K \qquad n=0, 1, 2, \cdots. \tag{43}$$

Referring to (41), this is equivalent to asserting that there exist a $t>0$ and a K' such that

$$|\rho_n(0)|\leqq K' \qquad n=0, 1, 2, \cdots. \tag{44}$$

We can prove this as follows. Suppose first that $q(z)$ is such that (42) holds for it. This function $q(z)$ may have a zero of the pth order at $z=0$. The function $f(z)=q(z)/z^p$ is then regular in D and is such that $f(0)\neq 0$. Now,

$$\begin{aligned} I_n(f(z)/f(0)) &= \sum_{j=0}^{n}\frac{t^j}{m_j^2}\int_C\left|\frac{q(z)}{f(0)z^p(z-1)^j}\right|^2 ds \\ &\leqq \sum_{j=0}^{n}\frac{t^j}{m_j^2}\frac{1}{|f(0)|^2}\frac{1}{d^{2p}}M\cdot m_j^2k^j \\ &\leqq \frac{M}{d^{2p}|f(0)|^2}\sum_{j=0}^{n}t^jk^j\leqq\frac{M}{d^{2p}|f(0)|^2(1-tk)}, \end{aligned} \tag{45}$$

provided we select $0<t<1/k$. Here d designates the minimum distance from $z=0$ to C. Now since

$$I_n(f_n^*)\leqq I_n(f(z)/f(0))\leqq\frac{M}{d^{2p}|f(0)|^2(1-tk)}\,, \qquad (n=0,\,1,\,\cdots) \tag{46}$$

then (43) is satisfied with K equal to the right hand constant in (46).

Conversely, suppose that there exists a $t>0$ and $K>0$ such that (43) holds. Then from (31),

$$\sum_{k=0}^{n}\frac{t^k}{m_k^2}\|f_n^*\|_k^2\leqq K \qquad (n=0,\,1,\,2,\,\cdots)\,. \tag{47}$$

In particular, taking the first term of (47) we obtain

$$\frac{1}{m_0^2}\|f_n^*\|_0^2<K \qquad n=0,\,1,\,2,\,\cdots\,. \tag{48}$$

Hence we have

$$\|f_n^*\|<\text{const.} \qquad (n=0,\,1,\,2,\,\cdots)\,. \tag{49}$$

The inequalities (49) imply that the sequence of minimizing functions $\{f_n^*\}$ form a normal family and therefore there exist indices $n_1,\,n_2,\,\cdots$ such that $f_{n_k}^*\to F(z)$ uniformly in any closed region interior to D. Again, using (47) we have, for fixed j and for all $n\geqq j$

$$\frac{t^j}{m_j^2}\|f_n^*\|_j^2\leqq K\,. \tag{50}$$

Now for any $0<\rho<1$, we have

$$\|f_n^*\|_j^2=\int_C\left|\frac{f_n^*(z)}{(z-1)^j}\right|^2ds\geqq\int_{C_\rho}\left|\frac{f_n^*(z)}{(z-1)^j}\right|^2ds\,, \tag{51}$$

so that from (50) and (51),

$$\int_{C_\rho}\left|\frac{f_n^*(z)}{(z-1)^j}\right|^2ds<Km_j^2t^{-j} \qquad (k=0,\,1,\,2,\,\cdots)\,. \tag{52}$$

Let n take on the values n_i in (52) and let j be fixed. Then since $f_n^*(z)\to F(z)$ uniformly in and on C_ρ,

$$\int_{C_\rho}\left|\frac{F(z)}{(z-1)^j}\right|^2ds\leqq Km_j^2t^{-j}\,. \tag{53}$$

This result is independent of ρ and hence we may allow $\rho\to1$. Thus,

(54) $$\int_C \left|\frac{F(z)}{(z-1)^j}\right|^2 ds < K m_j^2 t^{-j} \qquad (j=0, 1, 2, \cdots).$$

Since obviously $F(0)=1$, we have exhibited in $F(z)$ a function regular in D, which does not vanish identically, a constant $M(=K)$ and a constant $k(=t^{-1})$ for which (42) holds.

It remains to construct $\rho_n(z)$, to show that it does not vanish, and to compute $\rho_n(0)$. Designate by $t_n(z)$ the positive function

(55) $$t_n(z) = \left\{\sum_{k=0}^{n} \frac{t^k}{m_k^2} |(z-1)^{n-k}|^2\right\}^{1/2}$$

defined on C. Now $\log t_n(z)$ is continuous on C and hence

(56) $$u_n(z) = \frac{1}{2\pi}\int_C \log t_n(w) \frac{\partial g(z, w)}{\partial n} ds$$

where $g(z, w)$ is the Green's function for D, is harmonic in D and assumes on C the boundary values $\log t_n(z)$. Designate by v_n the harmonic conjugate of u_n. Then $u_n(z)+iv_n(z)$ is regular and single valued in D, as is

(57) $$p_n(z) = \exp[u_n(z)+iv_n(z)].$$

Now, $|p_n(z)|=e^{u_n}$, so that on C, $|p_n(z)|=t_n(z)$. Furthermore $p_n(z)\neq 0$, as is clear from (57). Thus we may use $\rho_n(z)=p_n(z)$. The condition (44) then becomes: there exists a $t>0$ and a $K'>0$ such that

(58) $$u_n(0) \leq K' \qquad (n\to\infty).$$

Finally, using the representation

(59) $$g(z, w) = \log\left|\frac{m(z)-m(w)}{1-m(z)\overline{m(w)}}\right|$$

with $z=0$ in (56), we obtain the stated condition (29).

4. Concluding remarks. Norms other than (6) might be contemplated. In particular, we might have used

(60) $$\|f\|^2 = \iint_D |f(z)|^2 dA.$$

However (60) has the disadvantage that the solution of the corresponding minimum problem P_n can not be so elegantly expressed in terms of an analytic function $\rho_n(z)$ and so the role of the sequence $\{m_n\}$ is not immediately evident as with (29).

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