A SPACE OF MULTIPLIERS OF TYPE $L^p(-\infty, \infty)$

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1. Introduction. Let $V(G)$ denote the set of all functions having finite variation on G. Set $G = (-\infty, \infty) = \hat{G}$, and let $V_\infty(G)$ be the Banach space of all functions in $V(G)$ which vanish at infinity. If $f \in V_\infty(G)$, then there exists a bounded linear operator $(t_p f)$ on $L^p(\hat{G})$ such that

$$(\mathrm{i}_0) \qquad (\text{Fourier transform of } (t_p f)x) = (\text{Fourier transform of } x) \cdot f$$

for all x in $L^p(\hat{G})$. This will be shown in 7.2. In the terminology of Hille [3, p. 566], functions f having property (i_0) are called "factor functions for Fourier transforms of type (L_p, L_p)".

Suppose $1 < p < \infty$. When $f \in L^1(G) \cap V(G) \subset V_\infty(G)$, then $(t_p f)$ is a singular integral operator: for all x in $L^p(\hat{G})$ it is found that $(t_p f)x$ has the form

$$[(t_p f)x]_\lambda = \frac{1}{2\pi i}\int_{-\infty}^{\infty} x(\theta)\frac{F(\theta - \lambda)}{\theta - \lambda}d\theta \qquad (\lambda \in \hat{G}) ,$$

where the integral is taken in the Cauchy principal value sense.

In 6.2 will be defined a set $\blacktriangle(L^p(\hat{G}))$ which contains all factor functions for Fourier transforms of type (L_p, L_p); the set $\blacktriangle(L^p(\hat{G}))$ is a slight extension of what Mihlin [6] calls "multipliers of Fourier integrals". We will find a number N_p such that

(i) *if* $f \in V_\infty(G)$ *then* $f \in \blacktriangle(L^p(\hat{G}))$ *and* $\|(t_p f)\| \leqq N_p \cdot \|f\|_v$,

where $\|f\|_v$ denotes the total variation on G of the function f. Let F_* be the mapping $\{x \to x * F\}$, where $x * F$ is the convolution of the functions x and F;

$$[x * F]_\lambda = \int_{-\infty}^{\infty} x(\theta) \cdot F(\theta - \lambda)d\theta \qquad (\lambda \in \hat{G}).$$

Let (Yf) denote the Fourier transform of the function f:

(ii) *if* $f \in L^1(G) \cap V(G)$, *then the transformation* $(Yf)_*$ *is a densely defined bounded operator, and* $(t_p f)$ *is its continuous linear extension to the whole space* $L^p(\hat{G})$.

Let us for a moment call $G = \{0, \pm 1, \pm 2, \cdots\}$ and $\hat{G} = [0, 1]$. In

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a sense, the following relations are duals of (i) and (ii), respectively:

(i′) *if* $F \in V(\hat{G})$ *then* $(\dot{Y}F) \in \blacktriangle(L^p(\hat{G}))$ *and* $\|t_p(YF)\| \leqq k_p \cdot \|F\|_v$

(ii′) *if* $F \in V(\hat{G})$ *then* $F_* = t_p(YF)$ *is a bounded operator on* $L^p(\hat{G})$.

When $\hat{G} = [0, 1]$ these properties are easily verified (see 8.1). We will not[1] prove (i′)-(ii′) for other choices of G.

When $G = [0, 1]$, then (ii) is seen to be a theorem due to Stečkin [10]; by means of appropriate definitions, it could be shown that (i) also holds for this particular choice of G.

2. Applications. If f belongs to the class S of members of $L^1(G) \cap V(G)$ such that $(Yf) \in L^1(\hat{G})$, then $(Yf)_* = (t_p f)$ is a bounded operator defined on all of $L^p(\hat{G})$; it is interesting to compare this result with the conclusion $F_* = t_p(YF)$ of (ii′). All the classical convolution operators (Poisson, Picard, Weierstrass, Stieltjes, Dirichlet, Fejér,..etc. [7]) are of the form $(t_p f)$, where $f \in S$. See § 8.

3. Preliminaries. We assume $1 < p < \infty$ throughout, and write $G = (-\infty, \infty)$. Denote by L^0 the set of step functions with compact support. Let V be the set of all functions a defined on G and such that $\|a\|_v \neq \infty$, where $\|a\|_v$ denotes the total variation on G.

3.1 Definitions. Let V_∞ be the set of all functions a in V such that $\lim a(\theta) = 0$ whenever $|\theta| \to \infty$. We will write L^p instead of $L^p(G)$. If $\iota = 0, 1$ and $f \in L^1$, then the Fourier transforms $[_\iota Yf]$ are the functions g_ι defined by

$$[_\iota Yf]_\lambda = g_\iota(\lambda) = \int_{-\infty}^{\infty} \exp(2\pi i\lambda(-1)^\iota\theta) \cdot f(\theta) d\theta \qquad (\lambda \in G) . \tag{1}$$

To lighten the notation, we will write Yf for $[_1 Yf]$ and Ψf for $[_0 Yf]$.

3.2 Lemma. *If* $a \in L^1 \cap V$, *then* $a \in V_\infty$ *and*

$$\int_{-\infty}^{\infty} e^{-2\pi i\theta t} da(t) = 2\pi i\theta \cdot [Ya]_\theta \qquad (\theta \in G) . \tag{2}$$

Proof. Since $a \in V$, the limits $a(\pm\infty) = \lim a(\theta)$ (when $\theta \to \pm\infty$) exist. Since $\|a\|_1 < \infty$ we have

$$\lim_{\theta \to \pm\infty} \int_\theta^{\theta+1} |a| = 0 . \tag{3}$$

The eventuality $a(\pm\infty) \neq 0$ implies a contradiction of (3). Therefore

[1] It would be of interest to determine the validity of (i)-(ii) and (i′)-(ii′) in the general case where G is a connected locally compact abelian group with dual group $\hat{G}$. It is mainly in order to suggest such an investigation that (i′)-(ii′) are mentioned here.

$a(\pm\infty) = 0$, which permits the integration of (1) by parts to obtain (2).

3.3 DEFINITIONS. Let $\delta_* = (-\infty, -\delta] \cup [\delta, \infty)$ and let $(T_\delta a)x$ be the function defined by

$$[(T_\delta a)x]_\lambda = \int_{\delta_*} d\theta \frac{x(\lambda - \theta)}{2\pi i\theta} \int_{-\infty}^{\infty} e^{-2\pi i\theta t}\, da(t) \tag{4}$$

for all λ in G. We denote by V_1 the set of all members a of V such that, for all x in L^0, the limit

$$[(Ta)x]_\lambda = \lim_{\delta \to 0+} [(T_\delta a)x]_\lambda$$

exists almost-everywhere on G. Let Ta be the operator $\{x \to (Ta)x\}$ defined on L^0.

3.4 LEMMA. *If $h(\theta) = i\theta/|\theta|$, then $h \in V_1$ and Th is the restriction to L^0 of the Hilbert transformation. Moreover $\|(T_\delta h)x\|_p \leqq c_p \cdot \|x\|_p$, where c_p is the norm of Th.*

Proof. This follows from the statement in [8, p. 241] that $\|(T_\delta h)x\|_p \leqq \|(Th)x\|_p$. Theorem G in [1, p. 251] yields a less precise result.

3.5. LEMMA. *If $a \in L^1 \cap V$ then $a \in V_1$ and $x * [Ya] = (Ta)x$ whenever $x \in L^0$.*

Proof. Suppose $\delta > 0$. By definition

$$(x * [Ya])_\lambda = \int_{-\infty}^{\infty} d\theta \cdot x(\lambda - \theta) \cdot [Ya]_\theta = E^\delta(\lambda) + G^\delta(\lambda) ,$$

where

$$G^\delta(\lambda) = \int_{\delta_*} d\theta \cdot x(\lambda - \theta) \cdot [Ya]_\theta \qquad (\lambda \in G) ,$$

while $E^\delta(\lambda)$ is the same integral over the interval $(-\delta, \delta)$. It is clear that $\lim E^\delta(\lambda) = 0$ when $\delta \to 0+$. On the other hand, $G^\delta = (T_\delta a)x$ follows immediately from (2) and (4). This concludes the proof.

3.6 LEMMA. *Suppose $a \in V_1$ and $x \in L^0$. If there exists a number k_p such that $\|(T_\delta a)x\|_p \leqq k_p$ for all $\delta > 0$, then $\|(Ta)x\|_p \leqq k_p$.*

Proof. Set $q = p/(p-1)$. Observe first that

$$\|g\|_p = \sup\left\{\left|\int g \cdot \varphi\right| : \varphi \in L^q \text{ and } \|\varphi\|_q \leqq 1\right\} . \tag{5}$$

Next, we infer from a theorem of F. Riesz ([8], p. 227 footnote 10) that the uniform boundedness of $\|(T_\delta a)x\|_p$ implies that, for all φ in L^q with $\|\varphi\|_q \leqq 1$:

$$\int[(Ta)x]\cdot\varphi = \lim_{\delta\to 0+}\int[T_\delta a)x]\cdot\varphi\ . \tag{6}$$

By (5) we have $\left|\int[(T_\delta a)x]\cdot\varphi\right| \leqq k_p$; this enables us to use (6) to deduce $\left|\int[(Ta)x]\cdot\varphi\right| \leqq k_p$. The conclusion is reached by another application of (5).

3.7 LEMMA. *If* $a \in L^1 \cap V$ *and* $x \in L^0$, *then*

$$\|(Ta)x\|_p \leqq 2^{-1}c_p\|a\|_v\|x\|_p\ .$$

Proof. Suppose $\delta > 0$. Apply Fubini's theorem to (4):

$$[(T_\delta a)x]_\lambda = \int_{-\infty}^{\infty} da(t)e^{-2\pi i\lambda t}\int_{\delta_*} d\theta\frac{x(\lambda-\theta)}{2\pi i\theta}e^{2\pi it(\lambda-\theta)}\ .$$

Set $x^t(\beta) = x(\beta)\ \exp(2\pi it\beta)$. Keeping both (4) and 3.4 in mind, we can therefore write

$$[(T_\delta a)x]_\lambda = (2i)^{-1}\int_{-\infty}^{\infty} da(t)\{e^{-2\pi i\lambda t}[(T_\delta h)x^t]_\lambda\}\ . \tag{7}$$

This implies

$$\|(T_\delta a)x\|_p \leqq 2^{-1}\|a\|_v \sup_{t\in G}\|(T_\delta h)x^t\|_p\ . \tag{8}$$

The derivation of (8) from (7) is obtained by a standard procedure (e.g. as in [3, Lemma 21.2.1]); it rests upon (5) and requires a single application of the Fubini theorem. On the other hand, 3.4 implies that

$$\|(T_\delta h)x^t\|_p \leqq c_p\cdot\|x^t\|_p \leqq c_p\cdot\|x\|_p\ .$$

In view of (8) therefore: $\|(T_\delta a)x\|_p \leqq 2^{-1}c_p\|a\|_v\|x\|_p$. Use now 3.6 to reach the conclusion.

4. The Banach space V_∞. Let V_s denote the set of all functions in V which have compact support. The norm $\{a \to \|a\|_v\}$ makes the set $\{a \in V\colon a(-\infty) = 0\}$ into a Banach space V_0. Note that $V_s \subset V_\infty \subset V_0$. Henceforth V_∞ will be given the topology of V_0. We will write $\|a\|_\infty = \sup\{|a(\theta)|\colon \theta \in G\}$; it is easily checked that

$$\|a\|_\infty \leqq \|a\|_v \qquad \text{(when } a \in V_0)\ . \tag{9}$$

Let χ_n denote the characteristic function of the interval $(-n, n)$, and set $a_n = \chi_n \cdot a$.

4.1 LEMMA. *If* $a \in V_\infty$, *then* $\lim_{n\to\infty} \|a - a_n\|_v = 0$.

Proof. Suppose $f \in V$. Using the notation δ_* of 3.3, we have

$$\|f\|_v = v(f; [-\delta, \delta]) + v(f; \delta_*)\ , \tag{iii}$$

where $v(f; I)$ denotes the total variation over I. Set $\delta = n$ and $h_n = a - a_n$; therefore $v(h_n; [-\delta, \delta]) = |a(-\delta)| + |a(\delta)|$ and $v(h_n; \delta_*) = v(a; \delta_*)$. From (iii) therefore $\|h_n\|_v = |a(-\delta)| + |a(\delta)| + v(a; \delta_*)$, and the conclusion follows by letting $\delta \to \infty$.

4.2 REMARK. The set V_s is dense in V_∞ (since 4.1 and the fact that $a_n \in V_s$).

4.3 THEOREM. *The set* V_∞ *is a Banach space.*

Proof. Since V_∞ is a metric subspace of the Banach space V_0, it will suffice to show that V_∞ is complete. To that effect, consider a Cauchy sequence $\{g_k\}$ in V_∞; since $\{g_k\}$ is also in V_0, it will converge to some function f in V_0; therefore $f(-\infty) = 0$ and we need only establish that $f(\infty) = 0$. From (9) we see that

$$|f(\theta) - g_k(\theta)| \leqq \|f - g_k\|_v \qquad (\theta \in G)\ .$$

In view of $g_k(\infty) = 0$, the conclusion is obtained by letting $\theta \to \infty$ and $k \to \infty$.

5. The bilinear operator B_p. From 3.2 results that $V_s \subset L^1 \cap V \subset V_\infty$; it follows from 4.2 that $L^1 \cap V$ is dense in V_∞. Consider the bilinear operator $B = \{(x, a) \to (Ta)x\}$ which maps $L^0 \times (L^1 \cap V)$ into L^p. From 3.7 we see that B is a continuous bilinear mapping of $L^0 \times (L^1 \cap V)$ into L^p. Since L^0 and $L^1 \cap V$ are dense in L^p and V_∞, respectively, it follows that B has a (unique) continuous extension B_p to $L^p \times V_\infty$. Accordingly, if $a \in V_\infty$, then

$$\|B_p(x, a)\|_p \leqq 2^{-1}c_p\|a\|_v\|x\|_p \qquad (\text{if } x \in L^p) \tag{10}$$

If $a \in L^1 \cap V$, then (from 3.5)

$$B_p(x, a) = x * Ya \qquad (\text{if } x \in L^0)\ . \tag{11}$$

5.1 NOTATION. We henceforth identify functions equal almost-everywhere on G. If the sequence $\{f_n\}$ converges in the mean of order p (i.e., in the topology of L^p), then its limit will be denoted $(L^p)\lim f_n$.

5.2 LEMMA. *Let* $\bar{\chi}_n$ *be the function defined by*

$$\bar{\chi}_n(\theta) = (\sin 2\pi n\theta)/\pi\theta \qquad (\theta \in G) .$$

If $f \in L^p$, *then* $f = (L^p) \lim f * \bar{\chi}_n$ *as* $n \to \infty$.

Proof. Observe that Dunford's proof [2, p. 51, Lemma 3] for the case $p = 2$ holds without alteration whenever $1 < p < \infty$.

6. The main result. Suppose $\iota = 0, 1$. When f is a locally integrable function, we set

$$[({}_\iota Y_p)f] = (L^p) \lim_{n \to \infty} [{}_\iota Y(\chi_n \cdot f)] . \tag{12}$$

As in 3.1, we lighten the notation by writing $Y_p f = [({}_1 Y_p)f]$ and $\Psi_p f = [({}_0 Y_p)f]$.

6.1 REMARK. If $f \in L^1$ then $[({}_\iota Y_p)f] = [{}_\iota Yf]$. The following definition is an extension of the one used by Mihlin ("Multipliers of Fourier integrals"[2]).

6.2 DEFINITION. A locally integrable function a is called a *"multiplier of type L^p"* if both the following conditions hold:

$$\begin{cases} \text{the transform } Y_p(a \cdot [\Psi x]) \text{ exists and belongs to } L^p \text{ whenever } x \in L^0 \\ \infty \neq \sup\{\| Y_p(a \cdot [\Psi x]) \|_p \colon x \in L^0 \text{ and } \|x\|_p \leqq 1\} . \end{cases}$$

Let $\blacktriangle(L^p)$ denote the set of all multipliers of type L^p. When $a \in \blacktriangle(L^p)$, then $(t_p a)$ is defined as the continuous extension to all of L^p of the transformation $\{x \to Y_p(a \cdot [\Psi x])\}$ defined on L^0.

6.3 THEOREM. *If* $a \in V_\infty$, *then* $a \in \blacktriangle(L^p)$ *and* $(t_p a)x = B_p(x, a)$ *for all* x *in* L^p.

Proof. Note first that $a_n = (\chi_n \cdot a) \in L^1 \cap V$. Suppose $x \in L^0$. From (11) we see that

$$[B_p(x, a_n)]_\lambda = \int d\theta \cdot x(\theta) \int dt \cdot e^{-2\pi i(\lambda - \theta)t} a_n(t) \qquad (\text{when } \lambda \in G) .$$

By Fubini's theorem

$$[B_p(x, a_n)]_\lambda = \int dt \cdot a_n(t) e^{-2\pi i \lambda t} [\Psi x]_t \qquad (\text{for all } \lambda \text{ in } G) .$$

Or, equivalently

$$B_p(x, a_n) = Y(\chi_n \cdot a \cdot [\Psi x]) .$$

[2] See [6]; in that article, Mihlin gives a condition which ensures that a differentiable function be in $\blacktriangle(L^p)$.

From (10) and 4.1 we can now infer that

$$B_p(x, a) = (L^p) \lim_{n\to\infty} Y(\chi_n \cdot \{a \cdot [\Psi x]\}) .$$

From the definition (12) now results that $B_p(x, a) = Y_p(a \cdot [\Psi x])$ for all x in L^0. This completes the proof, in view of (10) and 6.2.

7. **Hille's definition.** Set $q = p/(p-1)$. The following definition is found in [3, p. 566]: a function a is said to be a *factor function for Fourier transforms of type* (L_p, L_p) if and only if

$$a \cdot [\Psi_q x] \in \{\Psi_q z : z \in L^p\}$$

wherever $x \in L^p$. This definition seems to require the restriction $p \leqq 2$, since $[\Psi_q x]$ need not exist otherwise.

7.1 THEOREM. *Suppose* $1 < p \leqq 2$. *If* a *is a factor function for Fourier transforms of type* (L_p, L_p), *then* $a \in \blacktriangle(L^p)$.

Proof. If a is such a factor function, there exists a bounded linear mapping $(t'_p a)$ of $L^p(G)$ into itself (see [3, Theorem 21.2.1]); this operator is defined by

$$a \cdot [\Psi_q x] = \Psi_q((t'_p a)x) \qquad \text{for all } x \text{ in } L^p .$$

In view of [11, 5.17], this implies

$$Y_p(a \cdot [\Psi_q x]) = (t'_p a)x \qquad \text{for all } x \text{ in } L^p . \tag{13}$$

The conclusion follows from 6.1 and 6.2.

7.2 THEOREM. *Suppose* $1 < p \leqq 2$ *and* $a \in V_\infty$. *Then* a *is a factor function for Fourier transforms of type* (L_p, L_p); *moreover,*

$$\Psi_q(B_p(x, a)) = a \cdot [\Psi_q x] \qquad (\text{when } x \in L^p) . \tag{14}$$

Proof. Since $B_p(x, a) \in L^p$ when $x \in L^p$ (see §4), it will suffice to prove (14). Consider first the case $(x, a) \in L^0 \times V_s$. From (12) we see that

$$\Psi_q(B_p(x, a)) = (L^q) \lim_{n\to\infty} g_n , \tag{15}$$

where $g_n = \Psi[\chi_n \cdot B_p(x, a)]$. From (11):

$$g_n(\lambda) = \int_{-n}^{n} d\theta \cdot e^{2\pi i \lambda\theta} \int d\alpha \cdot x(\alpha)[Ya]_{\theta-\alpha} \qquad (\text{when } \lambda \in G) .$$

A repeated application of the Fubini theorem yields

$$g_n(\lambda) = \int dt \cdot a(t)[\Psi x]_t \int_{-n}^{n} d\theta \cdot e^{2\pi i(\lambda - t)\theta} \qquad \text{(when } \lambda \in G) .$$

In the notation of 5.2 we accordingly have

$$g_n = \{a \cdot [\Psi x]\} * \bar{\chi}_n .$$

Since $a \cdot [\Psi x]$ is in L^q, it can be inferred from 5.2 and (15) that

$$\Psi_q(B_p(x, a)) = (L^q) \lim_{n \to \infty} (\{a \cdot [\Psi x]\} * \bar{\chi}_n) = a \cdot [\Psi x] .$$

Keeping $\Psi x = \Psi_q x$ in mind (see 6.1), it is clear that (14) is now proved in the case $(x, a) \in L^0 \times V_s$. Consider the bilinear operator $R = \{(x, a) \to a \cdot \Psi_q x\}$ defined on $L^p \times V_\infty$; since $\|\Psi_q z\|_q \leqq \|z\|_p$, it follows that $\|R(x, a)\|_q \leqq \|x\|_p \|a\|_\infty$, and from (9) results that R is a bounded bilinear mapping of $L^p \times V_\infty$ into L^q. In view of (10), this remark also shows that the bilinear operator $J = \{(x, a) \to \Psi_q(B_p(x, a))\}$ is a bounded bilinear mapping of $L^p \times V_\infty$ into L^q.

Having shown that $R(x, a) = J(x, a)$ whenever $(x, a) \in L^0 \times V_s$, the desired conclusion $R = J$ can now be inferred from the denseness of L^0 and V_s in L^p and V_∞, respectively (see 4.2).

8. **Concluding remarks.** From 6.3, 3.2 and 3.5 follows that, if $f \in L^1 \cap V$ and $x \in L^p$, then $(t_p f)x = B_p(x, f) = Tf$; hence, if F is the Fourier-Stieltjes transform of f, we have (from 3.3) the relation

$$[(t_p f)x]_\lambda = \frac{1}{2\pi i} \int_{-\infty}^{\infty} x(\theta) \frac{F(\theta - \lambda)}{\theta - \lambda} d\theta \qquad (\lambda \in G)$$

which was announced in the introduction. Property (ii) of the introduction follows from (11) and 6.3. If $A \in L^1$ we denote by A_{*p} the bounded operator $\{x \to x * A\}$ defined on L^p. Let S be the set of all a in $L^1 \cap V$ such that $Ya \in L^1$, and observe that $(Ya)_{*p} = (t_p a)$ when $a \in S$. Again if $a \in S$, then $A = Ya \in L^1$ and $a = \Psi A$; from [4] it is seen that the spectrum of $(t_p a)$ is the closure of the range of a.

8.1 REMARK. Set $\hat{G} = [0, 1]$ and $G = \{0, \pm 1, \pm 2, \cdots\}$. We will now sketch a proof of the properties (i′)-(ii′) described in §1. Denote by $\|A\|_v$ the total variation of A on $\hat{G}$, and suppose $\|A\|_v \neq \infty$. Observe that, since $A \in L^1(\hat{G})$, we may borrow from [5, p. 10] the following conclusion: $a = YA \in \blacktriangle(L^p(\hat{G}))$ and $t_p(YA) = A_*$ is a bounded linear operator on $L^p(\hat{G})$.

This is all of (i′)-(ii′) except for the inequality. The main result of [5] can be stated as follows[3]:

[3] The definition of $V_\sigma(a)$ is given in [5, p. 8].

$$\|t_p(a)\| \leqq 2k_p \cdot V_\sigma(a) . \tag{16}$$

Note also that $|[YA]_n| \leqq |2\pi n|^{-1}\|A\|_v$ when $n \in G$ (this is obtained by integrating by parts, as in 3.2); consequently $V_\sigma(a) = V_\sigma(YA) \leqq m_p\|A\|_v$. In view of (16), the proof of the inequality in (i′) is completed.

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