AVERAGES OF FOURIER COEFFICIENTS

RICHARD R. GOLDBERG

We shall say the sequence $a_n (n = 1, 2, \cdots)$ is a p-sequence $(1 \leqq p < \infty)$ if there is a function $f \in L^p(0, \pi)$ such that

$$a_n = \int_0^\pi f(t) \cos nt \, dt \qquad n = 1, 2, \cdots ;$$

(i.e. the a_n are Fourier cosine coefficients of an L^p function).

A famous theorem of Hardy [1] states that if a_n is a p-sequence $(1 \leqq p < \infty)$ and $b_n = \frac{1}{n}(a_1 + a_2 + \cdots + a_n)$, then b_n is also a p-sequence.

In this paper we shall prove the following generalization of Hardy's theorem:

THEOREM 1. *Let $\psi(x)$ be of bounded variation on $0 \leqq x \leqq 1$, and let $1 \leqq p < \infty$. Then, if a_n is a p-sequence and*

$$b_n = \frac{1}{n} \sum_{m=1}^{n} \psi\left(\frac{m}{n}\right) a_m ,$$

b_n is also a p-sequence.

Hardy's theorem is the special case $\psi(x) = 1$ for $0 \leqq x \leqq 1$.

If the conclusion of Theorem 1 holds for each of two functions ψ it clearly holds for their difference. Hence it is sufficient to prove Theorem 1 in the case where $\psi(x)$ is non-decreasing for $0 \leqq x \leqq 1$. Further, since any non-decreasing function may be written as the difference of two non-negative non-decreasing functions (the second of which is constant) to prove Theorem 1 it is sufficient to prove

THEOREM 1A. *Let $\psi(x)$ be non-negative and non-decreasing on $0 \leqq x \leqq 1$ and let $1 \leqq p < \infty$. Then, if a_n is a p-sequence and*

$$b_n = \frac{1}{n} \sum_{m=1}^{n} \psi\left(\frac{m}{n}\right) a_m ,$$

b_n is also a p-sequence.

The proof of Theorem 1A will follow a sequence of lemmas.

LEMMA 1. *Let $B_t(x) = \int_0^x \cos yt \, d(y - [y])$. Then there is an $M > 0$*

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such that

$$|B_t(x)| \leqq M \qquad 0 \leqq t \leqq \pi;\ 0 \leqq x < \infty\ .$$

The symbol $[y]$ denotes the greatest integer not exceeding y.

Proof. Let n be any non-negative integer. Then for $t > 0$

$$\int_0^n \cos yt\, dy = \frac{\sin nt}{t}$$

and

$$\int_0^n \cos yt\, d[y] = \sum_{m=1}^{n} \cos mt = \frac{\sin (n+1/2)t}{2 \sin t/2} - \frac{1}{2}\ .$$

Hence

$$\begin{aligned} B_t(n) &= \frac{\sin nt}{t} - \frac{\sin (n+1/2)t}{2\sin t/2} + \frac{1}{2} \\ &= \sin nt\Big(\frac{1}{t} - \frac{1}{2}\cot\frac{t}{2}\Big) - \frac{\cos nt}{2} + \frac{1}{2} \end{aligned}$$

and so

$$(1) \qquad |B_t(n)| \leqq \left|\frac{1}{t} - \frac{1}{2}\cot\frac{t}{2}\right| + 1 \qquad n = 0, 1, 2, \cdots$$

The right side of (1) is bounded for $0 < t \leqq \pi$. Thus for some $M \geqq 1$

$$(2) \qquad |B_t(n)| \leqq M - 1 \qquad n = 0, 1, 2, \cdots;\ 0 < t \leqq \pi\ .$$

Now take any $x \geqq 0$ and let $n = [x]$. Then

$$B_t(x) = B_t(n) + \int_n^x \cos yt d(y - [y])$$

so that from (2) we have for any $x \geqq 0$

$$|B_t(x)| \leqq M - 1 + \int_n^x |d(y-[y])| \leqq M - 1 + x - n \leqq M,\ 0 < t \leqq \pi$$

and the proof is complete since $B_0(x) = x - [x] \leqq 1 \leqq M$.

(Henceforth we assume $\psi(x) \geqq 0$ and $\psi(x)$ non-decreasing for $0 \leqq x \leqq 1$.)

LEMMA 2. *There is an $M > 0$ such that*

$$\left|\int_0^n \psi\Big(\frac{x}{n}\Big)\cos xt\, d(x - [x])\right| \leqq M \qquad 0 \leqq t \leqq \pi;\ n = 1, 2, \cdots$$

Proof. With $B_t(x)$ as in Lemma 1 we have

$$\int_0^n \psi\left(\frac{x}{n}\right)\cos xt\, d(x-[x]) = \int_0^n \psi\left(\frac{x}{n}\right) dB_t(x)$$
$$= \psi(1)B_t(n) - \int_0^n B_t(x)d\psi\left(\frac{x}{n}\right) .$$

Thus with M as in Lemma 1

$$\left|\int_0^n \psi\left(\frac{x}{n}\right)\cos xt\, d(x-[x])\right| \leqq M\psi(1) + M\int_0^n d\psi\left(\frac{x}{n}\right) \leqq 2M\psi(1) ,$$

and the lemma is prove (with $2M\psi(1)$ instead of M).

LEMMA 3. *Let* $f \in L'(0, \pi)$ *and let*

$$d_n = \frac{1}{n}\int_0^\pi f(t)dt\int_0^n \psi\left(\frac{x}{n}\right)\cos xt\, d(x-[x]) \qquad n = 1, 2, \cdots .$$

Then

$$d_n = O\left(\frac{1}{n}\right) \qquad n \to \infty \tag{3}$$

and hence d_n *is a* p*-sequence for every* $p \geqq 1$.

Proof. By Lemma 2 there is an $M > 0$ such that $|d_n| \leqq \frac{M}{n}\int_0^\pi |f(t)|dt$ from which (3) follows. From (3) it follows that $\sum_{n=1}^\infty |d_n|^q < \infty$, for every $q > 1$. By the Hausdorff-Young theorem and the fact that $L^p \subseteqq L^{p'}$ if $1 \leqq p' \leqq p$, this implies that d_n is a p-sequence for every $p \geqq 1$. (See [2].)

From now on we shall write $f \sim a_n$ as an abbreviation for $a_n = \int_0^\pi f(t)\cos nt\, dt, n = 1, 2, \cdots$.

LEMMA 4. *Let* $1 \leqq p < \infty$, $f \in L^p(0, \pi)$ *and* $a(x) = \int_0^\pi f(t)\cos xt\, dt$ so *that*

$$f \sim a_n = a(n) .$$

Let

$$g(x) = \int_x^\pi \frac{1}{t}\psi\left(\frac{x}{t}\right)f(t)dt \qquad c_n = \frac{1}{n}\int_0^n \psi\left(\frac{x}{n}\right)a(x)dx .$$

Then $g \in L^p(0, \pi)$ *and*

$$g \sim c_n .$$

Proof. Since $|g(x)| \leqq \psi(1)\int_x^\pi \frac{|f(t)|}{t}dt$ it follows from the proof in [1] that $g \in L^p$. Also

$$\begin{aligned}\int_0^\pi g(x)\cos nx\,dx &= \int_0^\pi \cos nx\,dx\int_x^\pi \frac{1}{t}\psi\Big(\frac{x}{t}\Big)f(t)dt\\ &= \int_0^\pi \frac{1}{t}f(t)dt\int_0^t \psi\Big(\frac{x}{t}\Big)\cos nx\,dx = \int_0^\pi f(t)dt\int_0^1 \psi(x)\cos nxt\,dt\\ &= \frac{1}{n}\int_0^\pi f(t)dt\int_0^n \psi\Big(\frac{x}{n}\Big)\cos xt\,dt = \frac{1}{n}\int_0^n \psi\Big(\frac{x}{n}\Big)\int_0^\pi f(t)\cos xt\,dt = c_n\,.\end{aligned}$$

The changes in order of integration are valid since

$$\int_0^\pi |f(t)|dt\int_0^1 |\psi(x)\cos nxt|dx \leqq \psi(1)\int_0^\pi |f(t)|dt < \infty\,.$$

(Note $f \in L'(0,\pi)$ since $f \in L^p(0,\pi)$.) Thus $g \sim c_n$, which is what we wished to show.

We can now establish our principal result.

Proof of Theorem 1A. Let $f \in L^p(0,\pi)$ be such that $f \sim a_n$ and let $a(x)$, $g(x)$, c_n be as in Lemma 4. Then

$$b_n = \frac{1}{n}\sum_{m=1}^n \psi\Big(\frac{m}{n}\Big)a_m = \frac{1}{n}\int_0^n \psi\Big(\frac{x}{n}\Big)a(x)d[x]$$

so that

$$\begin{aligned}c_n - b_n &= \frac{1}{n}\int_0^n \psi\Big(\frac{x}{n}\Big)a(x)d(x-[x]) = \frac{1}{n}\int_0^n \psi\Big(\frac{x}{n}\Big)d(x-[x])\int_0^\pi f(t)\cos xt\,dt\\ &= \frac{1}{n}\int_0^\pi f(t)dt\int_0^n \psi\Big(\frac{x}{n}\Big)\cos xt\,d(x-[x])\,.\end{aligned}$$

The last iterated integral clearly converges absolutely, justifying the change in order of integration. By Lemma 3 $c_n - b_n$ is a p-sequence. Also c_n is a p-sequence since, by Lemma 4, $g \in L^p(0,\pi)$ and $g \sim c_n$. Hence $b_n = c_n - (c_n - b_n)$ is a p-sequence and the theorem is proved.

REMARK. Note that except for the result of Lemma 1 the only properties of the cosine function used were its boundedness and the fact that $O\Big(\frac{1}{n}\Big)$ is a p-sequence for all $p \geqq 1$.

LEMMA 5. *Let* $C_t(x) = \int_0^x \sin yt\,d(y-[y])$. *Then there is an* $M > 0$ *such that*

$$|C_t(x)| \leqq M \qquad 0 \leqq t \leqq \pi;\ 0 \leqq x < \infty\,.$$

Proof. Let n be any non-negative integer. Then for $t > 0$

$$\int_0^n \sin yt\, dy = \frac{1}{t} - \frac{\cos nt}{t}$$

and

$$\int_0^n \sin yt\, d[y] = \sum_{k=1}^{n} \sin kt = \frac{\cos t/2 - \cos (n + 1/2)t}{2 \sin t/2} .$$

Hence

$$\begin{aligned} C_t(n) &= \frac{1}{t} - \frac{\cos nt}{t} - \frac{\cos t/2 - \cos (n + 1/2)t}{2 \sin t/2} \\ &= (1 - \cos nt)\left(\frac{1}{t} - \frac{1}{2}\cot\frac{t}{2}\right) - \frac{\sin nt}{2} . \end{aligned}$$

The remainder of the proof follows as in Lemma 1.

In view of Lemma 5 and the remark preceding it the exact analogue of Theorem 1 for sine coefficients must hold. This we now state:

THEOREM 2. *Fix* $p \geqq 1$. *If, for some* $f \in L^p$,

$$a_n = \int_0^\pi f(t)\sin ntdt \qquad n = 1, 2, \cdots ,$$

and if $b_n = \frac{1}{n}\sum_{m=1}^{n} \psi\left(\frac{m}{n}\right)a_m$ *where* $\psi(x)$ *is of bounded variation on* $0 \leqq x \leqq 1$ *then there exists* $g \in L^p$ *such that*

$$b_n = \int_0^\pi g(t)\sin ntdt \qquad n = 1\,2, \cdots$$

REFERENCES

1. G. H. Hardy, *Notes on some points in the integral calculus*, Messenger of Mathematics **58** (1929), 50-52.
2. A. Zygmund, Trigonometrical Series, Warsaw 1935 p. 190.

NORTHWESTERN UNIVERSITY