ANALYTIC AUTOMORPHISMS OF BOUNDED SYMMETRIC COMPLEX DOMAINS

HELMUT KLINGEN

In a former paper [2] I determined the full group of one-to-one analytic mappings of a bounded symmetric Cartan domain [1]. Those investigations were incomplete, because it was impossible to treat the second Cartan-type of $n(n-1)/2$ complex dimensions for odd n by this method. The present note is devoted to a new shorter proof of the former result (n even), which furthermore covers the remaining case of odd n.

Take the complex $n(n-1)/2$-dimensional space of skew symmetric n-rowed matrices Z. The irreducible bounded symmetric Cartan space in question is the set $\mathcal{E}_n$ of those matrices Z, for which

$$I + Z\bar{Z} > 0 \ , \ Z' = -Z,$$

is positive definite. Here I is the n by n unit matrix. Obviously $\mathcal{E}_2$ is the unit circle. It is easy to see that analytic automorphisms of $\mathcal{E}_n$ are described by the group ϕ of the mappings

$$W = (AZ + B)(-\bar{B}Z + \bar{A})^{-1} \ , \tag{1}$$

where the n-rowed matrices A, B fulfill

$$M^*KM = K \quad \text{with} \quad M = \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}, \quad K = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

Here M^* denotes the conjugate transpose of M. For $n = 4$

$$W = \tilde{Z}$$

is a further analytic automorphism, where $\tilde{Z}$ arises from Z by interchanging the elements z_{14} and z_{23},

$$\tilde{Z} = \begin{pmatrix} 0 & z_{12} & z_{13} & z_{23} \\ -z_{12} & 0 & z_{14} & z_{24} \\ -z_{13} & -z_{14} & 0 & z_{34} \\ -z_{23} & -z_{24} & -z_{34} & 0 \end{pmatrix}.$$

For $W\bar{W}$ and $\tilde{Z}\bar{\tilde{Z}}$ have the same characteristic roots. But this mapping is not contained in ϕ, since $CZ = \tilde{Z}D$ cannot be satisfied identically in Z by non-singular constant matrices C, D. On the other hand the following theorem holds.

Received October 13, 1959.

THEOREM. *Each analytic automorphism of $\mathcal{E}_n$ can be written as $W = f(Z)$ or $W = f(\tilde{Z})$ (only for $n = 4$) with $f \in \phi$.*

Therefore the group ϕ is already the full group of analytic automorphisms for $n \neq 4$. Only in the exceptional case $n = 4$ there are the further mappings $W = f(\tilde{Z})$, which together with ϕ form the full group of analytic automorphisms. The proof of this theorem consists of two parts. The first analytic part is a reproduction of my former proof [2], which will be given here again for completeness, the second part is of algebraic character.

The group ϕ acts transitively on $\mathcal{E}_n$. For take an arbitrary point Z_1 of $\mathcal{E}_n$, choose the matrix A such that

$$A(I + Z_1\bar{Z}_1)A^* = I$$

and define $B = -AZ_1$. Then (1) maps Z into 0. Therefore it is sufficient to investigate the stability group of the zero matrix.

First we show that each analytic one-to-one mapping $W = W(Z)$ of $\mathcal{E}_n$ with the fixed point 0 is linear. For an arbitrary point $Z_1 \in \mathcal{E}_n$ let $r_1, \cdots, r_n, 0 \leqq r_1 \leqq \cdots \leqq r_n < 1$, be the characteristic roots of $Z_1 Z_1^*$. Then also tZ_1 belongs to $\mathcal{E}_n$, if t is a complex number with $t\bar{t}r_n < 1$. Consequently there exists a power series expansion

$$W(tZ_1) = \sum_{k=1}^{\infty} t^k W_k(Z_1)\,, \qquad t\bar{t}r_n < 1\,. \tag{2}$$

The elements of the skew-symmetric matrices $W_k(Z_1)$ are homogeneous polynomials of degree k in the independent elements of Z_1. Because of $I + W(tZ_1)\bar{W}(tZ_1) > 0$ for $\bar{t}t = 1$, one obtains from (2)

$$\frac{1}{2\pi i}\int_{t\bar{t}=1} (I + W(tZ_1)\bar{W}(tZ_1))\frac{dt}{t} = I + \sum_{k=1}^{\infty} W_k(Z_1)\bar{W}_k(Z_1) > 0 \tag{3}$$

and in particular $I + \bar{W}_1(Z_1)W_1(Z_1) > 0$. Therefore the linear function $W_1(Z)$ is an analytic mapping of $\mathcal{E}_n$ into itself. Its determinant D is at the same time the Jacobian of the function $W(Z)$ with respect to Z. By interchanging Z and W it can be assumed $D\bar{D} \geqq 1$. Consequently $W(Z)$ is an analytic automorphism of $\mathcal{E}_n$ and even maps the boundary onto itself. Take now in particular

$$Z_1 = U'PU\,, \quad P = [(0), p_1 F, \cdots, p_m F]\,, \quad F = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4}$$

with an unitary matrix U, $m = [n/2]$. P shall be the matrix, which is built up by the two-rowed blocks $p_1 F, \cdots, p_m F$ and possibly by the element 0 along the main diagonal. Z_1 belongs to the interior of $\mathcal{E}_n$, if $-1 < p_k < 1$ $(k = 1, \cdots, m)$, and to the boundary, if $-1 \leqq p_k \leqq 1$ $(k =$

$1, \cdots, m)$ and $p_k = \pm 1$ for at least one k. Now $|I + W_1(Z_1)\bar{W}_1|$ is a polynomial in $p_1, \cdots, p_m$ of total degree $4m$ and on the other hand (see [2], Lemma 4) the square of a polynomial. As $|I + W_1(Z_1)\bar{W}_1|$ vanishes on the boundary of $\mathcal{E}_n$, this polynomial is divisible by

$$|I + Z_1\bar{Z}_1| = \prod_{k=1}^{m}(1 - p_k^2)^2 .$$

Because the constant terms and the degrees of both polynomials are equal, one obtains

$$|I + W_1(Z_1)\bar{W}_1| = |I + Z_1\bar{Z}_1| \tag{5}$$

even identically in Z_1; for each skew-symmetric matrix Z_1 permits a representation (4) (see [2], Lemma 3). On account of (5) and the linearity of W_1 the matrices $W_1\bar{W}_1$ and $Z\bar{Z}$ always have the same characteristic roots and this implies

$$W_1(Z) = U'ZU \tag{6}$$

with unitary U, which for the present still depends on Z.

Put now

$$Z = uX, \quad X = U_1'[e^{i\zeta_1}F, \cdots, e^{i\zeta_r}F, (0)]U_1, \quad 0 \leqq u \leqq 1,$$

with real variables $\zeta_1, \cdots, \zeta_r$. Then $Z \in \mathcal{E}_n$ and by (6)

$$W_1W_1^* = u^2U'U_1'\begin{pmatrix} I^{(n-1)} & 0 \\ 0 & (0) \end{pmatrix}\bar{U}_1\bar{U}$$

for all u between 0 and 1. Because of (3) one obtains

$$\bar{U}_1\bar{U}(I + W_1\bar{W}_1 + W_k\bar{W}_k)U'U_1' > 0 \qquad (k = 2, 3, \cdots) .$$

If u tends to 1, one gets

$$\begin{pmatrix} 0 & 0 \\ 0 & (1) \end{pmatrix} + \bar{U}_1\bar{U}W_k\bar{W}_kU'U_1' > 0 ,$$

hence $W_k(X) = 0$. As W_k is a polynomial, $W_k(Z)$ even vanishes identically in Z. Therefore the stability group of $\mathcal{E}_n$ is linear.

The investigation of $W = W_1(Z)$ is now a purely algebraic problem. The representation (6) shows that rank W = rank Z and beyond this the equality of the characteristic roots of $W\bar{W}$ and $Z\bar{Z}$. These properties will be used in order to determine $W(Z)$ explicitly. We have to prove

$$W(Z) = U'ZU \quad \text{or} \quad W(Z) = U'\tilde{Z}U \tag{7}$$

with unitary constant U, where the second type only occurs for $n = 4$. The proof of this fact will be given by induction. The assertion (7) is trivial for the unit circle ($n = 2$). Let us assume its correctness for $2, 3, \cdots, n-1$ and consider $\mathcal{E}_n$. Write the linear mapping $W(Z)$ of $\mathcal{E}_n$ onto itself as

$$W = \sum_{k<l} z_{kl} A_{kl}$$

with constant skew-symmetric n by n matrices A_{kl}. Because of the equality of the characteristic roots of WW^* and ZZ^* the hermitian matrix $A_{kl}A_{kl}^*$ has $1, 1, 0, \cdots, 0$ as characteristic roots. Therefore after unitary transformation of W we can assume $A_{12} = E_{12}$, where in general E_{kl} denotes the skew-symmetric matrix the elements of which are all zero besides the element in the kth row and lth column and the element in the lth row and kth column, which are 1 respectively -1. Since $\operatorname{tr}(A_{12}\bar{A}_{kl}) = 0$ for $(k, l) \neq (1, 2)$, one obtains

$$A_{kl} = \begin{pmatrix} 0^{(2)} & * \\ * & * \end{pmatrix} \qquad (k, l) \neq (1, 2) .$$

$A_{12} = E_{12}$ does not change, if W is transformed by

$$\begin{pmatrix} U^{(2)} & 0 \\ 0 & V \end{pmatrix}$$

with unitary $U, V, |U| = 1$. Therefore

$$A_{13} = \begin{pmatrix} 0^{(2)} & B \\ -B' & C \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & 0 & \\ 0 & b_2 & 0 \end{pmatrix}$$

can be assumed. From rank W = rank Z identically in Z one obtains possibly after unitary transformation $A_{13} = E_{13}$.

For $A_{14} = (a_{kl})$ we get two possibilities. First the equation $\operatorname{tr}(A_{12}\bar{A}_{14}) = \operatorname{tr}(A_{13}\bar{A}_{14}) = 0$ implies $a_{12} = a_{13} = 0$. After unitary transformation all the elements of the first row besides a_{14} are zero. Then take only the elements z_{12}, z_{13}, z_{14} of Z distinct from zero; from rank W = rank $Z = 2$ one sees

$$A_{14} = E_{14} \quad \text{or} \quad A_{14} = E_{23} .$$

By a similar consideration $A_{1\nu}$ turns out to be $E_{1\nu}$ or E_{23}. But actually for $\nu > 4$ the second possibility $A_{1\nu} = E_{23}$ may not occur. For $A_{14} = A_{1\nu} = E_{23}$ is impossible because of $\operatorname{tr}(A_{14}\bar{A}_{1\nu}) = 0$. If $A_{14} = E_{14}$, $A_{1\nu} = E_{23}$, choose only the elements $z_{1\nu}, z_{14} \neq 0$, then rank $Z = 2$ but rank $W = 4$. Therefore $A_{1\nu} = E_{1\nu}$ ($\nu \neq 4$), $A_{14} = E_{14}$ or E_{23}. Furthermore $A_{14} = E_{23}$ may only happen if $n = 4$. For assume $A_{14} = E_{23}$, $A_{15} = E_{15}$ and take only the elements $z_{14}, z_{15} \neq 0$. This implies rank $Z = 2$ but rank $W = 4$.

Let us summarize our results. After a suitable unitary transformation W can be written as

$$W = \begin{pmatrix} 0 & z' \\ -z & L(QZ_0) \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & z' \\ -z & Z_0 \end{pmatrix},$$

besides the exceptional case $n = 4$, $A_{14} = E_{23}$. Now $L(Z_0)$ is an analytic automorphism of $\mathcal{E}_{n-1}$ with the fixed point 0. For $n = 3$ we know $L(Z_1) = e^{i\zeta}Z_1$ with a real constant ζ. Therefore $W = U'ZU$ with a constant unitary matrix U, which is the theorem for $n = 3$. For $n > 5$ the induction hypothesis shows

$$W = \begin{pmatrix} 0 & z'U' \\ -Uz & Z_0 \end{pmatrix}$$

with constant unitary U. From the equality

$$\operatorname{rank} W = \operatorname{rank} Z$$

U turns out to be a diagonal matrix. Finally consider the sum of the two-rowed principal minors of $W\bar{W}$ and $Z\bar{Z}$. These two quantities are equal identically in Z because of the fact that $W\bar{W}$ and $Z\bar{Z}$ have the same characteristic roots. By this identity one obtains $U = aI$ with a complex number a of absolute value 1, which again proves our theorem.

There still remain the cases $n = 4$ and 5. For $n = 4$, $A_{14} = E_{14}$ we can use the reasoning above. Let $A_{14} = E_{23}$; since

$$\operatorname{tr}(A_{1\nu}\bar{A}_{23}) = \operatorname{tr}(A_{1\nu}\bar{A}_{24}) = \operatorname{tr}(A_{1\nu}\bar{A}_{34}) = 0 \qquad (\nu = 2, 3, 4)$$

W only differs from $\tilde{Z}$ in the last row, where a linear combination of z_{23}, z_{24}, z_{34} appears. The identity between the ranks of Z and W shows $w_{14} = a_1 z_{23}$, $w_{24} = a_2 z_{24}$, $w_{34} = a_3 z_{34}$. Now it is easy to compute the sum of the two-rowed principal minors of $W\bar{W}$ and $Z\bar{Z}$. This computation shows again the assertion for $n = 4$.

For $n = 5$ we know by the induction hypothesis

$$L(Z_0) = U'Z_0U \quad \text{or} \quad L(Z_0) = U'\tilde{Z}_0U$$

with constant unitary U. The first case can be treated as above. In the second case one obtains

$$W = \begin{pmatrix} 0 & z'U' \\ -Uz & Z_0 \end{pmatrix}.$$

Choose once only $z_{14}, z_{24} \neq 0$, then only $z_{14}, z_{34}, z_{45} \neq 0$. In any case $\operatorname{rank} Z = 2$, hence $\operatorname{rank} W = 2$. But this implies that all the elements of the third column of U vanish, which is a contradiction to the unitary character of U. This final remark completes the proof.

References

1. E. Cartan, *Sur les domaines bornés homogènes de l'espace de n variables complexes*, Oeuvres, partie 1, 1259–1304.
2. H. Klingen, *Diskontinuierliche Gruppen in symmetrischen Räumen*, Mathematische Annalen **129** (1955), 470–488.

Mathematisches Institut der Universität
Göttingen, Germany