AUTOMORPHISMS OF CLASSICAL LIE ALGEBRAS

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1. Introduction. Starting with a simple Lie algebra over the complex field C, Chevalley [2] has given a procedure for replacing C by an arbitrary field K. Under mild restrictions on the characteristic of K, the algebra so obtained is simple over its center, and it is our purpose here to determine the automorphisms of each such quotient algebra $\mathfrak{g}$. In terms of the group G defined in [2] and also in § 3 below and the group A of all automorphisms of $\mathfrak{g}$, the principal result is that, with some exceptions, which occur only at characteristic 2 or 3, A/G is isomorphic to the group of symmetries of the corresponding Schläfli diagram. As might be expected, the main step in the development is the proof of a suitable conjugacy theorem for Cartan subalgebras (4.1 and 7.1 below). The final result then quickly follows.

Definitions of the algebras and automorphisms to be considered are given in § 2 and § 3. Sections 4, 5 and 6 contain the main development and § 7 treats some special cases. The last section contains some remarks on the extension of the preceding results to other algebras. In 4.6, 4.7, 4.8, 7.2 and 7.3 the results are interpreted for the various types of algebras occurring in the Killing-Cartan classification, thereby yielding results of other authors [4, 5, 6, 7, 8, 9, 12, 14, 15, 16, 18] who have worked on various types of algebras from among those usually denoted A, B, C, D, G and F. For other treatments in which all types are considered simultaneously, the reader is referred to [4; 16, Exp. 16] where the problem is solved over the complex field, however by topological methods which can not be used for other fields, and to [14] where general fields occur but only partial results are obtained. General references to the classical theory of Lie algebras over the complex field are [1, thesis; 3; 16; 19].

2. The algebras. Let us start with a simple Lie algebra $\mathfrak{g}_C$ over the complex field C, a Cartan subalgebra $\mathfrak{h}_C$, the (ordered) system Σ of (nonzero) roots relative to $\mathfrak{h}_C$, the set Φ of fundamental positive roots, and for each pair of roots r and s, define c_{rs} to be the Cartan integer $2(r, s)/(s, s)$, and p_{rs} to be 0 if $r + s$ is not a root and otherwise to be the least positive integer p for which $r - ps$ is not a root. Then Chevalley [2, Th. 1] has shown that there exists a set of root elements $\{X_r\}$ and a set $\{H_r\}$ of elements of $\mathfrak{h}_C$ such that the equations of structure of $\mathfrak{g}_C$ are:

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2.1 $H_{-r} = -H_r$, *and if* r, s *and* t *are roots such that* $r + s + t = 0$ *and* r *is at most as long as* s *or* t, *then*

$$H_r + (s, s)/(r, r)H_s + (t, t)/(r, r)H_t = 0 \ .$$

2.2 $H_r H_s = 0$.

2.3 $H_r X_s = c_{sr} X_s$.

2.4 $X_r X_{-r} = H_r$.

2.5 $X_r X_s = \pm\, p_{rs} X_{r+s}$ *if* $r + s \neq 0$.

The equations 2.1 imply that each H_r is an integral linear combination of the elements $H_a (a \in \Phi)$, which form a basis for $\mathfrak{h}_C$. Just as in [2] the base field C can now be replaced by an arbitrary field K (because the structural constants are all integers), yielding an algebra $\bar{\mathfrak{g}}$ over K, an Abelian subalgebra $\bar{\mathfrak{h}}$, a set of numbers $\{\bar{p}_{rs}, \bar{c}_{rs}\}$ in K, and a set of roots relative to $\bar{\mathfrak{h}}$ defined by $\bar{r}(H_s) = \bar{c}_{rs}$. We use the notation $\{X_r, H_r\}$ for the generating set of $\bar{\mathfrak{g}}$, the subscript r referring to a root of the original system Σ.

2.6 *Assume that* $\bar{\mathfrak{g}}$ *is one of the algebras just constructed, but if* Σ *has roots of unequal length or if* Σ *is of type* A_1 *assume that* K *is not of characteristic* 2, *and if* Σ *is of type* G_2 *assume further that* K *is not of characteristic* 3. *Then*

(1) *if* $p_{rs} \neq 0$, *then* $\bar{p}_{rs} \neq 0$, *whereas if* $c_{rs} \neq 0$, *then* $\bar{c}_{rs} \neq 0$ *unless* $r = \pm\, s$ *and* K *is of characteristic* 2;

(2) *no* H_r *is in the center of* $\bar{\mathfrak{g}}$;

(3) *the center* $\bar{\mathfrak{c}}$ *of* $\bar{\mathfrak{g}}$ *consists of those* H *in* $\bar{\mathfrak{h}}$ *such that* $\bar{r}(H) = 0$ *for all* r *in* Σ;

(4) *if* $\mathfrak{h} = \bar{\mathfrak{h}}/\bar{\mathfrak{c}}$ *and* $\mathfrak{g} = \bar{\mathfrak{g}}/\bar{\mathfrak{c}}$, *then* $\mathfrak{h}$ *is a Cartan subalgebra of* $\mathfrak{g}$;

(5) $\mathfrak{g}$ *is simple.*

Proof. (1) From known properties of root systems if $r \neq \pm\, s$ then p_{rs} and c_{rs} take on values other than $0, \pm 1$ only if Σ has roots of different lengths: the values ± 2 and ± 3 if Σ is of type G_2 and ± 2 if Σ is one of the other types. The possibility of these numbers becoming 0 in K has been ruled out by the assumptions. On the other hand $\bar{c}_{rr} = \bar{2}$ which is 0 if and only if K is of characteristic 2.

(2) If K is not of characteristic 2, then $H_r X_r = 2X_r \neq 0$, and if K is of characteristic 2, then there is a root s not orthogonal to r, whence $H_r X_s = c_{sr} X_s \neq 0$ by (1). Thus H_r is not in the center.

(3) Assume that $X = H + \Sigma c_s X_s$ is in the center. Then multiplication by X_{-r} yields $c_r = 0$ because of 2.4, whence

$$0 = XX_r = HX_r = \bar{r}(H) X_r$$

so that $\bar{r}(H) = 0$. The converse is easily checked.

(4) $\mathfrak{h}$ is Abelian, and if $X = H + \Sigma c_s X_s$ is in the normalizer of $\mathfrak{h}$, then $c_r = 0$ just as before and X is in $\mathfrak{h}$. Hence $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{h}$.

(5) Let $\mathfrak{m}$ be an ideal in $\mathfrak{h}$ and Y a nonzero element of $\mathfrak{m}$. Then by repeated multiplication by elements of the form X_a $(a \in \Phi)$ (now considered to be in $\mathfrak{g}$) we arrive at a nonzero X which is in $\mathfrak{m}$ and commutes with all X_a. By (1) this implies that X is a scalar multiple of X_d, d being the unique root such that $d + a$ is not a root for each a in Φ. Thus X_d is in $\mathfrak{m}$ and by repeated multiplication by elements of the form X_{-a} $(a \in \Phi)$ we get all X_r in $\mathfrak{m}$, whence $\mathfrak{m} = \mathfrak{g}$. Hence $\mathfrak{g}$ is simple.

In regard to the cases excluded by the assumptions of 2.6, let us observe first that if K is of characteristic 2 and Σ is of type A_1 then $\bar{\mathfrak{g}}$ is nilpotent while if Σ is of type G_2 then $\bar{\mathfrak{g}}$ is isomorphic to the algebra $\mathfrak{g}$ of type D_3, as is seen by an examination of the multiplication tables. In the other cases $\mathfrak{g}$ is not simple because those X_r and H_r for which r is a short root span an ideal as is seen from 2.1 to 2.5 and the following properties of Σ: if r is a long root and s is a short one, then c_{rs} is 0 or $\pm(r, r)/(s, s)$; if $r + s$ is also a root then it is a short one (because $(r + s, r + s) = (1 + c_{sr})(r, r) + (s, s)$ which is not a multiple of (r, r)); if r and s are short roots and $r + s$ is a long root, then

$$p_{rs} = (r + s, r + s)/(r, r)$$

(check for Σ of type B_2 or G_2).

In the sequel, each algebra $\mathfrak{g}$ of 2.6 is called a classical Lie algebra, and the algebra $\mathfrak{h}$ and the set of elements $\{X_r, H_r \mid r \in \Sigma\}$, now considered to be in $\mathfrak{g}$, which occur in the explicit mode of construction described are called standard Cartan subalgebra and standard set of generators, respectively. (Actually the subset $\{X_a \mid \pm a \in \Phi\}$ is enough to generate $\mathfrak{g}$.) In addition the notations $\bar{p}_{rs}$, $\bar{c}_{rs}$ and $\bar{r}$ are used in reference to $\mathfrak{g}$ rather than $\bar{\mathfrak{g}}$. Observe that $\bar{r}$ is defined in a natural way on $\mathfrak{h}$ because of (3) of 2.6.

A consequence of (4) of 2.6 which should be borne in mind is that $\bar{r} \neq 0$ if $r \neq 0$, although it may happen that $\bar{r} = \bar{s}$ with $r \neq s$.

3. The groups. Following Chevalley [2], let us now describe certain automorphisms of classical Lie algebras. Let $\mathfrak{g}$ be such an algebra and $\{X_r\, H_r \mid r \in \Sigma\}$ a standard set of generators. For each r in Σ and each k in K, let $x_r(k)$ be the automorphism of $\mathfrak{g}$ which has the same effect as $\exp \operatorname{ad} kX_r$ on each generator, with the sole exception: if K is of characteristic 2, then $x_r(k)X_{-r} = X_{-r} + kH_r + k^2X_r$ (see [2, p. 24]), and then let G' be the group generated by all such automorphisms as r runs through Σ and k through K. Then for each $w \in W$, the Weyl group of Σ, there is $\omega(w)$ in G' such that $\omega(w)X_r = \pm X_{wr}$ and $\omega(w)H_r = H_{wr}$ for each r in Σ [2, p. 35]. If χ is a homomorphism of the additive group generated by the roots of Σ into the multiplicative group K^* of K, then there is an automorphism h of $\mathfrak{g}$ such that $hX_r = \chi(r)X_r$ for each r in Σ. The group of such automorphisms is denoted $\mathfrak{H}$, and the subgroup corresponding to those homomorphisms which can be extended to the group of weights relative to Σ is denoted $\mathfrak{H}'$. Let G be the group generated by G' and $\mathfrak{H}$. One has [2]:

3.1. *G' is normal in G, $\mathfrak{H}' = \mathfrak{H} \cap G'$, $G = G'\mathfrak{H}$, and G/G' is isomorphic to $\mathfrak{H}/\mathfrak{H}'$.*

Finally, if $r \to r'$ is a permutation of Φ, the set of fundamental roots, such that $c_{a'b'} = c_{ab}$ for all a and b in Φ, then there is a *graph* automorphism g of $\mathfrak{g}$ defined by: $gX_a = X_{a'}$ if $\pm a$ is in Φ (see one of [3, p. 116; 16, p. 11–04; 19, p. 94] for the proof of existence and [1, p. 361] for an interesting discussion). Although the automorphisms of this paragraph are defined in [2] to act on $\bar{\mathfrak{g}}$, we can (and shall) think of them as acting on $\mathfrak{g}$. Nothing is lost in the passage from $\bar{\mathfrak{g}}$ to $\mathfrak{g}$: if x is an automorphism of $\bar{\mathfrak{g}}$ which induces the identity on $\mathfrak{g}$ then, in the notation prior to 2.6, $xX_r \equiv X_r \bmod \bar{\mathfrak{c}}$ for each r, whence $xH_r = H_r$ by 2.4 and then $xX_r = X_r$ by 2.3, implying that x is the identity.

The following observation will be used later:

3.2. *Let S be a standard set of generators of $\mathfrak{g}$ and x an automorphism of $\mathfrak{g}$. Let G' be the group defined above relative to S, and let G'' be the corresponding group defined relative to the standard set xS. Then $G'' = xG'x^{-1}$.*

Proof. Let B be a subset of S which is also a vector space basis for $\mathfrak{g}$. Then the matrices representing G' relative to B are the same as those representing G'' relative to xB, whence $G'' = xG'x^{-1}$.

4. Principal results. Throughout the next three sections, $\mathfrak{g}$ denotes a classical Lie algebra with a fixed standard set of generators

$$S = \{X_r, H_r \mid r \in \Sigma\}$$

and corresponding Cartan subalgebra $\mathfrak{h}$, K is the underlying field, the symbols G', G, $\mathfrak{H}'$, and "graph" refer to the automorphisms of $\mathfrak{g}$ defined relative to S as in § 3, and A denotes the group of all automorphisms of $\mathfrak{g}$. It is assumed that Σ is not of type A_2 if K is of characteristic 3 and not of type D_n if K is of characteristic 2. These exceptional cases are considered in § 7.

4.1. *Conjugacy theorem. If* $\mathfrak{h}_1$ *and* $\mathfrak{h}_2$ *are standard Cartan subalgebras of* $\mathfrak{g}$, *there is* x *in* G' *such that* $x\mathfrak{h}_1 = \mathfrak{h}_2$.

4.2. *Each* x *in* A *can be written uniquely* $x = ihg$, *with* i *in* G', h *in* $\mathfrak{H}$ *and* g *a graph automorphism.*

4.3. G' *and* G *are normal subgroups of* A.

4.4. G/G' *is isomorphic to* $\mathfrak{H}/\mathfrak{H}'$, *hence is Abelian.*

4.5. *The graph automorphisms form a system of coset representatives for* A *over* G. *Hence* A/G *is isomorphic to the group of symmetries of the Schläfli graph.*

These results may be amplified thus:

4.6. $G = G'$ *if* Σ *is of type* E_8, F_4 *or* G_2 *or if* Σ *is of arbitrary type and* K *is algebraically closed.* G/G' *is isomorphic to* K^*/K^{*f}, *with* $f = n + 1, 2, 3, 4, 3, 2$ *in the respective cases that* Σ *is of type* A_n, B_n, C_n, D_n *(n odd),* E_6, E_7, *and is isomorphic to the direct product of* 2 *copies of* K^*/K^{*2} *if* Σ *is of type* D_n *(n even).*

4.7. $A = G$ *with the exceptions:* A/G *is of order* 2 *if* Σ *if of type* $A_n (n \geqq 2)$, $D_n (n \geqq 5)$ *or* E_6, *and is isomorphic to the symmetric group on* 3 *objects if* Σ *is type* D_4.

4.8. $A = G'$, *hence is simple, if* Σ *is of type* E_8, F_4 *or* G_2 *and* K *is arbitrary or if* Σ *is of type* B_n, C_n *or* E_7 *and every element of* K *is a square.* $A \neq G'$ *otherwise.*

5. The theorem of conjugation. We first show that the group G' depends only on $\mathfrak{h}$, not on all of S.

5.1. *If* r *and* s *are in* Σ *and* $r \neq s$, *then* $\bar{r} = \bar{s}$ *if and only if both* $r = -s$ *and* K *is of characteristic* 2.

Proof. Let r and s be roots such that $r \neq s$ and $\bar{r} = \bar{s}$. Assume first that K is of characteristic other than 2. The equations

$$\overline{(-r)}(H_r) = -2 = -\bar{r}(H_r)$$

show that $r \neq -s$. Then since $c_{rs}c_{sr} = 0, 1, 2$ or 3 and $\bar{c}_{rs}\bar{c}_{sr} = \bar{c}_{rr}\bar{c}_{ss} = \bar{4}$, the only possibility is that $c_{rs}c_{sr} = 1$ and K is of characteristic 3. From $\bar{c}_{rs} = \bar{c}_{ss} = \bar{2} = -\bar{1}$, we see that r and s have the same length and form an angle of $2\pi/3$. Since Σ is not of type G_2, this implies that

r and s can be incorporated into a fundamental set [2, p. 19], and since Σ is not of type A_2, this set contains a third root t which can be taken orthogonal to one of r, s and not to the other. But then $\bar{c}_{rt} \neq \bar{c}_{st}$ by (1) of 2.6, contradicting $\bar{r} = \bar{s}$. Now assume that K is of characteristic 2. Then all roots have the same length. Thus if r is orthogonal to s, then r and s can be incorporated into a fundamental set, and since Σ is not of type D_n, one reaches a contradiction just as before. On the other hand if r is not orthogonal to s, then the equation $\bar{c}_{rs} = \bar{c}_{ss} = 0$ implies that $c_{rs} = \pm 2$, whence $r = \pm s$ because r and s have the same length. Since $\overline{(-r)} = \bar{r}$ if K is of characteristic 2, 5.1 is proved.

5.2. *Let $S = \{X_r, H_r \mid r \in \Sigma\}$ be the standard set of generators of $\mathfrak{g}$ introduced in § 4 and let $S' = \{X_q, H_q \mid q \in \Sigma'\}$ be a second standard set such that S and S' determine the same Cartan subalgebra $\mathfrak{h}$. Then there exists a bijective mapping $r \to r'$ of Σ onto Σ' such that*

(1) *if r, s and $r + s$ are in Σ, then $r' + s'$ is in Σ', $(r+s)' = r' + s'$, and $(-r)' = -r'$, and*

(2) *for each r in Σ, $H_{r'} = H_r$ and $X_{r'} = c_r X_r$ with c_r in K and $c_r c_{-r} = 1$.*

Proof. The nonzero root spaces of $\mathfrak{g}$ relative to $\mathfrak{h}$ are determined by 5.1 as $\{KX_r\}$ if K is not of characteristic 2 and $\{KX_r + KX_{-r}\}$ if K is of characteristic 2. In the latter case, if $X = kX_r + lX_{-r}$, then $\operatorname{ad} X$ is nilpotent only if either k or l is 0, as one sees by choosing a root s of Σ such that $r + s$ is also a root and then computing $(\operatorname{ad} X)^2 X_s = klX_s$. Thus in all cases $\mathfrak{h}$ determines $\{KX_r\}$ (and $\{KX_q\}$) and there exist a bijective mapping $r \to r'$ and scalars c_r such that $X_{r'} = c_r X_r$. Since $X_s X_r$ is a nonzero element of $\mathfrak{h}$ if and only if $s = -r$, one has $(-r)' = -r'$, and if r, s and $r + s$ are in Σ, then $X_r X_s$ is a nonzero element of KX_{r+s} by (1) of 2.6, which implies that $r' + s'$ is in Σ' and $(r+s)' = r' + s'$. Next $H_{r'} = c_r c_{-r} H_r$ by 2.4. Now one can find a root s such that $\bar{s}(H_{r'}) = \bar{s}(H_r) \neq 0$: if K is of characteristic 2, choose for s any root not orthogonal to r, and if K is not of characteristic 2, choose $s = r$. Thus $c_r c_{-r} = 1$, $H_{r'} = H_r$, and 5.2 is proved.

5.3. *Under the assumptions of 5.2 if G'' is the group defined relative to S' in the same way that G' is defined relative to S then $G'' = G'$.*

Proof. If either $r \neq -s$ or K is not of characteristic 2, then $x_{r'}(k)X_s = (\exp \operatorname{ad} kX_{r'})(c_s^{-1}X_{s'}) = (\exp \operatorname{ad} kc_r X_r)X_s = x_r(kc_r)X_s$, while if K is of characteristic 2, then

$$\begin{aligned} x_{r'}(k)X_{-r} &= x_{r'}(k)(c_{-r}^{-1}X_{-r'}) = c_{-r}^{-1}(X_{-r'} + kH_{r'} + k^2 X_{r'}) \\ &= X_{-r} + (kc_r)H_r + (kc_r)^2 X_r = x_r(kc_r)X_{-r}\ , \end{aligned}$$

by 5.2. Hence $x_{r'}(k) = x_r(kc_r)$ and $G'' = G'$.

Let us now turn to the proof of 4.1. Clearly it is enough to prove that $\mathfrak{h}_1$ is conjugate to $\mathfrak{h}$ under G'. For then by symmetry $\mathfrak{h}_2$ is also conjugate to $\mathfrak{h}$ and then to $\mathfrak{h}_1$. Let S_1 be a standard set of generators corresponding to $\mathfrak{h}_1$. Assume first that K is algebraically closed (so that $G' = G$ by 3.1) and let G_1 be the group defined relative to S_1 in the same way that G is defined relative to S. By a familiar argument of Harish-Chandra (see [16, Exp. 15] or [13]), there exist y in G and y_1 in G_1 such that $y\mathfrak{h} = y_1\mathfrak{h}_1$. Set $x = y_1^{-1}y$. Then $x\mathfrak{h} = \mathfrak{h}_1$, and by 3.2 and 5.3, $G_1 = xGx^{-1}$, whence $G_1 = y_1G_1y_1^{-1} = yGy^{-1} = G$ and x is in G. Now assume that K is not algebraically closed. Let $\hat{K}$ be its algebraic closure and let $\hat{\mathfrak{g}}$, etc., be the objects corresponding to $\mathfrak{g}$, etc., when K is replaced by $\hat{K}$. As has just been shown, there is y in $\hat{G}$ such that $y\hat{\mathfrak{h}} = \hat{\mathfrak{h}}_1$. By 5.2 the elements of yS are multiples of those of S_1. One can normalize y by multiplication by an element of $\hat{\mathfrak{H}}$ so that yX_a is in S_1 for each fundamental root a, and then by 5.2, yX_{-a} and yH_a are also in S_1. Since $\mathfrak{g}$ is generated over K by the elements H_a, and $\mathfrak{g}$ is generated by the elements X_a, X_{-a}, it follows that $y\mathfrak{h} = \mathfrak{h}_1$ and $y\mathfrak{g} = \mathfrak{g}$. Since y is in $\hat{G}$ and y induces an automorphism of $\mathfrak{g}$, a result of Ono [10] implies that y is is G. By 3.1 one can write $y = xh$ with x in G' and h in $\mathfrak{H}$. Thus $x\mathfrak{h} = xh\mathfrak{h} = y\mathfrak{h} = \mathfrak{h}_1$, and 4.1 is completely proved.

By combining 3.2, 4.1 and 5.3 we get:

5.4. *The group G' is independent of the standard set of generators used to define it.*

Finally, let use observe that the word standard may be omitted from 4.1 if K is algebraically closed and not of characteristic 2, 3 or 5 because then every Cartan subalgebra is standard (see [1, thesis; 12; 2]).

6. **Proofs of 4.2 to 4.8.** If x is an automorphism of $\mathfrak{g}$, then $x\mathfrak{h}$ is a standard Cartan subalgebra of $\mathfrak{g}$. Hence by 4.1 there is j in G' such that $j^{-1}x\mathfrak{h} = \mathfrak{h}$. Then 5.2 implies that there is a permutation $r \to r'$ on Σ such that (1′) if r, s and $r+s$ are in Σ, then $(r+s)' = r' + s'$ and $(-r)' = -r'$, and (2′) $j^{-1}xX_r = c_rX_{r'}$ and $c_rc_{-r} = 1$ for each r in Σ. By (1′), Φ' is a fundamental set of roots since Φ is. Hence [16, p. 16–05] there is w in W, the Weyl group, such that $w\Phi = \Phi'$. Then replacing j by $i = j\omega(w)$ we see that the refinement $\Phi' = \Phi$ is achieved. We can now choose h in $\mathfrak{H}$ so that $hX_{a'} = c_aX_{a'}$ for each a in Φ, whence $h^{-1}i^{-1}xX_a = X_{a'}$ and then $h^{-1}i^{-1}xX_{-a} = X_{-a'}$, because $c_ac_{-a} = 1$. Thus by 2.3 and 2.4 and the fact that $h^{-1}i^{-1}x$ is an automorphism $c_{a'b'} = c_{ab}$ for a and b in Φ. That is, $h^{-1}i^{-1}x$ is a graph automorphism, and 4.2 is proved.

From the definitions, it is easily checked that $hG'h^{-1} = G'$, $gG'g^{-1} = G'$ and $g\mathfrak{H}g^{-1} = \mathfrak{H}$ if h is in $\mathfrak{H}$ and g is a graph automorphism. Thus 4.2 implies 4.3.

As a restatement of part of 3.1, 4.4 is true.

Next assume that the graph automorphism g is in G. Let $\mathfrak{u}$ and $\mathfrak{U}$, respectively, be the subalgebra of $\mathfrak{g}$ and subgroup of G' generated by those X_r and $x_r(k)$ for which r is positive. Then by [2, Th. 2] there are u, u'' in $\mathfrak{U}$, h in $\mathfrak{H}$ and w in W such that $g = uh\omega(w)u''$, whence $\omega(w)\mathfrak{u} = h^{-1}u^{-1}gu''^{-1}\mathfrak{u} \subseteqq \mathfrak{u}$. This implies that w maps positive roots onto positive roots, whence $w = 1$. Then the equation

$$gX_r = uhu''X_r = c_rX_r + \sum_{s>r} c_sX_s ,$$

$c_r \neq 0$, in conjunction with the definition of graph automorphism, implies that $g = 1$, that 4.5 is true.

Let P and P_r be the additive groups generated by the weights and by the roots relative to Σ. By a basic theorem for free modules, there exist bases $\{b_i\}$ and $\{b'_i\}$ of P and P_r and a set of positive integers $\{f_i\}$ such that $b'_i = f_ib_i$ for each i. Then from the definitions $\mathfrak{H}/\mathfrak{H}'$ is isomorphic to the direct product of the groups K^*/K^{*f_i}. Now since Φ is a basis for P_r and $\{a' \mid a \in \Phi, (2a', b)/(b, b) = \delta_{ab}, b \in \Phi\}$ is a basis for P, the numbers f_i can be found by reducing the matrix $(c_{ab})(a, b \in \Phi)$ to diagonal form. In this way 4.6 is proved.

Finally, an examination of the various root systems yields 4.7, and then 4.6 and 4.7 imply 4.8.

7. The other algebras. Continuing with the previous notation, but dropping the assumption in the second sentence of § 4, we define G'' to be the group generated by the automorphisms of type $x_r(k)$ constructed relative to all standard sets of generators for which $\mathfrak{h}$ is the corresponding Cartan subalgebra. By 5.3, $G'' = G'$ for the algebras treated there, but this is not the case for the algebras yet to be considered.

7.1. *If $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are standard Cartan subalgebras of $\mathfrak{g}$, there is x in G'' such that $x\mathfrak{h}_1 = \mathfrak{h}_2$.*

7.2. *In the respective cases that Σ is of type A_2, D_4 or D_n $(n \neq 4)$ and K is of characteristic 3, 2 or 2, the group G'' is isomorphic to the group G' of type G_2, F_4 or C_n.*

7.3. *In the first two cases above $A = G''$ and in the third A/G'' is isomorphic to K^*/K^{*2}.*

The proofs of these results require suitable analogues of 5.1 and 5.2:

7.4. *In the respective cases of* 7.2, *the nonzero root spaces of* $\mathfrak{g}$ *relative to* $\mathfrak{h}$ *have dimensions* 3, 8 *or* 4.

7.5. *If* S *and* S'' *are standard sets of generators both of which have* $\mathfrak{h}$ *as the corresponding Cartan subalgebra, then there is* x *in* G'' *such that* S *and* $S' = xS''$ *satisfy the properties* (1) *and* (2) *of* 5.2.

The ideas in the proofs of these results are the same for all three types of algebras. However, the details are somewhat different. Hence we shall restrict ourselves to a discussion of the algebra of type A_2 over a field of characteristic 3.

Now the roots of a system of type G_2 may be so labelled that the set of short ones is $\Sigma = \{\pm a, \pm b, \pm(a+b)\}$ and the set of long ones is $\Lambda = \{\pm(a-b), \pm(a+2b), \pm(2a+b)\}$ (see any of [1, p. 93; 3, p. 141; 16, p. 14–06]). As has already been mentioned, the construction of 2.6 does not yield a simple algebra if a root system of type G_2 is combined with a field K of characteristic 3: the set $S = \{X_r, H_r \mid r \in \Sigma\}$ spans an ideal which is easily seen to be a classical Lie algebra of type A_2 with S as a standard set of generators. Let $\mathfrak{g}$ denote the ideal and $\mathfrak{m}$ the full algebra. First we observe that an automorphism of $\mathfrak{m}$ which is the identity on $\mathfrak{g}$ is the identity on $\mathfrak{m}$ because the adjoint action of $\mathfrak{m}$ on $\mathfrak{g}$ is faithful by 2.3 and 2.5. Thus the automorphisms of $\mathfrak{m}$ may be considered to act on $\mathfrak{g}$ (the unique minimal ideal) without any ambiguity. Now $\bar{c}_{ab} = \bar{c}_{ba} = -1 = 2 = \bar{c}_{aa} = \bar{c}_{bb}$. Hence $\bar{a} = \bar{b}$, and then $-\bar{a} - \bar{b} = -\bar{a} - \bar{b} = \bar{a}$. Thus $\bar{a}$ corresponds to a root space R^+ spanned by those X_r for which r is in $\Sigma^+ = \{a, b, -a-b\}$; a similar statement for $-\bar{a}$ establishes 7.4. Now each r in Λ can be written uniquely $r = t - s$ with t and s in Σ^+. Hence if u denotes the third element of Σ^+, $x_r(k)$ maps X_s, X_t, X_u onto $X_s \pm kX_t, X_t, X_u$, respectively. Here s, t, u run through the permutations of Σ^+ as r runs through Λ. Hence the group generated by $\{x_r(k) \mid r \in \Lambda, k \in K\}$ induces in R^+ the three-dimensional unimodular group. Now if $S'' = \{Y_q, J_q, q \in \Sigma'\}$ is a second standard set of generators of $\mathfrak{g}$ corresponding to the same Cartan subalgebra $\mathfrak{h}$ as S, then the root spaces, as determined by $\mathfrak{h}$, are three dimensional and Σ' is of type A_2, whence its roots can be labelled so that R^+ is spanned by $Y_{a'}$, $Y_{b'}$ and $Y_{-a'-b'}$. Thus by what has just been said there is x in G''', the group of type G' for $\mathfrak{m}$ such that, if we set $xY_r = X_r$ and $xJ_r = H_r$ for each r in Σ', then $X_{a'}$, $X_{b'}$, $X_{-a'-b'}$ are scalar multiples of X_a, X_b, X_{-a-b}, respectively. But then also $X_{a'+b'} = \pm X_{a'}X_{b'}$ is a scalar multiple of $X_aX_b = \pm X_{a+b}$, with similar statements for X_{-a} and X_{-b}, whence the properties $H_{r'} = H_r$ and $c_rc_{-r} = 1$ are proved as before. Now consider the identity [2, p. 63, 1.7]

$$x_{r+3s}(k)x_{-s}(1)x_{r+3s}(k)^{-1} = x_{-s}(1)x_{r+2s}(\pm k)x_{r+s}(\pm k)x_r(\pm k)x_{2r+3s}(\pm k^2)$$

which is valid if s and $r+s$ are in Σ, r is in Λ and k is in K. By 3.2 the left side is in G'' as are the first three terms on the right. Thus the product of the last two is also, and replacing k by $-k$, we conclude that $x_r(k)$ is in G''. Thus $G''' \subseteqq G''$, completing the proof of 7.5. We see by 3.2 that G'' is generated by elements of the form $xx_r(k)x^{-1}$, with r in Σ and x in G'''. Hence $G'' \subseteqq G'''$, whence $G'' = G'''$ and 7.2 is proved. The deduction of 7.1 and 7.3 now proceeds as before and details are left to the reader.

8. **Classification theorem.** By 4.1, 5.2, 7.1 and 7.5, if two classical Lie algebras are isomorphic, then they can be identified so that specified standard sets of generators satisfy conditions (1) and (2) of 5.2, whence the root systems are of the same type. Hence (see [13]).

8.1. *Two classical Lie algebras are isomorphic if and only if they have the same type.*

9. **Extensions.** If $\hat{\mathfrak{g}}$ is obtained from an algebra $\mathfrak{g}$ by extension of the base field, then any automorphism of $\mathfrak{g}$ has a unique extension to $\hat{\mathfrak{g}}$, whence the automorphisms of $\mathfrak{g}$ may be described as the restrictions to $\mathfrak{g}$ of those automorphisms of $\hat{\mathfrak{g}}$ which fix $\mathfrak{g}$. Thus if $\hat{\mathfrak{g}}$ turns out to be a direct sum of classical Lie algebras, the results above enable us to determine the automorphisms of $\mathfrak{g}$. For example, using well-known identifications [11], we infer from 4.2 to 4.8 for $\mathfrak{g}$ of type B_n or D_n that each automorphism of the Lie algebra of those linear transformations of a vector space of dimension not 8 over an algebraically closed field of characteristic not 2 which are skew relative to a non singular symmetric bilinear form is induced by an orthogonal transformation of the underlying space, and we then easily deduce if the field is not necessarily algebraically closed that every automorphism is induced by a similitude.

A procedure often used to construct a Lie algebra $\mathfrak{g}$ is to start with $\hat{\mathfrak{g}}$, a direct sum of classical Lie algebras, to then prescribe a group F of semiautomorphisms of $\hat{\mathfrak{g}}$, and finally to define $\mathfrak{g}$ as the set of fixed points of F. Let us assume that F is so chosen that $\hat{\mathfrak{g}}$ can be regarded as a field extension of $\mathfrak{g}$. Then the device stated above is applicable in the following easily proved form: the automorphisms of $\mathfrak{g}$ are the restrictions to $\mathfrak{g}$ of those automorphisms of $\hat{\mathfrak{g}}$ which commute with the elements of F. Examples here are the analogues over general fields of the real forms of Cartan [1, p. 399], and the algebras which can be constructed from those classical ones which admit graph automorphisms by naturally defined semiautomorphisms. For these latter algebras one can thus obtain explicit statements such as 4.2 to 4.5 with the rôle of G' taken by the simple groups considered in [17].

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