ON THE NUMBER OF PURE SUBGROUPS

PAUL HILL

A problem due to Fuchs [3] is to determine the cardinality of the set $\mathscr{P}$ of all pure subgroups of an abelian group. Boyer has already given a solution for nondenumerable groups G [1]; he showed that $|\mathscr{P}| = 2^{|G|}$ if $|G| > \aleph_0$, where $|A|$ denotes the cardinality of a set A. Our purpose is to complement the results of [1] by determining those groups for which $|\mathscr{P}|$ is finite, $\aleph_0$, and $c = 2^{\aleph_0}$. In the following, group will mean abelian group.

LEMMA 1. *If G is a torsion group with $|G| \leqq \aleph_0$, then $|\mathscr{P}| = c$ unless*

$$(1) \qquad G = p_1^\infty \oplus p_2^\infty \oplus \cdots \oplus p_n^\infty \oplus B\,,$$

a direct sum of (at most) a finite number of groups of type p^∞ and a finite group, where $p_i \neq p_j$ if $i \neq j$. If G is of the form (1), *then $|\mathscr{P}|$ is finite.*

Proof. The latter statements is clear, and if none of the following hold

(i) G decomposes into an infinite number of summands

(ii) G contains $p^\infty \oplus p^\infty$ for some prime p

(iii) $|B| = \aleph_0$, where B is the reduced part of G,

then G is of the form (1). Moreover, if (i) holds, then obviously $|\mathscr{P}| = c$. Every automorphism of p^∞ determines a pure subgroup of $p^\infty \oplus p^\infty$, and distinct automorphisms correspond to distinct subgroups. Since $|A(p^\infty) = \text{automorphism group}| = c$, it follows that $p^\infty \oplus p^\infty$ has c pure subgroups. Thus if (ii) holds, $|\mathscr{P}| = c$ since $p^\infty \oplus p^\infty$ is a direct summand of G. Finally, if (iii) holds and if (i) does not, then the following argument shows that $|\mathscr{P}| = c$. We may write[1] $B = C_1 \oplus B_1 = C_1 \oplus C_2 \oplus B_2$, and continuing in this way define an infinite sequence C_n of cyclic groups such that no C_i is contained in the direct sum of any of the others. The direct sum of any subcollection of these cyclic groups is a pure subgroup of B and, therefore, of G.

An interesting corollary is noted: there is no torsion group with exactly $\aleph_0$ pure subgroups.

LEMMA 2. *If $G = F \oplus B$ is the direct sum of a torsion free group*

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[1] This is precisely the proof of Boyer that such a group has c subgroups [2].

F of rank r and a finite group B with $|G| \leqq \aleph_0$, then $|\mathscr{P}|$ is finite, $\aleph_0$, or c, depending on whether $r = 1$, $1 < r < \infty$, or $r = \infty$.

Proof. First, assume that $B = 0$. Let H be the minimal divisible group containing G. The correspondence $D \to D \cap G$ is one-to-one between pure (divisible) subgroups D of H and pure subgroups of G. Thus only divisible groups G need be considered, and the proof is already clear except, possibly, the relation $|\mathscr{P}| \leqq \aleph_0$ for the case $1 < r < \infty$. However, let R^* denote the direct sum of $r - 1$ copies of R, the additive rationals. Since $G = R^* \oplus R$, any pure subgroup P of G is a subdirect sum of a subgroup S^* of R^* and a subgroup S of R. Moreover, S^* and $S^* \cap P$ are pure in R^*; S and $S \cap P$ are pure in R. Since $|A(R)| = \aleph_0$, it follows by induction that $|\mathscr{P}| \leqq \aleph_0$.

Now consider the case $B \neq 0$. The lemma has already been proved if $r = \infty$, so assume that r is finite. Any pure subgroup P of $G = F \oplus B$ is a subdirect sum of a pure subgroup E of F and a subgroup A of B. Since $E \cap P$ has index in E which divides the order of B, there are only a finite number of choices of $E \cap P$ for a given E (and consequently only a finite number of choice of P). Thus the lemma is proved.

The theorem follows almost immediately from the lemmas.

THEOREM. *For any group G, $|\mathscr{P}| \leqq \aleph_0$ if and only if: $G = F \oplus T$ where T is torsion of the form* (1) *and F is torsion free of finite rank $r \geqq 0$; further if the prime p is in the collection $\pi = \{p_1, p_2, \cdots, p_n\}$ of the decomposition* (1) *of T, then F has no pure subgroup which can be mapped homomorphically onto p^∞. In all other cases, $|\mathscr{P}| = 2^{|G|}$. Moreover, $|\mathscr{P}|$ is finite if and only if either $r = 0$ or $r = 1$ and T is finite.*

Proof. Suppose that $|\mathscr{P}| \neq 2^{|G|}$. Then $|G| \leqq \aleph_0$ and the torsion part T of G is of the form (1). Hence G splits into its torsion and torsion free components, $G = F \oplus T$. Also, F is of finite rank $r \geqq 0$. And there exists no homomorphism of a pure subgroup of F onto p^∞ where $p \in \pi$ (since there would be c such homomorphisms, each determining a pure subgroup of G). But suppose that $G = F \oplus T$, where F and T satify the given conditions. Let T' denote the divisible part of T and set $F' = F \oplus B$, where $T = T' \oplus B$. Since B is finite, $|\mathscr{P}(F')| \leqq \aleph_0$ is given by Lemma 2. Evidently, a pure subgroup P of G is the direct sum of a divisible subgroup of T' and a subdirect sum of a pure subgroup of F' and a finite subgroup of T'. Thus $|\mathscr{P}| \leqq \aleph_0$.

If $r = 1$, then $|\mathscr{P}(F \oplus p^\infty)| \geqq \aleph_0$, for there are at least $\aleph_0$ homomorphisms of F into p^∞, each determining a pure subgroup. In view of Lemmas 1 and 2, this completes the proof of the theorem.

REFERENCES

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