OPERATORS OF FINITE RANK IN A REFLEXIVE BANACH SPACE

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1. Let X be a reflexive Banach space and $F(X)$ the Banach algebra of all uniform limits of operators of finite rank, in X. Bonsall [1] has characterized $F(X)$ as a simple, $B^\sharp$-annihilator algebra: $F(X)$ contains no proper closed two-sided ideals, every proper, closed right (left) ideal of $F(X)$ has a nonzero left (right) annihilator, and, given any $T \in F(X)$, there exists $T^\sharp \in F(X)$ such that

$$||T||\,||T^\sharp|| = ||(TT^\sharp)^n||^{1/n}\,, \qquad n = 1, 2, 3, \cdots .$$

In this note, we obtain a new characterization for $F(X)$ (Theorem 3.2): a Banach algebra A is the algebra $F(X)$ of all uniform limits of operators of finite rank in a reflexive Banach space X if and only if A is a simple, weakly compact, $B^\sharp$-algebra with minimal ideals (A is weakly compact if left- and right-multiplications by every $a \in A$ are weakly compact operators). In the process of proving this result, we obtain a characterization of reflexive Banach spaces which seems to be of some independent interest (Theorem 2.2): a Banach space X is reflexive if and only if every operator in X of rank 1 is a weakly compact element of $B(X)$.

2. Let X be a Banach space and $B = B(X)$ the Banach algebra of all bounded operators in X with the uniform topology. For $T \in B$, let R_T denote the operator in B obtained by multiplying elements of B on the right by T: $R_T(A) = AT$ for $A \in B$.

Suppose that T is a fixed operator of rank 1 in X with $H = [x \in X\colon Tx = 0]$. Then H is a closed hyperplane in X and if x_0 is an element of X such that $Tx_0 \neq 0$, then $X = H \oplus (x_0)$ and we may assume that $||x_0|| = 1$. Write $B' = [S \in B\colon S(H) = (0)]$. For each $S \in B'$, we define an element x_S of X by setting $x_S = S(x_0)$. The mapping $S \to x_S$ is clearly linear.

LEMMA 2.1. *The linear mapping $S \to x_S$ is a homeomorphism of B' onto X.*

Proof. It is clear that the mapping is one-to-one and, since $||S(x_0)|| \leqq ||S||$, it is continuous. It is also onto; in fact, let $\varphi \in X^*$ be such that $\varphi(H) = (0)$, $\varphi(x_0) = 1$. Then for given $x \in X$, the operator S_x defined by setting $S_x(y) = \varphi(y)x$, $y \in X$ belongs to B' and is mapped into x by the mapping $S \to S(x_0)$. Hence, by the closed graph theorem, the

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mapping is bicontinuous and the proof is complete.

Let B_1 denote the unit ball in B, so that $R_T(B_1) = [PT \in B: ||P|| \leqq 1]$.

LEMMA 2.2. $R_T(B_1) = [A \in B': ||Ax_0|| \leqq ||Tx_0||]$.

Proof. It is clear that $R_T(B_1) \subset [A \in B': ||Ax_0|| \leqq ||Tx_0||]$. Now let $A \in B'$ with $||Ax_0|| \leqq ||Tx_0||$; we find $P \in B_1$ such that $A = PT$. There exists $\psi \in X^*$ such that $||\psi|| = 1$ and $\psi(Tx_0) = ||Tx_0||$. We define P by setting $Px = \psi(x)Ax_0/||Tx_0||$. Then $PTx = 0$ if $x \in H$ and $PTx_0 = Ax_0$. Thus PT and A coincide in the subspace (x_0) and must therefore coincide everywhere in X. Finally $||P|| = \sup_{||x|| \leqq 1} ||\psi(x)Ax_0||/||Tx_0|| \leqq 1$; hence $P \in B_1$ and $R_T(B_1) = [A \in B': ||Ax_0|| \leqq ||Tx_0||]$.

LEMMA 2.3. *Let F be any subset of B'. If $F^{B'}$ denotes the closure of F with respect to the weak topology of B' and F^B the closure of F with respect to the weak topology of B, then $F^{B'} = F^B$.*

Proof. Let $P_0 \in F^{B'}$ and let

$$\begin{aligned} N &= N(P_0; \Phi_1, \Phi_2, \cdots, \Phi_n; \varepsilon) \\ &= [P \in B: |\Phi_k(P - P_0)| < \varepsilon; k = 1,2, \cdots, n; \Phi_k \in B^*] \end{aligned}$$

be an arbitrary neighborhood of P_0 in B. Then the neighborhood $N' = N(P_0; \Phi'_1, \Phi'_2, \cdots, \Phi'_n; \varepsilon)$ of P_0 obtained by taking the restriction of Φ_k to B' for each k, contains a point P of F. Since P must therefore belong to N, it follows that $F^{B'} \subseteqq F^B$.

Now suppose that $P_0 \in F^B$. Then $P_0 \in B'$ since B' is closed with respect to the weak topology of $B(X)$ (being linear and strongly closed). Let $N' = [P \in B': |\varphi_k(P - P_0)| < \varepsilon, k = 1, 2, \cdots, n; \varphi_k \in (B')^*]$ be an arbitrary neighborhood of P_0 in B'. Then again, by considering the neighborhood $N = [P \in B: |\Phi_k(P - P_0)| < \varepsilon, k = 1, 2, \cdots, n, \Phi_k \in B^*]$ obtained by extending φ_k to Φ_k, for each k, on the whole of B, we can find $P \in F$ such that $P \in N'$. Hence $F^B \subseteqq F^{B'}$. This completes the proof.

THEOREM 2.1. *A Banach space X is reflexive if and only if every operator in X of rank 1 is a right weakly compact element of $B(X)$.*

Proof. If X is reflexive and T is of rank 1, then by Lemma 2.1, B' is homeomorphic with X under the correspondence $S \leftrightarrow S(x_0)$. Now the image of B_1 under R_T is a bounded subset of B' which is therefore contained in a set U which is compact with respect to the weak topology of B' and by Lemma 2.3, with respect to the weak topology of $B(X)$. Thus R_T is a weakly compact operator in $B(X)$ and T is a right weakly compact element of $B(X)$.

Now suppose that R_T is weakly compact in $B(X)$. Then $R_T(B_1)$ is contained in a set $V \subset B'$ which is compact with respect to the weak topology of $B(X)$ and hence also with respect to the weak topology of B'. Now the ball $Q = [A \in B': ||A|| \leqq ||Tx_0||/||x_0||]$ is contained in $R_T(B_1) \subset V$ and is weakly closed. Hence Q is compact with respect to the weak topology of B' and therefore B' is reflexive. Since B' is homeomorphic with X, it follows that X is reflexive and the proof is complete.

COROLLARY 2.1. *If X is a reflexive Banach space, then the algebra $F(X)$ of all uniform limits of operators of finite rank in X is a weakly compact algebra.*

COROLLARY 2.2. (Ogasawara [2] Theorem 4.) *Let H be a Hilbert space and $B(H)$ the Banach algebra of all bounded operators in H. If T is a compact operator in H, then T is a weakly compact element of $B(H)$.*

3. This section is devoted to the study of simple, weakly compact, B^*-algebras with minimal ideals.

LEMMA 3.1. *Let A be a simple Banach algebra with minimal ideals. Then every maximal regular left ideal M of A has a nonzero right annihilator.*

Proof. Since A is a simple Banach algebra, there exists an idempotent $e \in A$ such that $M \cap Ae = (0)$ and $M \oplus Ae = A$. Since M is regular, there is $j \in A$ such that $xj - x \in M$ for every $x \in A$. For some $a_0 \in A$ and $m_0 \in M$, $j = m_0 + a_0e$, $a_0e \neq 0$. Suppose now that m is an arbitrary element in M. We have $mj - m \in M$ and $mj - ma_0e = mm_0 \in M$, from which it follows that $m - ma_0e \in M$. Now, $m \in M$ and hence $ma_0e \in M$. However, $ma_0e \in Ae$ since Ae is a left ideal, thus $ma_0e \in M \cap Ae = (0)$ and since m is arbitrary in M, the lemma is proved.

LEMMA 3.2. *Let A be a simple Banach algebra with minimal right ideals. If $j \in A$ and j has no left reverse, then there exists $a \neq 0$ such that $ja = a$.*

Proof. Let $J = [yj - y: y \in A]$. Then J is a regular left ideal of A which is proper since $j \notin J$. Hence by Lemma 3.1, there exists $a \in A$, $a \neq 0$ such that $Ja = (0)$, i.e. such that $yja - ya = 0$ for all $y \in A$ or $A(ja - a) = (0)$. Since $(A)_r = (0)$, this implies that $ja = a$.

LEMMA 3.3. *Let A be a simple B^*-algebra with minimal right* *ideals. If* $|\ \ |$ *is any other norm in* A *with* $|a| \leqq \|a\|$ *for each* $a \in A$, *then* $|\ \ | = \|\ \ \|$.

Proof. Lemma 3.2 implies that if $|\ \ |$ is any other norm in A, then $\lim_{n\to\infty} |a^n|^{1/n} = \lim_{n\to\infty} \|a^n\|^{1/n}$ for every $a \in \mathrm{A}$ (Cf [4], Lemma 3.1). Then since A is a $B^\#$-algebra, we have

$$|a^\#||a| \geqq |a^\# a| \geqq \lim_{n\to\infty} |(a^\# a)^n|^{1/n}$$
$$= \lim_{n\to\infty} \|(a^\# a)^n\|^{1/n} = \|a\|\,\|a\|\ ,$$

and since $|a^\#| \leqq \|a^\#\|$ and $|a| \leqq \|a\|$, the result follows.

THEOREM 3.1. *A Banach algebra* A *is the algebra* $F(X)$ *of all uniform limits of operators of finite rank in a reflexive Banach space* X *if and only if* A *is a simple, weakly compact,* $B^\#$*-algebra with minimal right ideals.*

Proof. Let A be a simple, weakly compact, $B^\#$-algebra with eA a minimal right ideal, e a primitive idempotent. We represent A as an algebra of operators $\mathscr{A}$ in eA, the latter regarded as a Banach space. Corresponding to each $a \in A$, we define an operator $\bar{a} \in \mathscr{A}$ by $\bar{a}: x \to xa$ for $x \in eA$. The correspondence $a \to \bar{a}$ is obviously an isomorphism and if we take $\|\bar{a}\| = \sup_{\|x\|\leqq 1} \|xa\|$, $x \in eA$, the correspondence is an isometry in view of Lemma 3.3. Thus A is isomorphic and isometric to $\mathscr{A}$ and A is the uniform closure of $\mathscr{A}$.

Next we show that eA is a reflexive Banach space. Now e has no left reverse in A; hence by Lemma 3.2, there exists $a \in A$, $a \neq 0$ such that $ea = a$. The set $P = [a \in A: ea = a]$ is a right ideal of A and since $P \subseteqq eA$, we must have $P = eA$ since eA is minimal. If e is now regarded as a left weakly compact operator on A, then it is clear that the set $P = eA$ is a reflexive Banach space.

Our next step is to show that in the representation described above, $\mathscr{A}$ contains all operators of finite rank in eA. Corresponding to each $a \in Ae$, there exists a continuous linear functional φ_a on eA satisfying $\varphi_a(x)e = xa$, $x \in eA$. Let $G = [\varphi_a \in (eA)^*: a \in A]$; then G is a linear subspace of $(eA)^*$. We show that G is closed with respect to the usual norm in $(eA)^*$ defined by $\|\varphi\| = \sup_{\|x\|\leqq 1} |\varphi(x)|\ x \in eA$. For $a \in Ae$, we have $xa = \varphi_a(x)e$, $x \in eA$, and since $\|a\| = \|\bar{a}\|$ for each $a \in A$, we have

$$\|a\| = \|\bar{a}\| = \sup_{\|x\|\leqq 1} \|xa\| \qquad a \in Ae$$
$$= \sup_{\|x\|\leqq 1} \|\varphi_a(x)e\|$$
$$= \sup_{\|x\|\leqq 1} |\varphi_a(x)|\,\|e\|$$

$$= \|\varphi_a\| \cdot \|e\| .$$

Thus G is topologically equivalent to Ae and hence closed. Having proved that G is a closed linear subspace of $(eA)^*$, we now show that G is in fact the whole of $(eA)^*$. Suppose that there exists $\varphi' \in (eA)^*$ such that $\varphi' \notin G$. Since G is closed, there exists $\Phi \in (eA)^{**}$ such that $\Phi(\varphi_a) = 0$ for all $\varphi_a \in G$ and $\Phi(\varphi') = 1$. However, eA is a reflexive Banach space: hence there exists $u_0 \in eA$, $u_0 \neq 0$ such that $\Phi(\varphi) = \varphi(u_0)$ for all $\varphi \in (eA)^*$. In particular, for $\varphi_a \in G$, this implies that $0 = \varphi_a(u_0)e = u_0 a$ for all $a \in Ae$, which in turn implies that $u_0 \in (Ae)_l = (0)$ which is absurd. Hence $G = (eA)^*$. From this it follows that $\mathscr{A}$ contains all operators of rank 1 and hence all operators of finite rank in eA, since if T is an operator of rank 1 in eA, then there exists $\varphi \in (eA)^*$ and $u_0 \in eA$ such that $xT = \varphi(x)u_0$, $x \in eA$. Since $\varphi \in G$, there exist $a \in Ae$ and $\varphi_a \in (eA)^*$ such that $\varphi = \varphi_a$ and $xa = \varphi_a(x)e$. Let $u_0 = ea_0$ for some $a_0 \in A$; we have $xT = \varphi_a(x)u_0 = \varphi_a(x)ea_0 = xaa_0$, and since $aa_0 \in A$, the operator $aa_0 = T$ belongs to $\mathscr{A}$.

Finally, the uniform closure of the set of all operators of finite rank in eA is a closed two-sided ideal of $\mathscr{A}$ which must coincide with $\mathscr{A}$ since $\mathscr{A}$ is simple. Thus the "if" part of the theorem is proved.

That $F(X)$ is a simple, weakly compact $B^\sharp$-algebra with minimal ideals follows form corollary 1 and a result due to Bonsall and Goldie [1], Theorem 2. This completes the proof of the theorem.

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