BOUNDS IN THE NEUMANN PROBLEM FOR SECOND ORDER UNIFORMLY ELLIPTIC OPERATORS

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1. Introduction. In this paper we derive certain a priori inequalities which are useful for obtaining bounds in the interior Neumann problem for second order elliptic partial differential equations. In establishing these inequalities by our methods it is necessary to obtain lower bounds for the inverse of the Poincaré constant (μ_2 of eq. 3.3) and the first nonzero Steklov eigenvalue (p_2 of eq. 3.11). An optimal Poincaré inequality for convex domains in n-dimensions was given by Payne and Weinberger [5], and a method for obtaining lower bounds for p_2 for n-dimensional star-shaped regions was indicated by Payne and Weinberger [3]. However, to the authors' knowledge, no explicit lower bounds for p_2 and μ_2 for general n-dimensional regions have previously been given. Lower bounds for these constants which lead to the above mentioned inequalities in the Neumann problem are of interest in themselves and should prove useful in other applications.

For the special case of the Laplace equation other methods for deriving bounds for the Dirichlet integral in the Neumann problem appear in the literature (see [2], [6]). For starshaped regions a method similar to that proposed here was obtained in [3]. Bounds in exterior Neumann problems were given in [4].

2. Preliminary inequalities. Let R be a simply connected bounded region with boundary C in Euclidean n-space. In R we assume that the operator L given by

$$Lu \equiv (a^{ij}u_{,i})_{,j} \tag{2.1}$$

is a uniformly elliptic operator, defined for sufficiently smooth functions u. In (2.1) $,i$ denotes partial differentiation with respect to the coordinate x_i and the summation convention is assumed. The coefficient matrix a^{ij} is symmetric and the condition of uniform ellipticity may be stated as follows: There exist positive constants a_0 and a_1 such that for every real vector $(\xi_1, \cdots, \xi_n)$, the relation

$$a_0\sum_{i=1}^{n}\xi_i\xi_i \leqq a^{ij}\xi_i\xi_j \leqq a_1\sum_{i=1}^{n}\xi_i\xi_i \tag{2.2}$$

is valid uniformly in R.

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We consider now an arbitrary point of R which we choose to be the origin. Let S_a be the interior of a sphere of radius a, with center at the origin and such that $S_a \subset R$. The surface of the sphere will be called Σ_a. We denote by R_a the region $R - \bar{S}_a$, where $\bar{S}_a$ is the closure of S_a.

Let u be any sufficiently smooth function in $R + C$ and let f^i be a sufficiently smooth vector field defined in $\bar{R}_a$. Then, by the divergence theorem, we have

$$\oint_C f^i n_i u^2 ds = -\oint_{\Sigma_a} f^i n_i u^2 ds + \int_{R_a} f^i_{,i} u^2 dv + 2\int_{R_a} f^i u u_{,i} dv \,, \tag{2.3}$$

where n_i is the component of the unit normal directed outward on C. An application of the arithmetic-geometric mean inequality applied to the last term on the right of (2.3) yields

$$\oint_C f^i n_i u^2 ds \leqq -\oint_{\Sigma_a} f^i n_i u^2 ds + \int_{R_a}\left(f^i_{,i} + \frac{1}{\alpha} f^i f^i\right) u^2 dv + \int_{R_a} \alpha u_{,i} u_{,i} dv \tag{2.4}$$

where α is some positive function in R_a.

We assume now that f^i and α have been chosen so that

$$\begin{gathered} t \equiv f^i n_i \geqq K_1 > 0 \text{ on } C \\ -f^i n_i \leqq K_2 \text{ on } \Sigma_a \\ f^i_{,i} + \frac{1}{\alpha} f^i f^i \leqq 0 \text{ in } R_a \,, \end{gathered} \tag{2.5}$$

where K_1 and K_2 are constants. (We shall in a subsequent section construct vectors f^i satisfying (2.5) for certain domains.) Using conditions (2.5) with (2.4) we have that

$$\oint_C t u^2 ds \leqq K_2 \oint_{\Sigma_a} u^2 ds + \bar{\alpha} \int_{R_a} u_{,i} u_{,i} dv \,, \tag{2.6}$$

where $\bar{\alpha}$ is an upper bound for α in R_a. Suppose that u is normalized such that

$$\oint_{\Sigma_a} u ds = 0 \,. \tag{2.7}$$

Then

$$\oint_{\Sigma_a} u^2 ds \leqq p_2 \int_{S_a} u_{,i} u_{,i} dv \tag{2.8}$$

where p_2 is the first nonzero eigenvalue in the Steklov problem for the sphere S_a. That is

(2.9) $$p_2 = \min \frac{\int_{S_a} \chi_{,i}\chi_{,i} dv}{\oint_{\Sigma_a} \chi^2 ds}$$

where the minimum is taken over all sufficiently smooth functions in S_a which satisfy (2.7). For the sphere of radius a, p_2 is explicitly given by

(2.10) $$p_2 = 1/a \ .$$

Combining (2.6) and (2.8) it follows that

(2.11) $$\oint_C tu^2 ds \leqq K_3 D(u, u) \equiv K_3 \int_R u_{,i} u_{,i} dv \ ,$$

where $K_3 = \max(aK_2, \bar{\alpha})$, or using (2.5)

(2.12) $$\oint_C u^2 ds \leqq (K_3/K_1)\, D(u, u) \ .$$

Now from the divergence theorem

(2.13) $$\oint_C x^i n_i u^2 ds = n \int_R u^2 dv + 2 \int_R x^i u u_{,i} dv \ .$$

Using the arithmetic-geometric mean inequality it follows easily that

(2.14) $$\int_R u^2 dv \leqq \frac{2r_M}{n} \oint_C u^2 ds + \frac{4r_M^2}{n^2} D(u, u) \ ,$$

where r_M is the maximum distance from the origin to C. Inequality (2.12) with (2.14) yields

(2.15) $$\int_R u^2 dv \leqq K_4 D(u, u) \ ,$$

where

$$K_4 = \frac{2r_M}{n}\left[K_3/K_1 + \frac{2r_M}{n}\right] .$$

The preceding inequalities depended entirely on the existence of a vector field f^i satisfying (2.5). In certain cases, as will be shown in a subsequent section, such a vector field can be explicitly constructed so as to yield explicit, easily computable constants K_1, K_2 and K_3.

In some cases it may be that for the region R the vector field f^i is quite difficult to construct. We can make use of an additional inequality to reduce the problem to that of obtaining an inequality of the form (2.12) for a subregion of R.

Let us divide the region R into two disjoint subregions R_1 and R_2.

These regions are to be separated by a surface C'. The portion C which is part of the boundary of R_i will be denoted by C_i, $i = 1, 2$. Thus the boundary of R_i will be $C_i + C'$. We further assume that the subdivision has been made in such a way that C_1 is star shaped with respect to some point P not in $R_1 + C'$. We choose P to be the origin and apply the divergence theorem is R_1 to obtain

$$(2.16) \qquad \int_{C_1+C'} x^i r^{-(n+1)} n_i u^2 ds = -\int_{R_1} r^{-(n+1)} u^2 dv + 2\int_{R_1} x^i r^{-(n+1)} u u_{,i} dv$$

for any function u sufficiently smooth in R. Defining $\hat{t} = x_i n_i / r$ and using the arithmetic-geometric mean inequality we obtain

$$(2.17) \qquad \oint_{C_1} u^2 ds \leqq \frac{1}{\hat{t}_m}\left(\frac{r_M}{r_m}\right)^n \oint_{C'} u^2 ds + \frac{r_m}{\hat{t}_m}\left(\frac{r_M}{r_m}\right)^n D_1(u, u) .$$

In (2.17) r_M and r_m denote upper and lower bounds for r in $\bar{R}_1$ and $\hat{t}_m$ a lower bound for $\hat{t}$ on $C_1 \cdot D_1(u, u)$ denotes the Dirichlet integral over R_1.

Now suppose that for R_2 we could find a vector field f^i satisfying (2.5) relative to R_2 and obtain the inequality

$$(2.18) \qquad \oint_{C_2+C'} u^2 ds \leqq K_3/K_1\, D_2(u, u) .$$

Then clearly (2.17), together with (2.18) would yield

$$(2.19) \qquad \oint_C u^2 ds \leqq K_5 D(u, u)$$

where of course u is assumed normalized with respect to S_a in R_2, and K_5 is a constant.

It is now obvious that such a procedure could be repeated a finite number of times, finally reducing the region to one for which the inequality (2.12) may be more easily obtained. In particular if we iterate this procedure until the qth region R_q is star shaped, then, as we shall see in § 4, a vector field f^i for R_q is easily constructed.

3. Lower bounds for eigenvalues. The first nonzero eigenvalue μ_2 in the free membrane problem for R satisfies

$$(3.1) \qquad \Delta\tilde{v} + \mu_2\tilde{v} = 0 \text{ in } R$$

and

$$(3.2) \qquad \frac{\partial\tilde{v}}{\partial n} = 0 \text{ on } C ,$$

where $\tilde{v}$ is the corresponding eigenfunction. It is well known that μ_2

may be characterized by the minimum principle

$$\mu_2 = \min \frac{D(\varphi, \varphi)}{\int_R \varphi^2 dv} \tag{3.3}$$

for sufficiently smooth functions φ satisfying

$$\int_R \varphi \, dv = 0 \,, \tag{3.4}$$

and that $\tilde{v}$ is the minimizing function. That is

$$\mu_2 = \frac{D(\tilde{v}, \tilde{v})}{\int_R \tilde{v}^2 dv} \,. \tag{3.5}$$

Now let $u = \tilde{v} + c_1$ where

$$c_1 = -\frac{1}{\omega_n a^{n-1}} \int_{\Sigma_a} v \, ds \tag{3.6}$$

ω_n denoting the surface area of the n-dimensional unit sphere. Then u satisfies

$$\int_{\Sigma_a} u \, ds = 0 \,, \tag{3.7}$$

and hence by (2.15)

$$\int_R u^2 dv \leqq K_4 D(u, u) = K_4 D(\tilde{v}, \tilde{v}) \,. \tag{3.8}$$

But

$$\int_R u^2 dv = \int_R \tilde{v}^2 dv + c_1^2 \int_R dv \geqq \int_R \tilde{v}^2 dv \,. \tag{3.9}$$

Thus

$$\frac{1}{K_4} \leqq \frac{D(\tilde{v}, \tilde{v})}{\int_R \tilde{v}^2 dv} = \mu_2 \tag{3.10}$$

or $1/K_4$ is a lower bound for μ_2.

A lower bound is also easily obtained for p_2, the first nonzero eigenvalue in the Steklov problem for R. Let w be the corresponding eigenfunction. Then we have that

$$p_2 = \frac{D(w, w)}{\oint_C w^2 ds} \tag{3.11}$$

and

$$\oint_{\sigma} w ds = 0 . \tag{3.12}$$

Now let $u = w + c_2$ where

$$c_2 = -\frac{1}{\omega_n a^{n-1}} \oint_{\Sigma_a} w ds . \tag{3.13}$$

Then

$$\oint_{\Sigma_a} u ds = 0 \tag{3.14}$$

and we may apply (2.12) to u. But by (3.14) we have that

$$\oint_{\sigma} u^2 ds = \oint_{\sigma} w^2 ds + c_2^2 \oint_{\sigma} ds \geqq \oint_{\sigma} w^2 ds . \tag{3.15}$$

Thus from (2.12) and (3.15)

$$K_1/K_3 \leqq \frac{D(w,w)}{\oint_{\sigma} w^2 ds} = p_2 ,$$

which gives the desired lower bound for p_2.

4. Bounds in the Neumann problem for L. We assume now that ψ is any sufficiently smooth function in $R + C$. We shall obtain bounds for the generalized Dirichlet integral, $A(\psi, \psi)$, given by

$$A(\psi, \psi) = \int_R a^{ij} \psi_{,i} \psi_{,j} dv \tag{4.1}$$

in terms of $L\psi$ in R and

$$\frac{\partial \psi}{\partial \nu} \equiv a^{ij} n_i \psi_{,j} \text{ on } C . \tag{4.2}$$

We take $u = \psi + c_3$ where

$$c_3 = -\frac{1}{\omega_n a^{n-1}} \oint_{\Sigma_a} \psi ds . \tag{4.3}$$

As before

$$\oint_{\Sigma_a} u ds = 0 , \tag{4.4}$$

Now by Green's identity

$$(4.5) \qquad A(u, u) = \oint_C u \frac{\partial \psi}{\partial \nu} ds - \int_R u L\psi dv .$$

We have used the fact that u and ψ differ only by a constant. By Schwarz's inequality we have that

$$(4.6) \qquad A(u, u) \leqq \left(\oint_C tu^2 ds\right)^{1/2} \left(\oint_C t^{-1}\left(\frac{\partial \psi}{\partial \nu}\right)^2 ds\right)^{1/2} + \left(\int_R u^2 dv\right)^{1/2} \left(\int_R (L\psi)^2 d\nu\right)^{1/2} .$$

Because of (4.4) inequalities (2.11) and (2.15) are applicable and we obtain that

$$(4.7) \quad A(\psi, \psi)^{1/2} \leqq \left(\frac{K_3}{a_0}\right)^{1/2} \left(\oint_C t^{-1}\left(\frac{\partial \psi}{\partial \nu}\right)^2 ds\right)^{1/2} + \left(\frac{K_4}{a_0}\right)^{1/2} \left(\int_R (L\psi)^2 dv\right)^{1/2}$$

since

$$D(u, u) \leqq \frac{1}{a_0} A(u, u) = \frac{1}{a_0} A(\psi, \psi) .$$

The inequalities of this section and § 2 together with a mean value inequality given in [1] give immediately interior pointwise bounds for $\psi + c_3$.

As an application of the results of this section we note here that (4.7) may be used in conjunction with the Rayleigh-Ritz procedure to yield close bounds for the Dirichlet integral in a specific Neumann problem c.f. [1].

5. Construction of the vector field. We shall show in some cases how to construct a vector field satisfying (2.5).

(a) Star shaped regions.

We consider the case where C is star shaped with respect to some point. We choose this point to be the origin. Then if we take

$$(5.1) \qquad f^i = x^i r^{-(n+1)}$$

and

$$(5.2) \qquad \alpha = r^{-(n-1)} .$$

We have that

$$(5.3) \qquad t = f^i n_i = x^i n_i r^{-(n+1)} \geqq h_m r_M^{-(n+1)} \text{ on } C$$

where $h(P)$ is the distance from the origin to the tangent plane at a point P on C and h_m is the minimum of this function. The condition of star-shapedness insures that $h_m > 0$. Since $n_i = -x^i/a$ on Σ_a it follows that

$$(5.4) \qquad -f^i n_i = a^{-n} \text{ on } \Sigma_a$$

and

$$f^i_{,i} + \frac{1}{\alpha} f^i f^i = 0 \text{ in } R_a .$$

In this case, taking $a = r_m$, we obtain

$$(5.5) \qquad \mu_2 \geqq \frac{n}{2r_M^2\left[\left(\frac{r_M}{r_m}\right)^{n-1}\frac{r_M}{h_m} + \frac{2}{n}\right]}$$

and

$$(5.6) \qquad p_2 \geqq \frac{1}{r_M}\left[\left(\frac{r_m}{r_M}\right)^{n-1}\frac{h_m}{r_M}\right].$$

A different method for obtaining a lower bound for p_2 for star shaped regions has been indicated by Payne and Weinberger [3]. For convex region Payne and Weinberger [5] also obtained the optimal lower bound $\mu_2 \geqq \pi^2 d^{-2}$ where d is the diameter of R.

(b) *Smooth boundaries.*

Let R be a region whose boundary C has continuous curvature. Call the largest principal curvature at a point P of C, $K_M(p)$. Let $\rho(p)$ be the radius of a sphere which is tangent to C at P and contained in R. In addition we require $\rho(p)$ to be less than $K_M(p)^{-1}$. Denote by $\bar{K}$ a bound for the maximum of $\rho(p)^{-1}$ for $P \in C$. We consider the family of parallel surfaces

$$(5.7) \qquad N(x) = N(x^1, \cdots, x^n) = \text{constant}$$

with C given by

$$(5.8) \qquad N(x) = 0$$

and

$$(5.9) \qquad 0 \leqq N(x) \leqq 1/\bar{K} .$$

The outward normal vector n_i is defined in this strip and satisfies

$$(5.10) \qquad n_{i,i}(x) = J(x)$$

where $J(x)$ is the average curvature of the surface given by $N(x)$ at the point $x = (x^1, \cdots, x^n)$ c.f. [7, p. 3]. We assume also that $\bar{K}$ is chosen so that

$$J(x) \leqq \bar{K}\,. \tag{5.11}$$

The above conditions and definitions involve the smoothness of C and essentially the thickness of R. We impose a further condition on the shape of R.

We assume that there is a point, which we choose to be the origin, such that

$$\frac{x^i n_i}{r} \geqq -p + \beta > -p > -1 \tag{5.12}$$

for some constants p and $\beta > 0$ in the strip $0 \leqq N(x) \leqq 1/\bar{K}$. In this case f^i may be chosen as

$$f^i = \begin{cases} [pn_i(1 - \bar{K}N(x)) + x^i/r]r^{-q},\ 0 \leqq N(x) \leqq 1/\bar{K} \\ \dfrac{x^i}{r} r^{-q},\ \text{otherwise} \end{cases} \tag{5.13}$$

with q to be determined. Condition (5.12) means that there is an open subset Ω of R which has the property that no ray from the boundary in the direction of the outward normal intersects Ω.

Let a now be chosen so that S_a does not intersect the boundary strip.

Now on C

$$f^i n_i = \left[p + \frac{x^i n_i}{r}\right] r^{-q} \geqq \beta r_M^{-q}\,. \tag{5.14}$$

For $0 \leqq N(x) \leqq 1/\bar{K}$

$$f^i_{,i} = \Big\{p[J(1 - \bar{K}N(x)) + \bar{K}]r + n - 1 - q \\ \left[p\frac{x_i n_i}{r}(1 - \bar{K}N(x)) + 1\right]\Big\} r^{-(q+1)}\,, \tag{5.15}$$

since $n_i(\partial/\partial x_i)N = -1$. Because of (5.12) and the fact that $0 \leqq 1 - \bar{K}N(x) \leqq 1$ we have that

$$f^i_{,i} \leqq \{2\bar{K}r_M + n - 1 - q(1 - p^2)\} r^{-(q+1)}\,. \tag{5.16}$$

Now if we choose

$$q = \frac{2\bar{K}r_M + n + 3}{1 - p^2} \tag{5.17}$$

it follows that

$$f^i_{,i} \leqq -4r^{-(q+1)} \leqq -4r_M^{-(q+1)} \tag{5.18}$$

in the boundary strip. In the remaining part of R_a we have, since

$q \geqq n + 3$

$$f^i_{,i} = [n - (q + 1)]r^{-(q+1)} \leqq -4r_M^{-(q+1)} . \tag{5.19}$$

Now choose $\alpha = r^{1-q}$. Then

$$f^i_{,i} + \frac{1}{\alpha} f^i f^i \leqq 0 \text{ in } R_a . \tag{5.20}$$

On Σ_a

$$-f^i n_i = a^{-q} . \tag{5.21}$$

In this case we have

$$\mu_2 \geqq \frac{n}{2r_M^2\left[\frac{1}{\beta}\left(\frac{r_M}{a}\right)^{q-1} + \frac{2}{n}\right]} \tag{5.22}$$

and

$$p_2 \geqq \beta\left(\frac{a}{r_M}\right)^{q-1} 1/r_M . \tag{5.23}$$

(c) *Boundaries with star-shaped irregularities.*

Suppose now that the boundary C consists of two parts C_1 and C_2 where C_1 is smooth and C_2 is star-shaped with respect to the chosen origin. We assume that the closure of the interior of C_2 contains C_2. For example, in two dimensions, the components of C_2 cannot be isolated points. Let $\bar{K}$ now be defined relative to C_1. Denote by R_1 the region consisting of R minus the strip adjacent to C_1. We suppose that $\bar{K}$ is large enough (the strip small enough) to make R_1 connected.

We assume that β is such that on C_2

$$\frac{x^i n_i}{r} \geqq \beta . \tag{5.24}$$

Then in place of (5.13) we have

$$f^i = \begin{cases} \left[pn_i(1 - \bar{K}N(x)) + \dfrac{x^i}{r}\right]r^{-q}, & \text{in } R - R_1 \\ \dfrac{x^i}{r} r^{-q} , & \text{in } R_1 . \end{cases} \tag{5.25}$$

Since in the identity (2.3) it is only necessary that f^i have a continuous normal component on the boundaries of subregions of R_a this definition of f^i has sufficient smoothness properties.

In this case we again have the inequalities (5.22) and (5.23).

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