ON THE GREEN'S FUNCTION OF AN N-POINT BOUNDARY VALUE PROBLEM

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1. Introduction. In a recent paper [3], D. V. V. Wend made use of the Green's functions $g_2(x, s)$, $g_3(x, s)$ for the boundary value problems

$$u'' = 0;\ u(a_1) = u(a_2) = 0, \qquad (a_1 < a_2)\ ,$$
$$u''' = 0;\ u(a_1) = u(a_2) = u(a_3) = 0, \quad (a_1 < a_2 < a_3)\ .$$

In particular, he showed that if $a_1 \geqq 0$, then

$$|g_2(x, s)| < a_2\ , \quad |g_3(x, s)| < a_3^2$$

for $a_1 \leqq x, s \leqq a_2$ or $a_1 \leqq x, s \leqq a_3$ respectively. He conjectured that if $g_n(x, s)$ is the Green's function for the boundary value problem

$$u^{(n)} = 0;\ u(a_1) = \cdots = u(a_n) = 0,\ (a_1 < a_2 < \cdots < a_n)\ , \tag{1.1}$$

then

$$|g_n(x, s)| < a_n^{n-1}, \quad a_1 \leqq x, s \leqq a_n\ ,$$

(if $a_1 \geqq 0$) and states in a footnote that this conjecture has been verified for $n < 6$. Assuming this conjecture valid he uses the inequality to obtain a lower bound for the mth positive zero of a solution of the differential equation

$$y^{(n)} + f(x)y = 0 \tag{1.2}$$

where $f(x)$ is continuous and complex-valued on $0 \leqq x < \infty$. In this proof, all zeros of the solution are *counted* as though they were *simple* zeros.

In this paper, we consider a more general boundary value problem allowing for multiple zeros of $y(x)$. Let $g_n(x, s)$ now denote the Green's function of the differential system

$$\begin{cases} y^{(n)} = 0\ , \\ y(a_i) = y'(a_i) = y''(a_i) = \cdots = y^{(k_i)}(a_i) = 0\ . \qquad 1 \leqq i \leqq r\ , \end{cases} \tag{1.3}$$

where $a_1 < a_2 < \cdots < a_r$, $0 \leqq k_i$, $k_1 + k_2 + \cdots + k_r + r = n$. In §2, we shall prove that

$$|g_n(x, s)| \leqq \frac{\prod_{i=1}^{r} |x - a_i|^{k_i+1}}{(n-1)!\,(a_r - a_1)} \leqq \left(\frac{n-1}{n}\right)^{n-1} \frac{(a_r - a_1)^{n-1}}{n!} \tag{1.4}$$

for $a_1 < x, s < a_r$. In the case $r = n$, Wend's conjecture is thus verified,

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and improved. In §3, we apply this inequality to differential equations of the form (1.2), and more generally to nonlinear differential equations

$$y^{(n)} + f(x, y, y', \cdots, y^{(n-1)}) = 0 , \tag{1.5}$$

to obtain lower bounds for the mth zero of solutions. The inequality (1.4) also leads to an extension of an oscillation criterion of Liapounoff for the case $n = 2$.

2. The Green's function. If $g_n(x, s)$ denotes the Green's function of the system (1.3), then g_n satisfies—and in fact, is defined by—the three conditions:

1°. $g_n, g'_n, \cdots, g_n^{(n-2)}$ are continuous functions of (x, s) on the square $a_1 \leqq x,\ s \leqq a_r$, while $g_n^{(n-1)}$ is a continuous function of (x, s) in each of the two triangles $a_1 \leqq x \leqq s \leqq a_r$ and $a_1 \leqq s \leqq x \leqq a_r$ with

$$g_n^{(n-1)}(s+, s) - g_n^{(n-1)}(s-, s) \equiv -1, \quad a_1 < s < a_r .$$

2°. $g_n^{(n)}(x,s) \equiv 0$ in each of the two triangles above.

3°. For each s, with $a_1 < s < a_r$, $g_n(x, s)$ satisfies the n boundary conditions of the system (1.3).

In the above statements (and throughout this paper), all derivatives are taken with respect to x. For a thorough discussion of Green's functions for much more general systems than (1.3), see Toyoda [2]. The existence of g_n depends on the fact that the system (1.3) is incompatible. We need not verify this directly since the result will follow from our method of proof which is by induction.

Although the conditions 1°–3° define g_n on the square $a_1 \leqq x,\ s \leqq a_r$, we want to extend the domain of definition of g_n to the entire plane. We assert that this can be done in such a way that

(I_n) $g_n, g'_n, \cdots, g_n^{(n-2)}$ are continuous for all (x, s), while $g_n^{(n-1)}$ is continuous in each of the half-planes $x \leqq s$ and $s \leqq x$, with $g_n^{(n-1)}(s+, s) - g_n^{(n-1)}(s-, s) \equiv -1,\ -\infty < s < \infty$.

(II_n) $g_n^{(n)}(x, s) \equiv 0$ in each of the above half-planes.

(III_n) For each s, $(-\infty < s < \infty)$, $g_n(x, s)$ satisfies the n boundary conditions of the system (1.3).

(IV_n) $g_n(x, s) \equiv 0$ if $s \leqq \min(a_1, x)$, or $s \geqq \max(a_r, x)$.

We proceed to prove these assertions by induction. Suppose they are valid for any system of the form (1.3). If a_j is any zero of a boundary value problem of this form for the equation $y^{(n+1)} = 0$, the corresponding set of boundary conditions is either of the form (1.3) with k_j replaced by $k_j + 1$ (in case a_j is *not* a simple zero for the new system). or is of the form

$$\begin{cases} y(a_i) = y'(a_i) = \cdots = y^{(k_i)}(a_i) = 0,\ 1 \leqq i \leqq r,\ i \neq j , \\ y(a_j) = 0 , \end{cases} \tag{2.1}$$

where now $k_1 + \cdots + k_{j-1} + k_{j+1} + \cdots + k_r + r = n + 1$. Let $g_{n+1}(x, s)$ be the Green's function for this new system. We assert that

$$(2.2)\quad g_{n+1}(x, s) = \frac{1}{n}\Big\{(x-s)g_n(x, s) - \frac{(a_j - s)g_n^{(k_j+1)}(a_j, s)}{(k_j+1)!}(x-a_j)^{k_j+1}\prod_{\substack{i=1\\ i\neq j}}^{r}\left(\frac{x-a_i}{a_j-a_i}\right)^{k_i+1}\Big\}$$

in the first case noted above, while

$$(2.3)\quad g_{n+1}(x, s) = \frac{1}{n}\Big\{(x-s)g_n(x, s) - (a_j - s)g_n(a_j, s)\prod_{\substack{i=1\\ i\neq j}}^{r}\left(\frac{x-a_i}{a_j-a_i}\right)^{k_i+1}\Big\}$$

in the second case. Note that (2.3) is formally included in (2.2) by setting $k_j = -1$ in (2.2). In the sequel we work with (2.2) only; (2.3) will follow by making use of this formal identity. We remark that g_n is defined by the conditions 1°–3° in $2(r-1)$ "pieces", an explicit determination of any "piece" requiring the solution of n nonhomogeneous linear equations. For this reason, the recursion relations (2.2), (2.3) may be of some interest in themselves.

For brevity, set

$$P(x, s) = \frac{(a_j - s)g_n^{(k_j+1)}(a_j, s)}{(k_j+1)!}(x-a_j)^{k_j+1}\prod_{i=1, i\neq j}^{r}\left(\frac{x-a_i}{a_j-a_i}\right)^{k_i+1}.$$

For each s, $P(x, s)$ is a polynomial of degree n in x. If $k_j < n-2$ it follows from our induction assumptions that P, as well as all its derivatives with respect to x, is a continuous function of (x, s) in the entire plane. This also holds if $k_j = n-2$ because of the factor $(a_j - s)$, provided we define $P(x, a_j) \equiv 0$. We also note that

$$(2.4)\quad \begin{aligned} &P^{(m)}(a_i, s) \equiv 0, \quad 0 \leqq m \leqq k_i,\ i \neq j\,, \\ &P^{(m)}(a_j, s) \equiv 0, \quad 0 \leqq m \leqq k_j\,, \\ &P^{(k_j+1)}(a_j, s) \equiv (a_j - s)g_n^{(k_j+1)}(a_j, s)\,. \end{aligned}$$

(In the case $k_j = -1$, the second of the identities (2.4) does not appear.) Differentiating (2.2) partially with respect to x, we obtain

$$g'_{n+1}(x, s) = \frac{1}{n}\{(x-s)g'_n(x, s) + g_n(x, s) - P'(x, s)\}\,,$$

$$g''_{n+1}(x, s) = \frac{1}{n}\{(x-s)g''_n(x, s) + 2g'_n(x, s) - P''(x, s)\}\,,$$

$$\vdots$$

$$g^{(m)}_{n+1}(x, s) = \frac{1}{n}\{(x-s)g^{(m)}_n(x, s) + mg^{(m-1)}_n(x, s) - P^{(m)}(x, s)\}\,,$$

$$1 \leqq m \leqq n+1\,.$$

By our induction assumptions, together with the preceding remarks concerning $P(x, s)$, it follows that $g_{n+1}, g'_{n+1}, \cdots, g_{n+1}^{(n-2)}$ are continuous in the entire plane. The same is true of $g_{n+1}^{(n-1)}$, because of the factor $(x-s)$. Finally, for $x \neq s$,

$$g_{n+1}^{(n)}(x, s) = \frac{1}{n}\{(x-s)g_n^{(n)}(x, s) + ng_n^{(n-1)}(x, s) - P^{(n)}(x, s)\} ,$$

so that

$$g_{n+1}^{(n)}(s+, s) - g_{n+1}^{(n)}(s-, s) = \frac{1}{n}\cdot n\{g_n^{(n-1)}(s+, s) - g_n^{(n-1)}(s-, s)\} \equiv -1 ,$$

and condition (I_{n+1}) is thus satisfied. Condition (II_{n+1}) is also satisfied since $g_n^{(n)} \equiv 0$ and $P^{(n+1)} \equiv 0$ in each of the two half-planes $x \leqq s$ and $s \leqq x$. For the boundary conditions we have

$$g_{n+1}^{(m)}(a_i, s) = \frac{1}{n}\{(a_i - s)g_n^{(m)}(a_i, s) + mg_n^{(m-1)}(a_i, s) - P^{(m)}(a_i, s)\} \equiv 0$$

for $1 \leqq i \leqq r$ and $0 \leqq m \leqq k_i$, using the first two of (2.4). Using the last of (2.4), we also see that $g_{n+1}^{(k_j+1)}(a_j, s) \equiv 0$, and (III_{n+1}) is satisfied. Finally, suppose $s \leqq \min\ (a_1, x)$ so that $g_n(x, s) \equiv 0$, and hence also $g_n^{(k_j+1)}(a_j, s) \equiv 0$ since $s \leqq a_1 \leqq a_j$. Thus, by (2.2), the first of conditions (IV_{n+1}) is satisfied, and similarly, so is the second.

For $n = 2$, we have explicitly

$$(2.5) \qquad g_2(x, s) = \begin{cases} x - s, & x \leqq s, \\ 0, & s \leqq x, \end{cases} \quad -\infty < s \leqq a_1 , \\ \qquad\qquad \begin{cases} \dfrac{(x-a_1)(a_2-s)}{a_2-a_1}, & x \leqq s, \\ \dfrac{(s-a_1)(a_2-x)}{a_2-a_1}, & s \leqq x, \end{cases} \quad a_1 \leqq s \leqq a_2 , \\ \qquad\qquad \begin{cases} 0, & x \leqq s, \\ s - x, & s \leqq x, \end{cases} \quad a_2 \leqq s < \infty ,$$

from which one easily verifies that the conditions (I_2)–(IV_2) are satisfied, thus completing the induction for all $n \geqq 2$.

Our goal now is to obtain an upper bound for $|g_n(x, s)|$. It will, however, be easier to work with the related function $G_n(x, s)$ defined by

$$(2.6) \qquad \begin{cases} g_n(x, s) = G_n(x, s) \prod_{i=1}^{r} (x - a_i)^{k_i+1} \text{ for } x \neq a_i . \\ g_n^{(k_i+1)}(a_i, s) = (k_i + 1)! \prod_{m \neq i} (a_i - a_m)^{k_m+1} G_n(a_i, s) \text{ for } x = a_i . \end{cases}$$

We note that for each fixed $s \neq a_i$, $G_n(x, s)$ is continuous for all x. If

$k_i < n-2$, $G_n(x, a_i)$ is also continuous for all x, while if $k_i = n-2$, $G_n(x, a_i)$ has a finite jump at $x = a_i$ and is otherwise continuous. (The case $k_j = -1$ is again included in (2.6), the factor $(x-a_j)^{k_j+1}$ becoming unity in this case.) We now have

$$g_{n+1}(x, s) = G_{n+1}(x, s)(x - a_j)\prod_{i=1}^{r}(x-a_i)^{k_i+1}, \quad x \neq a_i \,.$$

Using (2.2) and (2.6), we also have

$$\begin{aligned} g_{n+1}&(x, s) \\ &= \frac{1}{n}\left\{(x-s)\prod_{i=1}^{r}(x-a_i)^{k_i+1}G_n(x, s) - (a_j - s)\prod_{i=1}^{r}(x-a_i)^{k_i+1}G_n(a_j, s)\right\} \\ &= \frac{1}{n}\prod_{i=1}^{r}(x-a_i)^{k_i+1}\{(x-s)G_n(x, s) - (a_j - s)G_n(a_j, s)\}\,, \end{aligned}$$

so that

$$(2.7) \qquad (x-a_j)G_{n+1}(x, s) = \frac{1}{n}\{(x-s)G_n(x, s) - (a_j - s)G_n(a_j, s)\}\,.$$

We note that for all $n \geqq 2$, (IV_n) gives

$$(2.8) \qquad G_n(x, s) \equiv 0 \text{ for } s \leqq \min(a_1, x) \text{ and } s \geqq \max(a_r, x)\,.$$

We now prove by induction that

$$(2.9) \quad |G_n(x, s)| \leqq \begin{cases} \dfrac{1}{(n-1)!}\,\dfrac{1}{a_r - x}, & =\infty < s < a_1,\ x \leqq s\,, \\ \dfrac{1}{(n-1)!}\,\dfrac{1}{a_r - a_1}, & a_1 < s < a_r,\ -\infty < x < \infty\,, \\ \dfrac{1}{(n-1)!}\,\dfrac{1}{x - a_1}, & a_r < s < \infty,\ s \leqq x. \end{cases}$$

For $n = 2$, (2.5) gives

$$|G_2(x, s)| = \begin{cases} \dfrac{(s-x)}{(a_1 - x)(a_2 - x)}, & -\infty < s < a_1\ x \leqq s\,, \\ \dfrac{(a_2 - s)}{(a_2 - x)(a_2 - a_1)}, & x \leqq s, \quad a_1 < s < a_2\,, \\ \dfrac{(s - a_1)}{(x - a_1)(a_2 - a_1)}, & s \leqq x, \quad a_1 < s < a_2\,, \\ \dfrac{(x - s)}{(x - a_1)(x - a_2)}, & a_2 < s < \infty,\ s \leqq x\,, \end{cases}$$

from which (2.9) is immediately verified for $n = 2$.

We will *first prove* (2.9) *under the assumption that* g_{n+1} *has at* *least three distinct zeros.* Taking $j = 1$ in (2.2) and (2.7), we have for $-\infty < s < a_1$, $x \leqq s$,

$$G_{n+1}(x, s) = \frac{1}{n}\left\{\frac{x-s}{x-a_1}G_n(x, s) - \frac{a_1 - s}{x - a_1}G_n(a_1, s)\right\}$$
$$= \frac{1}{n}\frac{x-s}{x-a_1}G_n(x, s) ,$$

by (2.8); hence

$$|G_{n+1}(x, s)| = \frac{1}{n}\frac{s-x}{a_1-x}|G_n(x, s)| \leqq \frac{1}{n!}\frac{1}{a_r - x} .$$

Similarly, taking $j = r$ in (2.2) and (2.7) we obtain for $a_r < s < \infty$, $s \leqq x$,

$$|G_{n+1}(x, s)| = \frac{1}{n}\frac{x-s}{x-a_r}|G_n(x, s)| \leqq \frac{1}{n!}\frac{1}{x - a_1} .$$

Note that the above work is valid whether a_1 or a_r are simple zeros of g_{n+1} or not. Also, the first inequality is valid even when $r = 2$ provided a_1 is *not* a simple zero of g_{n+1}, and similarly for the second inequality provided $a_r = a_2$ is not a simple zero of g_{n+1}.

In order to complete the induction on the middle inequality of (2.9), we suppose first that $a_2 < s < a_r$ and $s \leqq x$. Taking $j = 2$ in (2.2) and (2.7), we obtain

$$|G_{n+1}(x, s)| \leqq \frac{1}{n}\left\{\frac{x-s}{x-a_2}|G_n(x, s)| + \frac{s-a_2}{x-a_2}|G_n(a_2, s)|\right\}$$
$$\leqq \frac{1}{n!}\frac{1}{a_r - a_1} .$$

If, however, $a_1 < s \leqq a_2$ and $s \leqq x$, we again take $j = 1$, whence

$$|G_{n+1}(x, s)| \leqq \frac{1}{n}\left\{\frac{x-s}{x-a_1}|G_n(x, s)| + \frac{s-a_1}{x-a_1}|G_n(a_1,s)|\right\} \tag{2.10}$$
$$\leqq \frac{1}{n!}\frac{1}{a_r - a_1} ,$$

if a_1 is not a simple zero of g_{n+1}, or

$$|G_{n+1}(x, s)| = \frac{1}{n}\frac{s-a_1}{x-a_1}|G_n(a_1, s)| \leqq \frac{1}{n!}\frac{1}{a_r - a_1}$$

if a_1 is a simple zero of g_{n+1}. (In this latter case, we used (2.8) and the first of inequalities (2.9).)

Similarly, if $a_1 < s < a_{r-1}$ and $x \leqq s$, we take $j = r - 1$ in (2.7) to obtain

$$|G_{n+1}(x, s)| \leqq \frac{1}{n}\left\{\frac{s-x}{a_{r-1}-x}|G_n(x, s)| + \frac{a_{r-1}-s}{a_{r-1}-x}|G_n(a_{r-1}, s)|\right\}$$
$$\leqq \frac{1}{n!}\frac{1}{a_r - a_1} .$$

If $a_{r-1} \leqq s < a_r$ and $x \leqq s$, we again take $j = r$, whence

$$|G_{n+1}(x, s)| \leqq \frac{1}{n}\left\{\frac{s-x}{a_r-x}|G_n(x, s)| + \frac{a_r-s}{a_r-x}|G_n(a_r, s)|\right\} \tag{2.11}$$
$$\leqq \frac{1}{n!}\frac{1}{a_r - a_1}$$

if a_r is not a simple zero of g_{n+1}, or

$$|G_{n+1}(x, s)| = \frac{1}{n}\frac{a_r - s}{a_r - x}|G_n(a_r, s)| \leqq \frac{1}{n!}\frac{1}{a_r - a_1}$$

if a_r is a simple zero of g_{n+1}.

We now complete the induction in the case that g_{n+1} has only two distinct zeros. For $n \geqq 2$, at least one of a_1, a_2 must be a multiple zero of g_{n+1}. Suppose a_2 is a multiple zero of g_{n+1}. Let $g_n(x, s)$ denote the Green's function for the boundary conditions

$$y(a_1) = y'(a_1) = \cdots = y^{(k_1)}(a_1) = 0 ,$$
$$y(a_2) = y'(a_2) = \cdots = y^{(k_2)}(a_2) = 0 ,$$

and $g_{n+1}(x, s)$ the Green's function for these boundary conditions with k_2 replaced by $k_2 + 1$. For any α with $a_1 < \alpha < a_2$, let $g_{n+1}(x, s; \alpha)$ denote the Green's function for the boundary conditions of g_n together with the condition $y(\alpha) = 0$. Let $G_n(x, s)$, $G_{n+1}(x, s)$ $G_{n+1}(x, s; \alpha)$ denote the related functions defined by (2.6). By (2.7)

$$G_{n+1}(x, s) = \frac{1}{n}\left\{\frac{x-s}{x-a_2}G_n(x, s) - \frac{a_2-s}{x-a_2}G_n(a_2, s)\right\}, \quad x \neq a_2 ,$$
$$G_{n+1}(x, s; \alpha) = \frac{1}{n}\left\{\frac{x-s}{x-\alpha}G_n(x, s) - \frac{\alpha-s}{x-\alpha}G_n(\alpha, s)\right\}, \quad x \neq \alpha .$$

For each $s \neq a_2$ and $x \neq a_2$, we have

$$\lim_{\alpha \to a_2} G_{n+1}(x, s; \alpha) = G_{n+1}(x, s)$$

since $G_n(\alpha, s)$ is a continuous function of α for any $s \neq a_2$. Since we have already established

$$|G_{n+1}(x, s; \alpha)| \leqq \begin{cases} \dfrac{1}{n!}\dfrac{1}{a_2 - x}, & -\infty < s < a_1,\ x \leqq s\,, \\ \dfrac{1}{n!}\dfrac{1}{a_2 - a_1}, & a_1 < s < a_2,\ -\infty < x < \infty\,, \\ \dfrac{1}{n!}\dfrac{1}{x - a_1}, & a_2 < s < \infty,\ s \leqq x\,, \end{cases}$$

the same result holds for $|G_{n+1}(x, s)|$. The proof of (2.9) is now complete.

From (2.6) and (2.9) it follows that, for each $n \geqq 2$, we have

$$(2.12) \qquad |g_n(x, s)| \leqq \frac{\prod_{i=1}^{r} |x - a_i|^{k_i+1}}{(n-1)!\,(a_r - a_1)}, \quad a_1 < s < a_r,\ -\infty < x < \infty\,.$$

We now prove that

$$(2.13) \qquad \prod_{i=1}^{r} |x - a_i|^{k_i+1} \leqq \left(\frac{n-1}{n}\right)^{n-1} \frac{(a_r - a_1)^n}{n} \quad \text{for } a_1 \leqq x \leqq a_r\,,$$

which will complete the proof of (1.4). Here, $r \geqq 2$, $0 \leqq k_i$, $a_1 < a_2 < \cdots < a_r$, and $k_1 + k_2 + \cdots + k_r + r = n$. We note that equality is attained in (2.13) for $r = 2$, $k_1 = 0$, $k_2 = n - 2$, $x = [(n-1)a_1 + a_2]/n$, or for $r = 2$, $k_1 = n - 2$, $k_2 = 0$, and $x = [(n-1)a_2 + a_1]/n$.

Instead of proving (2.13) in the form stated, we will prove

$$(2.13_1) \qquad \left|\prod_{i=1}^{n} (x - a_i)\right| \leqq \left(\frac{n-1}{n}\right)^{n-1} \frac{(a_n - a_1)^n}{n} \quad \text{for } a_1 \leqq x \leqq a_n\,,$$

where $a_1 \leqq a_2 \leqq \cdots \leqq a_n$. As a first step, we prove

$$(2.14) \qquad \left|\prod_{i=1}^{n} (x - a_i)\right| \leqq \max \begin{cases} (x - a_1)(a_n - x)^{n-1}, \\ (x - a_1)^{n-1}(a_n - x), \end{cases} \quad a_1 \leqq x \leqq a_n\,.$$

To this end, suppose $a_j < x < a_{j+1}$. If $x - a_1 \geqq a_n - x$, then

$$\left|\prod_{i=1}^{n} (x - a_i)\right| \leqq (x - a_1)^j (a_n - x)^{n-j} \leqq (x - a_1)^{n-1}(a_n - x)\,;$$

if $x - a_1 \leqq a_n - x$ (and $a_j < x < a_{j+1}$), then

$$\left|\prod_{i=1}^{n} (x - a_i)\right| \leqq (x - a_1)^j (a_n - x)^{n-j} \leqq (x - a_1)(a_n - x)^{n-1}\,,$$

proving (2.14). Now, setting $f_1(x) = (x - a_1)(a_n - x)^{n-1}$, we see that $f_1(x)$ has an absolute maximum on $a_1 < x < a_n$ when $x = [(n-1)a_1 + a_n]/n$. Similarly, $f_2(x) = (x - a_1)^{n-1}(a_n - x)$ has an absolute maximum on $a_1 < x < a_n$ when $x = [(n-1)a_n + a_1]/n$. The inequalities (2.13_1) and (2.13) now follow by computation.

There remains the question as to whether the above inequalities are best possible. The inequality (2.9), or rather the restriction

$$(2.15) \quad |G_n(x, s)| \leqq \frac{1}{(n-1)!\,(a_r - a_1)}, \quad a_1 < s < a_r, \quad -\infty < x < \infty ,$$

is best possible. Indeed, equality holds in (2.15) for precisely the cases in which equality was attained in (2.13), that is, for $r = 2$, $k_1 = 0$, $k_2 = n - 1$ and for $r = 2$, $k_1 = n - 2$, $k_2 = 0$. For the first of these cases we shall show that

$$(2.16) \qquad \lim_{x \to a_2+} |G_n(x, a_2)| = \frac{1}{(n-1)!\,(a_2 - a_1)}$$

which will prove that (2.15) is best possible. Indeed, taking $j = 2$ and $s = a_2$ in (2.7), we have

$$G_{n+1}(x, a_2) = \frac{1}{n} G_n(x, a_2), \quad x \neq a_2 ,$$

so that (2.16) holds for $(n + 1)$ if it holds for n. One easily verifies that (2.16) is valid for $n = 2$. Similarly, for the second case noted above, we have

$$\lim_{x \to a_1-} |G_n(x, a_1)| = \frac{1}{(n-1)!\,(a_2 - a_1)} .$$

It seems likely that equality is possible in (2.15) *only* in these two cases.

Nevertheless, the inequality (1.4) is *not the best possible, even in the simple case* $r = 2$, $k_1 = 0$, $k_2 = 1$, when (2.15) *is* best possible. We leave it to the reader to verify that in this case

$$|g_3(x, s)| \leqq \frac{5\sqrt{5} - 11}{4} (a_2 - a_1)^2 ,$$

with equality holding for $s = 1/2\,\{(3 - \sqrt{5})a_1 + (\sqrt{5} - 1)a_2\} = s_0$, and $x = (a_2 s_0 - a_1^2)/(a_2 + s_0 - 2a_1)$. This is an improvement over our estimate (1.4) which, for this case, is

$$|g_3(x, s)| \leqq \frac{2}{27} (a_2 - a_1)^2 .$$

3. Applications. Consider the ordinary differential equation

$$(3.1) \qquad y^{(n)} + f(x, y, y', \cdots, y^{(n-1)}) = 0 ,$$

where we assume that f is a continuous, complex-valued function for $a_1 \leqq x \leqq a_r$, and for all $y, y', \cdots, y^{(n-1)}$, and

$$|f(x, y, \cdots, y^{(n-1)})| \leqq h(x)\,|y| \tag{3.2}$$

in this domain, where $h(x)$ is a nonnegative continuous function with $h(x) \not\equiv 0$ on $a_1 \leqq x \leqq a_r$. Suppose (3.1) has a nontrivial solution $y(x)$ satisfying the boundary conditions

$$y(a_i) = y'(a_i) = \cdots = y^{(k_i)}(a_i) = 0, \quad 1 \leqq i \leqq r\,, \tag{3.3}$$

where $a_1 < a_2 < \cdots < a_r$, $0 \leqq k_i$, $k_1 + k_2 + \cdots + k_r + r = n$. Then $y(x)$ is a solution of the linear nonhomogeneous equation

$$y^{(n)} = -f[x, y(x), y'(x), \cdots, y^{(n-1)}(x)]$$

which satisfies the linear homogeneous boundary conditions (3.3). By Theorem 1 of [2] it follows that $y(x)$ satisfies the integral equation

$$y(x) = \int_{a_1}^{a_r} g_n(x, s) f[s, y(s), \cdots, y^{(n-1)}(s)]ds, \quad a_1 \leqq x \leqq a_r\,, \tag{3.4}$$

where $g_n(x, s)$ is the Green's function of the system (1.3). Taking x to be the point—or one of the points—at which $|y(x)|$ assumes its maximum value on $a_1 \leqq x \leqq a_r$, we obtain

$$1 < \int_{a_1}^{a_r} |g_n(x, s)|\, h(s)ds\,, \tag{3.5}$$

by (3.2). Hence, using the inequality (1.4),

$$1 < \left(\frac{n-1}{n}\right)^{n-1} \frac{(a_r - a_1)^{n-1}}{n!} \int_{a_1}^{a_r} h(s)ds\,. \tag{3.6}$$

The inequality (3.6) is thus a *necessary condition* for the existence of a solution of the boundary value problem (3.1), (3.3). If the system (1.3) is self-adjoint, we may improve this necessary condition. The system (1.3) is self-adjoint if $n = 2m$, $r = 2$, and the boundary conditions are

$$\begin{cases} y(a_1) = y'(a_1) = \cdots = y^{(m-1)}(a_1) = 0\,, \\ y(a_2) = y'(a_2) = \cdots = y^{(m-1)}(a_2) = 0\,. \end{cases} \tag{3.7}$$

The Green's function is now symmetric, and by (2.12) we have

$$|g_n(x, s)| = |g_n(s, x)| \leqq \frac{(s - a_1)^m (a_2 - s)^m}{(2m-1)!\,(a_2 - a_1)}, \quad a_1 < s < a_2\,.$$

On substituting this in (3.5), we obtain

$$1 < \frac{1}{(2m-1)!\,(a_2 - a_1)} \int_{a_1}^{a_2} (s - a_1)^m (a_2 - s)^m h(s)ds \tag{3.8}$$

as a necessary condition for the existence of a solution of the boundary value problem consisting of (3.1)—with $n = 2m$—and (3.7).

We may adopt a different point of view and use (3.6) or (3.8) to obtain an extension of the following oscillation criterion due originally to Liapounoff (cf. [1]): *If* $y''(x)$ *and* $y''(x)y^{-1}(x)$ *are continuous for* $a_1 \leqq x \leqq a_2$, *with* $y(a_1) = y(a_2) = 0$, *then*

$$\int_{a_1}^{a_2} |y''y^{-1}|\, dx > \frac{4}{a_2 - a_1}\,. \tag{3.9}$$

By taking $f \equiv -\, y^{(n)}(x)y^{-1}(x)y$ in (3.1), $h(x) = |y^{(n)}(x)y^{-1}(x)|$ in (3.2), (3.6) leads to the following extension: *If* $y^{(n)}(x)$ *and* $y^{(n)}(x)y^{-1}(x)$ *are continuous for* $a_1 \leqq x \leqq a_r$, *and* $y(x)$ *has* n *zeros (counting multiplicity) including* a_1 *and* a_r, *on* $a_1 \leqq x \leqq a_r$, *then*

$$\int_{a_1}^{a_r} |y^{(n)}(x)y^{-1}(x)|\, dx > \left(\frac{n}{n-1}\right)^{n-1} \frac{n!}{(a_r - a_1)^{n-1}}\,. \tag{3.10}$$

This reduces to (3.9) when $n = 2$. Similarly, using (3.8) in the self-adjoint case: *If* $y^{(2m)}(x)$ *and* $y^{(2m)}(x)y^{-1}(x)$ *are continuous for* $a_1 \leqq x \leqq a_2$, *with* $y^{(k)}(a_1) = y^{(k)}(a_2) = 0$ *for* $0 \leqq k \leqq m - 1$, *then*

$$\int_{a_1}^{a_2} (x - a_1)^m (a_2 - x)^m\, |y^{(2m)}(x)y^{-1}(x)|\, dx > (2m-1)!\,(a_2 - a_1)\,, \tag{3.11}$$

$$\int_{a_1}^{a_2} |y^{(2m)}(x)y^{-1}(x)|\, dx > \frac{(2m-1)!\, 2^{2m}}{(a_2 - a_1)^{2m-1}}\,. \tag{3.12}$$

The inequality (3.12) also reduces to (3.9) when $m = 1$, but is better than (3.10) for $n = 2m \geqq 4$.

Next we turn to the question of obtaining a lower bound for the mth zero of solutions of the linear equation

$$y^{(n)} + h(x)y = 0 \tag{3.13}$$

on an interval I: $x_0 \leqq x < \infty$. cf. [3, Theorem 5]. *We suppose that* $h(x)$ *is continuous, complex-valued, with* $h(x) \not\equiv 0$ *on* I, *and*

$$\int_{x_0}^{\infty} |h(x)|\, dx = K\,. \tag{3.14}$$

If $a_1 \leqq a_2 \leqq \cdots \leqq a_m$ *are* m *consecutive zeros of any solution of* (3.13) *on the interval* I, *then for* $m \geqq n$

$$a_m > a_1 + \frac{n}{n-1} \sqrt[n-1]{\frac{(m-n+1)\,[(n-1)!]}{K}}\,. \tag{3.15}$$

To prove this, we first note that for the equation (3.13)—but not necessarily for (3.1)—no solution can have a zero of multiplicity greater than $(n-1)$ at any point of I. Hence, if $a_i \leqq a_{i+1} \leqq \cdots \leqq a_{i+n-1}$ are n consecutive zeros of a solution of (3.13) on I then $a_i < a_{i+n-1}$, and (3.6) applies to give

$$\left(\frac{n}{n-1}\right)^{n-1} n! < (a_{i+n-1} - a_i)^{n-1} \int_{ai}^{a_{i+n-1}} |h(x)|\, dx . \tag{3.16}$$

Suppose $m = qn + s$, where $q \geqq 1, 0 \leqq s \leqq n-1$, so $a_m \geqq a_{qn}$. Taking $i = 1, n+1, \cdots, (q-1)n+1$ in (3.16) and adding these inequalities gives

$$\left(\frac{n}{n-1}\right)^{n-1} qn! < \sum_{i=1}^{q} [a_{in} - a_{(i-1)n+1}]^{n-1} \int_{a_{(i-1)n+1}}^{a_{in}} |h(x)|\, dx ,$$

whence, since $qn = m - s \geqq m - n + 1$,

$$\left(\frac{n}{n-1}\right)^{n-1} (m-n+1)[(n-1)!] < (a_m - a_1)^{n-1} \int_{a_1}^{a_m} |h(x)|\, dx . \tag{3.17}$$

The inequality (3.15) follows at once from (3.17). As in [3], (3.17) can be used to obtain a lower bound for a_m even when $K = \infty$.

In case $m > 2n-1$, these inequalities can be improved slightly, as follows. If $m = qn + s$, with $q \geqq 1, 0 \leqq s \leqq n-1$, there exists precisely one integer $r \geqq 1$ such that

$$rn - (r-1) \leqq m < (r+1)n - r .$$

Now taking $i = 1, n, 2n-1, \cdots, (r-1)n - (r-2)$ in (3.16), and proceeding as above, we obtain

$$\left(\frac{n}{n-1}\right)^{n-1} rn! < [a_{rn-(r-1)} - a_1]^{n-1} \int_{a_1}^{a_{rn-(r-1)}} |h(x)|\, dx .$$

Since $r(n-1) + n > m$ and $a_m \geqq a_{rn-(r-1)}$, we have

$$\left(\frac{n}{n-1}\right)^{n-1} \frac{m-n}{n-1} n! < (a_m - a_1)^{n-1} \int_{a_1}^{a_m} |h(x)|\, dx ;$$

this yields the estimate

$$a_m > a_1 + \frac{n}{n-1} \sqrt[n-1]{\frac{(m-n)n!}{(n-1)K}} \tag{3.18}$$

which is a slight improvement on (3.15) for $m > 2n-1$.

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