RINGS IN WHICH SEMI-PRIMARY IDEALS ARE PRIMARY

ROBERT W. GILMER

Every ring considered in this paper will be assumed to be commutative and to have a unit element. An ideal A of a ring R will be called semi-primary if its radical $\sqrt{A}$ is prime. That a semi-primary ideal need not be primary is shown by an example in [3; p. 154]. This paper is a study of rings R satisfying the following condition: (*) Every semi-primary ideal of R is primary. The ring Z of integers clearly satisfies (*). More generally, if A is a semi-primary ideal of a ring R such that $\sqrt{A}$ is a maximal ideal of R, then A is primary. [3; p. 153]. Hence, every ring having only maximal nonzero prime ideals satisfies (*).

An ideal A of a ring R is called P-primary if A is primary and $P = \sqrt{A}$. If ring R satisfies (*), then A is $\sqrt{A}$-primary if and only if $\sqrt{A}$ is prime. Some well-known properties of a ring R satisfying (*) are listed below.

Property 1. If R satisfies (*) and A is an ideal of R, then R/A satisfies (*). [3; p. 148].

Property 2. If R satisfies (*), if A and B are ideals of R such that $A \subseteqq B \subseteqq \sqrt{A}$, and if A is $\sqrt{A}$-primary then B is $\sqrt{A}$-primary. [3; p. 147].

THEOREM 1. *If ring R satisfies (*) and P, A, and Q are ideals of R such that P is prime, $P \subset A$, and Q is P-primary, then $QA = Q$.*

Proof. Since $\sqrt{QA} = P$, QA is P-primary. Thus $Q \cdot A \subseteqq QA$ and $A \not\subseteqq P$ imply that $Q \subseteqq QA \subseteqq Q$. Hence $QA = Q$ as asserted.

THEOREM 2. *If P is a nonmaximal prime ideal in a ring R satisfying (*) and if Q is P-primary, then $Q = P$.*

Proof. We let P_1 be a proper maximal ideal properly containing P. If $p_1 \in P_1$ such that $p_1 \notin P$ and if $p \in P$, then $Q \subseteqq Q + (pp_1) \subseteqq P$. By property 2, $Q + (pp_1)$ is P-primary. Since $pp_1 \in Q + (pp_1)$ and $p_1 \notin P$, $p \in Q + (pp_1)$. Then for some $q \in Q$, $r \in R$, $p(1 - rp_1) = q$. Now $1 - rp_1 \notin P_1$ since $P_1 \subset R$ so that $1 - rp_1 \notin P$. Thus $p \in Q$ and $P \subseteqq Q \subseteqq P$. Hence $P = Q$ and our proof is complete.

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COROLLARY 2.1. *If ring R satisfies* (*), *if P_1 and P_2 are prime ideals of R with $P_1 \subset P_2$, and if Q is P_2-primary, then $P_1 \subset Q$.*

Proof. Since $\sqrt{QP_1} = P_1$, QP_1 is P_1-primary. By Theorem 2, $P_1 = QP_1 \subseteqq Q$. Now Q is P_2-primary so that $P_1 \neq Q$. Hence $P_1 \subset Q$.

COROLLARY 2.2. *If ring R satisfies* (*) *and P is a nonmaximal prime ideal of R, then P is idempotent.*

Proof. The ideal P^2 has radical P and is therefore P-primary. By Theorem 2, $P^2 = P$.

THEOREM 3. *If R is a ring satisfying* (*), *if d is not a zero divisor or unit of R, and if P is a minimal prime ideal of* (d), *then P is maximal in R.*

Proof. Suppose that P is not maximal in R. Let M denote the complement of P in R. We define A to be the set of all those elements x of R such that there exists $m \in M$ such that $xm \in (d)$. Since P is prime, A is an ideal and $A \subseteqq P$. We wish to show that $P = A$. Thus if $p \in P$ and if N is the set of all elements of R of the form $p^k m$ where k is a nonnegative integer and $m \in M$, then N is a multiplicatively closed set containing M and p and hence properly containing M. Because P is a minimal prime ideal of (d), M is a maximal multiplicatively closed subset of R not meeting (d). [2; p. 106]. Therefore $N \cap (d) \neq \phi$ so that there exists an integer $k > 0$ and an element m of M such that $p^k m \in (d)$. That is, $p^k \in A$ so that $p \in \sqrt{A}$. Hence $P \subseteqq \sqrt{A} \subseteqq \sqrt{P} = P$ which implies $P = \sqrt{A}$. This means that A is P-primary. Under the assumption that P is nonmaximal, we conclude that $P = A$ by Theorem 2. Now P is also a minimal prime ideal of (d^2) so that if B is the set of elements y of R such that $ym \in (d^2)$ for some $m \in M$, we likewise have $P = B$. Since $d \in P$, there exist $m \in M$ and $r \in R$ such that $dm = rd^2$. The element d is not a zero divisor so that $m = rd \in (d) \subseteqq P$ which is a contradiction to our choice of m. Therefore P is maximal as the theorem asserts.

COROLLARY 3.1. *If ring R satisfies* (*) *and if P is a proper prime ideal of R containing a nonzero divisor d, then P is maximal in R.*

Proof. There is a minimal prime ideal P_1 of (d) contained in P. [1; p. 9]. By Theorem 3, P_1 is maximal. Hence P is also maximal.

COROLLARY 3.2. *If J is an integral domain satisfying* (*), *then nonzero proper prime ideals of J are maximal.*

COROLLARY 3.3. *If ring R satisfies* (*) *and if P is a proper prime ideal of R, then P is either maximal or minimal.*

Proof. Suppose that P is not minimal and let P_1 be a prime ideal properly contained in P. Now R/P_1 is an integral domain satisfying (*) by property 1. By Corollary 3.2, P/P_1 is maximal in R/P_1. Thus P is maximal in R. [3; p. 151].

THEOREM 4. *If ring R satisfies* (*) *and P is a finitely generated nonmaximal prime ideal of R then P is a direct summand of R. If P_1 is a prime ideal not containing P, then P and P_1 are relatively prime.*

PROOF. By Corollary 2.2, $P = P^2$. Since P is finitely generated, there exists an element $e \in P$ such that $(1 - e)P = (0)$. [3; p. 215]. Evidently $e^2 = e$, $P = (e)$ and $R = P \oplus (1 - e)$. Now $e(1 - e) \in P_1$ and $e \notin P_1$ so that $1 - e \in P_1$. Therefore $1 = e + (1 - e) \in P + P_1$ so that P and P_1 are relatively prime.

THEOREM 5. *If the Noetherian ring S satisfies* (*), *S is a finite direct sum of Noetherian primary rings and Noetherian integral domains in which nonzero proper prime ideals are maximal. Conversely if T is a finite direct sum of Noetherian primary rings and Noetherian integral domains in which nonzero proper prime ideals are maximal, then T is a Noetherian ring satisfying* (*).

Proof. Since S is Noetherian, every ideal of S is finitely generated. Let $(0) = Q_1 \cap \cdots \cap Q_s$ be an irredundant representation of (0) as an intersection of greatest primary components where $P_i = \sqrt{Q_i}$. If P_1, $P_2, \cdots, P_k$ are the nonmaximal prime ideals of S in this collection, $P_i = Q_i$ for $1 \leqq i \leqq k$ by Theorem 2. If $1 \leqq i < j \leqq s$, $P_i + P_j = S$. This follows from Theorem 4 if P_i and P_j are nonmaximal. If P_j, say, is maximal, then $P_j \not\supseteq P_i$ by Corollary 2.1, for $Q_j \not\supseteq P_i$ from the irredundance of the representation. Therefore, $P_i + P_j = S$. Thus the P_i's, and hence the Q_i's, are pairwise relatively prime. [3; p. 177]. This means that $S \cong S/P_1 \oplus \cdots \oplus S/P_k \oplus S/Q_{k+1} \oplus \cdots \oplus S/Q_s$. [3; p. 178]. Each S/P_i in this representation is a Noetherian integral domain in which nonzero prime ideals are maximal. Since Q_j for $k + 1 \leqq j \leqq s$ is P_j-primary with P_j maximal, S/Q_j is a Noetherian primary ring. [3; p. 204].

The converse follows from elementary facts concerning the ideal theory in a finite direct sum since it is apparent that each summand satisfies (*).

We give the following example of ring which is not a finite direct sum of indecomposable summands and which satisfies (*).

Let $S = \sum_{i=1}^{\infty w} Z_i$, where each Z_i is the ring of integers and $\sum_{i=1}^{\infty w}$ designates the weak direct sum. Let $R = S + Z$ be the usual extension of S to a ring with unit element. [2; p. 87]. Clearly S is a prime ideal of R, as is $I_p = S + pZ$ for every prime p of Z. In fact, each I_p is a maximal ideal of R. It is easy to show that there is no prime ideal P between S and I_p.

Next, assume that P is a prime ideal of R that does not contain all of S. Then some $e_k \notin P$, where e_k is the unity of Z_k. However, since $e_j e_k = 0$ for every $j \neq k$, evidently $Z_k \subset P$ for every $j \neq k$. By the same reasoning, $(1 - e_k)R \subseteqq P$. As before, it is easily shown that either $P = (1 - e_k)R$ or $P = (1 - e_k)R + pe_kR$ for some prime p.

Knowing precisely what the prime ideals of R are, it is just a routine matter to check that R satisfies (*).

The author is not able to give necessary and sufficient conditions which he feels are adequate that an arbitrary ring satisfy (*). The condition of Corollary 3.3, while necessary, is not sufficient to imply that a ring satisfy (*) as is shown by the following example.

If S is the ring of polynomials in two indeterminates X and Y over a field K, then every nonzero proper prime ideal of S has height 1 or 2. [4; p. 193]. Therefore if $A = (XY)$ and if $R = S/A$, R is a Noetherian ring in which every prime ideal is maximal as minimal. The nonmaximal prime ideal $(X)/A$ of R, however, is not idempotent so that R does not satisfy (*).

BIBLIOGAPHY

1. W. Krull, *Idealtheorie,* (New York, 1948).
2. Neal H. McCoy, *Rings and Ideals,* (Menasha, Wisconsin, 1948).
3. O. Zariski, and Pierre Samuel, *Commutative Algebra.* V.I. (Princeton, 1958).
4. O. Zariski, and Pierre Samuel, *Commutative Algebra.* V. II. (Princeton, 1960).