# A JORDAN-HÖLDER THEOREM

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1. The purpose of this note is to present a certain general theorem, of the Jordan-Holder type, for finite groups. This theorem, although a simple and natural extension of the classical theorem, has we believe passed unnoticed before. The technique of proof is foreign to the usual methods of finite group theory, but seems well-suited to the situation.

2. A nonempty class $\mathscr{D}$ of finite groups will be called a *genetic class* provided:

(1) If $G_1$ belongs to $\mathscr{D}$ and if $G_2$ is isomorphic to $G_1$, then $G_2$ belongs to $\mathscr{D}$.

(2) If $G$ belongs to $\mathscr{D}$, then every normal subgroup and every quotient group of $G$ also belongs to $\mathscr{D}$.

The following examples of genetic classes will be used as illustrations in the sequel:

The class $\mathscr{G}$ of all finite groups.

The class $\mathscr{E}$ of all one-element groups.

The class $\mathscr{A}$ of all finite abelian groups.

The class $\mathscr{O}$ of all groups of odd order.

The class $\mathscr{G}_n$ of all groups of order $\leqq n$.

Given any genetic class $\mathscr{D}$, we shall construct a "Grothendieck group" in the following way. Let $\Sigma$ be the (countable) set of all isomorphism classes of finite groups, and let $F$ be the free abelian group generated by $\Sigma$. If $G$ is any finite group, its isomorphism class will be denoted by $[G]$, so that elements of $F$ are finite sums

$$\Sigma\lambda_i[G_i], \quad \lambda_i \in Z ,$$

where $Z$ denotes the ring of integers. We let $N(\mathscr{D})$ be the subgroup of $F$ generated by all elements of the form

$$[G] - [H] - [G/H]$$

such that $H$ is a normal subgroup of $G$ and $G/H$ belongs to the genetic class $\mathscr{D}$. Finally we set $K(\mathscr{D}) = F/N(\mathscr{D})$ and let $k$: $F \rightarrow K(\mathscr{D})$ be the natural epimorphism. Our object is to determine the structure of the abelian group $K(\mathscr{D})$.

3. Let $\mathscr{D}$ be a genetic class and let $G$ be an arbitrary finite

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group. We say $G$ is *$\mathscr{D}$-simple* provided that $G$ has more than one element and that no proper quotient group of $G$ belongs to $\mathscr{D}$, i.e., if $H$ is normal in $G$ and if $G/H$ belongs to $\mathscr{D}$, then $H = G$ or $H = 1$. In particular, if $G$ itself belongs to $\mathscr{D}$ and is $\mathscr{D}$-simple, then it is simple because of the second axiom for genetic classes. The following illustrations are based on the examples given in § 2 above.

$G$ is $\mathscr{G}$-simple if and only if it is simple in the classical sense.

Every finite group is $\mathscr{E}$-simple.

$G$ is $\mathscr{A}$-simple if and only if either $G$ is cyclic of prime order, or else $\neq 1$ and equal to its commutator subgroup.

$G$ is $\mathscr{O}$-simple if and only if $G$ is simple and has odd order, or else has even order and no proper normal subgroups of odd index.

$G$ is $\mathscr{G}_n$-simple if and only if $G$ is simple and has order $\leqq n$ or else has order $>n$ and no proper normal subgroups of index $\leqq n$.

Having given the definition of $\mathscr{D}$-simplicity, we can now state the theorem referred to in § 1:

THEOREM. *Given any genetic class $\mathscr{D}$, $K(\mathscr{D})$ is the free abelian group freely generated by the elements $k\ [S]$, where $S$ is $\mathscr{D}$-simple.*

4. In this section we begin to prove the theorem above and show its relation to the Jordan-Holder theorem.

Let $\mathscr{D}$ be a genetic class, $G$ any finite group. If $G$ is not $\mathscr{D}$-simple and if $G \neq 1$, we can find a normal subgroup $G_1'$ of $G$ such that $1 \neq G_1' \neq G$ and $G/G_1'$ belongs to $\mathscr{D}$. Let $G_1$ be a maximal proper normal subgroup containing $G_1'$. Then $G/G_1$, being a quotient of $G/G_1'$, is in $\mathscr{D}$ and is simple (and a fortiori $\mathscr{D}$-simple). Now if $G_1$ is not $\mathscr{D}$-simple and is $\neq 1$, we repeat the process to find a normal subgroup $G_2$ of $G_1$ such that $G_1/G_2$ is in $\mathscr{D}$ and is simple. Eventually we shall get a sequence

(1) $G = G_0, G_1, G_2, \cdots, G_n$ ,

where $G_{i+1}$ is normal in $G_i$, where $G_i/G_{i+1}$ is in $\mathscr{D}$ and is simple ($i = 0, 1, \cdots, n-1$), and where either $G_n$ is $\mathscr{D}$-simple and not in $\mathscr{D}$ or else $G_n = 1$. Since $G_i/G_{i+1}$ belongs to $\mathscr{D}$, we have

$$\begin{aligned} k[G] &= k[G_0] = k[G_0/G_1] + k[G_1] = \cdots \\ &= k[G_0/G_1] + k[G_1/G_2] + \cdots + k[G_{n-1}/G_n] + k[G_n] \, . \end{aligned}$$

Clearly if $G_n = 1$, then $k[G_n] = 0$. Thus we have shown that the elements $k[S]$, $S$ $\mathscr{D}$-simple, generate the group $K(\mathscr{D})$.

We remark at this point that once we have shown the linear independence (over $Z$) of these generators, it will follow that the $\mathscr{D}$-simple groups $G_i/G_{i+1}$, $G_n$ are uniquely determined (up to isomorphism) by $G$, and are independent of the sequence 1) used to compute them. Thus in the case $\mathscr{D} = \mathscr{G}$, we get precisely the classical Jordan-Holder theorem. In the general case, the groups $G_i/G_{i+1}$ are of course among the composition factors of $G$, but the group $G_n$ (if it is not 1) is something new. It is a subnormal subgroup of $G$ which depends, up to isomorphism, only on $G$ and on $\mathscr{D}$.

Continuing our digression from the proof, let us say that two finite groups $G$ and $G'$ are *$\mathscr{D}$-equivalent* if they represent the same element of $K(\mathscr{D})$. Thus $G$ and $G'$ are $\mathscr{G}$-equivalent if and only if they have the same composition factors, while to be $\mathscr{E}$-equivalent it is clear that they must be isomorphic. In general, the smaller the genetic class $\mathscr{D}$, the sharper is the notion of $\mathscr{D}$-equivalence.

5. We return to the proof of the theorem; it remains to show that the generators $k[S]$, $S$ $\mathscr{D}$-simple, are linearly independent over $Z$. We shall show that for each $\mathscr{D}$-simple group $S$ there exists an integer-valued function $f$ defined on $\Sigma$ (and depending on $S$) such that:

(1) $f[S] = 1$;
(2) $f[T] = 0$ if $T$ is any $\mathscr{D}$-simple group not isomorphic to $S$;
(3) If $H$ is a normal subgroup of $G$ and if $G/H$ is in $\mathscr{D}$, then $f[G] = f[H] + f[G/H]$.

Because of (3) such a function induces a homomorphism $K(\mathscr{D}) \to Z$, vanishing on $k[T]$ if $T$ is as in (2), but equal to 1 on $k[S]$. The linear independence of the generators is an immediate consequence of the existence of such homomorphisms.

We construct $f$ inductively. Let $\Sigma_r$ be the set of isomorphism classes of groups of order $\leqq r$. Define $f = 0$ on $\Sigma_1$. Now suppose that $f$ has been defined on $\Sigma_r$ in such a way that (1), (2), (3) hold whenever $S, T, G$ have orders $\leqq r$. Next suppose that $G$ has order $r + 1$. If $G$ is $\mathscr{D}$-simple, then the value of $f[G]$ is forced by (1) or by (2). If $G$ is not $\mathscr{D}$-simple, then it has a normal subgroup $H$ with $G/H$ in $\mathscr{D}$ and with $H$ and $G/H$ in $\Sigma_r$. Consequently, the value of $f[G]$ must be given by (3), and it remains to show that $f[H] + f[G/H]$ is independent of the choice of $H$ as long as $H$ has order $\leqq r$ and $G/H$ is in $\mathscr{D}$.

Thus let $K$ be another such subgroup. Then $G/HK$ is in $\mathscr{D}$, since it is a quotient of $G/H$, and $H/H \cap K$ is in $\mathscr{D}$, since it is isomorphic to $HK/H$, which is normal in $G/H$. Hence using the Noether isomorphisms we get

$$\begin{aligned} &f[H] + f[G/H] = f[H] + f[G/HK] + f[HK/H] \\ &= f[H] + f[G/HK] + f[K/H \cap K] \\ &= f[H/H \cap K] + f[H \cap K] + f[G/HK] + f[K/H \cap K]\,. \end{aligned}$$

This last expression being symmetric in $H$ and $K$, it follows that $f[H]+f[G/H]=f[K]+f[G/K]$. Thus we have shown how to extend $f$ unambiguously to $\Sigma_{r+1}$ in such a way that (1), (2), (3) still hold on this enlarged domain. Therefore $f$ can be defined on all of $\Sigma$ so as to have the desired properties, and this completes the proof.

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