EXTREMAL ELEMENTS OF THE CONVEX CONE B_n OF FUNCTIONS

E. K. McLachlan

Let B_0 be the set of nonnegative real continuous on $[0, 1]$, let B_1 be the set of functions belonging to B_0 such that $\Delta_h^1 f(x) = f(x+h) - f(x) \geqq 0$, $h > 0$, for $[x, x+h] \subset [0, 1]$, and let B_n, $n > 1$ be the set of functions belonging to B_{n-1} such that $\Delta_h^n f(x) \geqq 0$ for $[x, x+nh] \subset [0, 1]$ [1]. Since the sum of two functions in B_n belongs to B_n and since a nonnegative real multiple of a B_n function is a B_n function, the set of B_n functions form a convex cone. It is the purpose of this paper to give the extremal elements [2] of this cone, to prove that they are not dense in a compact convex set that does not contain the origin but meets every ray of the cone, and to show that for the functions of the cone an integral representation in terms of extremal elements is possible. The intersection of the B_n cones is the well-known class of functions, the absolutely monotonic functions. Thus the set of these functions form a convex cone also. The extremal elements for this convex cone are given too.

In some correspondence with the author relative to the convex cone B_2, Professor F. F. Bonsall noted that the extremal elements of B_2 were the indefinite integrals of the characteristic functions that are extremal elements of the weak closure of B_1. Professor Bonsall guessed that successive integration would give the extremal elements of B_n. This proved to be a very good guess, and the author gratefully acknowledges the assistance of these comments.

In the following discussion the vertex of the convex cone is not considered as an extremal element.

1. The convex cone B_0. For $f \in B_0$, then take $f_1(x) = x f(x)$ and $f_2 = f - f_1$. Then f is the sum of functions in B_0 that are not proportional to f. Therefore, B_0 has no extremal elements.

2. The convex cone B_1. For $f = c > 0$ and $f = f_1 + f_2$ where f_1 and $f_2 \in B_1$ then $0 = \Delta_h^1 f(x) = \Delta_h^1 f_1(x) + \Delta_h^1 f_2(x)$ implies $\Delta_h^1 f_i(x) = 0$ for $i = 1, 2$ and $[x, x+h] \subset [0, 1]$. Therefore $f_i = c_i$, $c_i > 0$, $i = 1, 2$, where $c_1 + c_2 = c$. Hence f is an extremal element of B_1. Now $f = c > 0$ belongs also to B_n for $n > 1$. The set B_n is a subcone of B_1 and hence $f = c$ is again an extremal element of B_n.

If f is not constant then $f(0) = m$ and $f(1) = M$ and a non-proportional decomposition can be given by taking $f_1(x) = \min(f(x), (1/2)(M + m))$

Received October 23, 1963.

and $f_2 = f - f_1$.

3. **The convex cone B_2.** The functions of B_2 are exactly the non-negative, nondecreasing and convex functions on $[0, 1]$ [5].

Again the positive constant functions are extremal functions. If $f \in B_2$, f is not constant and $f(0) > 0$ then take $f_1 = f(0)$ and $f_2 = f - f_1$. In so doing f_1 and $f_2 \in B_2$ and f_1 and f_2 are not proportional to f. Since this same technique still can be used for B_n, $n > 2$, the only extremal elements of B_n such that $f(0) > 0$ are the positive constant functions.

If $f(x) = 0$, $x \in [0, \xi]$ and $m(x - \xi)$ for $x \in (\xi, 1]$ where $0 \leqq \xi < 1$ and $m > 0$, then for $f = f_1 + f_2$ it follows that f_1 and f_2 are zero where f is zero and f_1 and f_2 are linear where f is linear. Thus f_1 and f_2 are proportional to f and f is therefore extremal.

If $f(x) = 0$, $x \in [0, \xi_1]$, $m_1(x - \xi_1)$ for $x \in (\xi_1, \xi_2]$, $\cdots$,

$$\sum_{i=1}^{k} m_i(x - \xi_i)$$

for $x \in (\xi_k, 1]$ where $0 < \xi_1 < \xi_2 < \cdots < \xi_k < 1$ and $m_i > 0$ for $i = 1, 2, \cdots, k$, for $k > 1$ then $f \in B_2$. Let $f_1(x) = 0$, for $x \in [0, \xi_1]$, $f_1(x) = m_1(x - \xi_1)$ for $(\xi_1, 1]$ and $f_2 = f - f_1$. Then f_1 and $f_2 \in B_2$ and both are not proportional to f.

Finally, if f is not any of the above functions, but f belongs to B_2, let $\xi_1 = \inf \{x: f(x) > 0\}$. Then $0 \leqq \xi_1 < 1$. On $[\xi_1, 1]$, f is convex, $f(\xi_1) = 0$ and $f(1)$ is finite. Furthermore, the right-hand derivative at ξ_1, $f'_+(\xi_1)$ is finite and in $[\xi_1, 1] f'_-$, the left-hand derivative, must take on more than a finite number of values since f is not polygonal on $[\xi_1, 1]$. Thus there exist ξ_2, $\xi_1 < \xi_2 \leqq 1$ such that on $[\xi_1, \xi_2] f'_+$ is not piecewise linear on three or more non-overlapping segments whose union is $[\xi_1, \xi_2]$ and $f'_-(\xi_2)$ is finite. By Lemma 4 of a paper by the author [4], there exist convex, nonnegative and nondecreasing functions f_1 and f_2 different from f on $[\xi_1, \xi_2]$ such that f_1 and f_2 have the same values and the same derivatives at the end-points as f and $f = \alpha f_1 + (1 - \alpha) f_2$ for some α, $0 < \alpha < 1$. Thus define f_1 and f_2 equal to f on the complement of $[\xi_1, \xi_2]$ relative to $[0, 1]$ and then αf_1 and $(1 - \alpha) f_2$ belong to B_2 and both are not proportional to f.

Thus the extremal elements of B_2 are positive constant functions and those f such that $f(x) = 0$, $x \in [0, \xi]$ and $f(x) = m(x - \xi)$ for $x \in [\xi, 1]$ where $0 \leqq \xi < 1$ and $m > 0$. Designate this latter function by $f(\xi, 1;)$ for $m = 1$.

4. **The convex cone B_n, $n > 2$.** The function f, such that $f(x) = 0$, $x \in [0, \xi]$, $f(x) = m(x - \xi)^{n-1}$, $x \in [\xi, 1]$, $0 \leqq \xi < 1$ and $m > 0$, that is $m f(\xi, n - 1;)$ belongs to B_n and is an extremal element of B_n.

Already $m f(\xi, 1;)$ belongs to B_2. Now by induction it shall be shown that $m f(\xi, n-1;) \in B_n$ for $n > 2$. In fact, it is true in general that if $f \in B_{n-1}$ and if

$$F(x) = \int_0^x f(t)\,dt ,$$

then $F \in B_n$. For if $\Delta_h^k f(x) \geqq 0$ for $k = 0, \cdots, n-1$ then

$$\Delta_h^k F(x) = \Delta_h^{k-1} \int_x^{x+h} f(t)\,dt = \Delta_h^{k-1} f(\xi) > 0$$

where $x < \xi < x - h$ and $k = 0, \cdots, n$. Thus since

$$m f(\xi, n-1; x) = \int_0^x (n-1) m f(\xi, n-2; t)\,dt$$

and since by the induction hypothesis $(n-2) m f(\xi, n-2;) \in B_{n-1}$, it follows that $m f(\xi, n-1;) \in B_n$.

Similarly, by induction it shall be shown that $f = m f(\xi, -1;)$ is an extremal element of B_n It has already been shown that $m f(\xi, 1;)$ is an extremal element of B_{n-1} for any $m > 0$ and for $0 \leqq \xi < 1$. Now let $f = m f(\xi, n-1;) = f_1 + f_2$ where f_1 and f_2 belong to B_n. For $n > 2$, functions in B_n have derivatives, f_1' and f_2' on $[0, 1)$ (See [5] Chapter IV) and the functions f_1' and f_2' belong to B_{n-1} on $[0, \delta]$ for any δ, $0 < \delta < 1$. Take $\delta < 1$ such that $\xi < \delta$, then by the induction hypothesis it follows that f_1' and f_2' are proportional to $f' = (n-1)m$ $f(\xi, n-2;)$ on $[0, \delta]$. Hence $f_i(x) = \lambda_i f(x) + c_i$, $x \in [0, \delta]$, $0 \leqq \lambda_i$, where c_i is a constant for $i = 1, 2$. Since $f_1(0) = f_2(0) = (n-1)m$ $f(\xi, n-2; 0) = 0$ it follows that $c_i = 0$, $i = 1, 2$ and hence f_1 and f_2 are proportional to f on $[0, \delta]$ for any δ, $0 < \delta < 1$. However, since f, f_1 and f_2 are continuous on $[0, 1]$, it follows then that f_1 and f_2 are proportional to f on $[0, 1]$. Therefore, $m f(\xi, n-1;)$ is an extremal element of B_n.

Notice that like the positive constant functions these functions $m f(\xi, n-1;)$ for $\xi = 0$, that is the functions $m f(0, n-1;)$ belong to B_n for all n since its derivatives of all orders exist and are nonnegative on $[0, 1]$. However, if $\xi > 0$, let s and k be integers such that $s > k$ and let x and h be such that $x + (s-2)h = \xi$, $0 \leqq x < x + sh \leqq 1$. Then

$$\Delta_h^s m f(\xi, k; x) = m[(2h)^k - s(h)^k] = mh^k(2^k - s) .$$

Hence, if $s > 2^k$, then the expression on the right is negative and thus $m f(\xi, k;) \notin B_s$. This means that whereas $m f(\xi, n-1;) \in B_n$ it does not belong to B_j for $j > 2^{n-1}$.

It remains only to show that the functions of B_n other than the positive constant functions of the form $m f(\xi, k;)$, $0 \leqq \xi < 1, m > 0$, $k = 1, 2, \cdots, n-1$ that belong to B_n are not extremal elements of B_n.

It is known that f' exists and is a continuous function on $[0, 1)$. If f' can be extended to be a continuous function on $[0, 1]$, that is, if $\lim f'(x)$ as $x \to 1^-$ exists and is finite, then $f' \in B_{n-1}$. By assuming the induction hypothesis on n, there exist functions g_1 and g_2 belonging to B_{n-1} such that $f' = g_1 + g_2$ and g_1 and g_2 are not proportional to f'. Let $f_i(x) = \int_0^x g_i(t)\,dt$, $i = 1, 2$. Thus f_1 and f_2 belong to B_n and they are not proportional to f. For if $f_1 = \lambda_1 f$, $\lambda_1 \geqq 0$, then $f_1' = \lambda_1 f' = g_1$. This clearly violates what is known about g_1. Hence such a function f is not an extremal element of B_n.

Finally, suppose that $f \in B_n$ and $\lim f'(x) = +\infty$ as $x \to 1^-$. Then the following must be true: $f', f'', \cdots, f^{(n-2)}$ and $f_+^{(n-1)}$, the right-hand derivative of $f^{(n-2)}$ are defined on $[0, 1)$; each of them approaches $+\infty$ as x approaches one from the left; and $\Delta_h^k f^{(j)}(x) \geqq 0$ for $0 \leqq x < 1$, $j = 1, 2, \cdots, n-1$, (with the special understanding for $j = n-1$), $k = 0, 1, 2, \cdots, n-j$. Denote by $B_{n-j}[0, 1)$ the set of real functions ϕ of $[0, 1)$ $\Delta_h^k \phi(x) \geqq 0$, $0 \leqq x < 1$, $k = 0, 1, \cdots, n-j$ for $j = 1, 2, \cdots, n-1$ such that $\phi(x) \to +\infty$ as $x \to 1^-$. The functions $B_{n-j}[0, 1)$ form a convex cone and $f^{(j)} \in B_{n-j}[0, 1)$ for $j = 1, \cdots, n-1$. By an argument similar to the one given earlier, the indefinite integral of a function F in $B_m[0, 1)$ belongs to $B_{m+1}[0, 1)$ if $\int_0^x F(t)\,dt \to +\infty$ as $x \to 1^-$. Also if g, g_1 and $g_2 \in B_m[0, 1)$, $g = g_1 + g_2$, and g_1 and g_2 are not proportional to g, then the indefinite integrals of g_1 and g_2 are not proportional to g. Not that if $g = g_1 + g_2$ as above and if $\int_0^{1-} S(t)\,dt$ is finite, then the same will be true of $\int_0^{1-} g_i(t)\,dt$ for $i = 1, 2$. If the $\lim g(t) = +\infty$ as $t \to 1^-$ and $\int_0^{1-} g_i(t)\,dt = +\infty$ then the same will be true of $\int_0^{1-} g_i(t)\,dt$ for $i = 1, 2$ if there exists constants $\gamma_i > 0$, $i = 1, 2$ such that $g_i(t) \geqq \gamma_i\, g(t)$ for some δ, $0 < \delta < 1$. For the case when $\int_0^{1-} g_i(t)\,dt$ is finite then f_i where $f_i(x) = \int_0^x g_i(t)\,dt$, $i = 1, 2$ can be extended into a function that is continuous on $[0, 1]$. Hence f_1 and f_2 will belong to B_{m+1}.

Thus the object is to find two functions g_1 and g_2 that belong to $B_1[0, 1)$, such that $f_+^{(n-1)} = g_1 + g_2$, g_1 and g_2 are not proportional to $f_+^{(n-1)}$, and such that $g_i(t) \geqq \lambda_i f_+^{(n-1)}(t)$, $\delta \leqq t < 1$, $\delta > 0$. Then f_1 given by

$$f_i(x) = \int_0^x \int_0^{t_{n-2}} \cdots \int_0^{t_2} \int_0^{t_1} g_i(t)\,dt\,dt_1 \cdots dt_{n-2}\,,$$

$i = 1, 2$ belong to B_n and give a nonproportional decomposition of f. The lemma below shows how the functions g_1 and g_2 with the desired properties can be constructed.

LEMMA. *Given f on $[0, 1)$ such that f is right continuous, nonnegative, nondecreasing and $f(x) \to +\infty$ as $x \to 1^{\cdot}$. There exist two functions f_1 and f_2 on $[0, 1)$ that are right continuous, nonnegative and nondecreasing, $f = f_1 + f_2$, f_1 and f_2 are not proportional to f, and $f_i(x) \geqq \gamma_i f(x)$ on $[\delta, 1)$ for some $0 < \delta < 1$ and $\gamma_i > 0$, $i = 1, 2$.*

Proof. All the discontinuities of f must be jump discontinuities. If the point $x = 1$ is an accumulation point of the discontinuities of f, then there exist c_1, c_2 and c_3, $0 < c_1 < c_2 < c_3 < 1$ such that f has a jump of θ_i at c_i, $\theta_i > 0$, $i = 1, 2, 3$. Take $\theta = (1/2) \min (\theta_1 \theta_2 \theta_3)$. Let f_1 be such that $f_1(x) = (1/2)(f(x) - \theta)$, $c_1 \leqq x < c_2$, $f_1(x) = (1/2)(f(x) + \theta)$, $c_2 \leqq x < c_3$ and $f_1(x) = (1/2) f(x)$ otherwise. Take $f_2 = f - f_1$. Then f_1 and f_2 have the required properties.

If the point $x = 1$ is not an accumulation point of the discontinuities then there exists δ, $0 < \delta < 1$ such that f is continuous on $[\delta, 1)$. Let ξ be a point such that $f(\xi) = f(\delta) + 1$, then $\delta \leqq \xi < 1$. Take f_1 such that $f_1(x) = (1/2)f(x)$, $0 \leqq x < \xi$ and $f_1(x) = (1/3)(f(x) - f(\delta) - 1) + (1/2)(f(\delta) + 1)$, $\xi \leqq x < 1$. Let $f_2 = f - f_1$. Then again f_1 and f_2 have the required properties.

5. **Absolutely monotonic functions.** The continuous functions f on $[0, 1]$ such that $f^{(k)}(x) \geqq 0$ for $0 < x < 1$, $k = 0, 1, 2, \cdots$ were called *absolutely monotonic* functions by Bernstein. These functions clearly form a convex cone of functions on $[0, 1]$. Since the functions f belonging to B_n, $n > 2$, have $f^{(k)}(x) \geqq 0$, $k = 0, 1, \cdots, n - 2$, it follows that $\bigcap_{n=0}^{\infty} B_n$ is contained in the set of absolutely monotonic functions. Since the continuous functions f on $[0, 1]$ such that $f^{(k)}(x) \geqq 0$, $k \leqq n$ on $(0, 1)$ have $\Delta_h^k f(x) \geqq 0$ for $k \leqq n$, then $\bigcap_{n=0}^{\infty} B_n$ is the set of absolutely monotonic functions. Denote this set by B_∞

From the earlier remarks it is clear that c_0, $c_1 x$, $c_2 x^2$, $\cdots$ belong to B_∞ for $c_i > 0$, $i = 0, 1, 2, \cdots$ and they are indeed extremal elements of B_∞. Since any $f \in B_\infty$ is absolutely monotonic on $[0, 1)$ it follows that

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(0)(x^n/n!) , \quad 0 \leqq x < 1 .$$

Consequently, if as many as two terms are nonzero in the series expansion, then take f_1 equal to one of the two nonzero terms and $f_2 = f - f_1$. Then clearly f_1 and f_2 belong to B_∞ and f has a nonproportional decomposition. Hence the only extremal elements of B_∞ are the functions $c_i x^i$, $c_i > 0$, $i = 0, 1, 2, \cdots$.

The following theorem summarizes all of the results up to this point.

THEOREM. *The convex cone B_0 has no extremal elements. The* *functions* $f = c > 0$, *where* c *is a constant, are extremal elements of* B_n, $n = 1, 2, 3, \cdots$. *The function* $m f(\xi, n-1; x) = 0$ *for* $0 \leqq x < \xi$ *and* $m(x-\xi)^{n-1}$ *for* $\xi \leqq x \leqq 1$, $m > 0$, $0 \leqq \xi < 1$ *are extremal elements of* B_n, $n = 2, 3, \cdots$. *The only other extremal elements of* B_n, $n = 2, 3, \cdots$ *are those functions* $m f(\xi, k;)$, $k = 1, 2, \cdots, n-2$ *that belong to* B_n. *The extremal elements of the convex cone* B_∞, *the absolutely monotonic functions, are the functions of the form* $c_i x^i$, $c_i > 0$, $i = 0, 1, 2, \cdots$.

6. Integral representations. The set of functions $B_n - B_n$, $n \geqq 1$, form a linear space containing the convex cone B_n. Using the topology of simple convergence $B_n - B_n$ becomes a locally convex space. Let C_n be the set of functions f of B_n such that $f(1) = 1$. Clearly, C_n meets every ray of C_n once and only once and does not meet the origin in $B_n - B_n$, that is the zero function. Furthermore, C_n is convex. Each function f of C_n is such that $0 \leqq f(x) \leqq 1$ for all $0 \leqq x \leqq 1$ since f is nonnegative and nondecreasing. It follows by use of the Tychonoff theorem that C_n is contained in a compact set in $B_n - B_n$, namely $\{f : f \in B_n - B_n, 0 \leqq f(x) \leqq 1, 0 \leqq x \leqq 1\}$. Thus C_n is compact, if it can be shown that C_n is closed. This will be done by showing the complement of C_n is open.

If $g \in B_n \backslash C_n$ then $g(1) \neq 1$. The set

$$V(1; \varepsilon) + g = \{f : f \in B_n - B_n, |f(1) - g(1)| < \varepsilon\}$$

where $\varepsilon = (1/2)|1 - g(1)|$ is an open set about g that fails to meet C_n. If $g \notin B_n$ then there exists x_0, k and h such that $\Delta_h^k g(x_0) = \delta < 0$. Now

$$\Delta_h^k g(x_0) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} g(x_0 + (k-j)h) .$$

Consider

$$\begin{aligned} V &= V(x_0, x_0 + h, \cdots, x_0 + kh; \varepsilon) + g \\ &= \{f : f \in B_n - B_n, |f(x_0 + jh) - g(x_0 + jh)| < \varepsilon, j = 0, 1, \cdots, k\} . \end{aligned}$$

where $\varepsilon = 2^{-(k+1)}(-\delta)$. Then V does not meet C_n since for if $f \in V$

$$\begin{aligned} \Delta_h^k f(x_0) &= \Delta_h^k(f(x_0) - g(x_0)) + \Delta_h^k g(x_0) \\ &< |\Delta_h^k(f(x_0) - g(x_0))| + \Delta_h^k g(x_0) \\ &< \sum_{j=0}^{k} \binom{k}{j} |f(x_0 + (k-j)h) - g(x_0 + (k-j)h)| + \delta \\ &< \varepsilon \sum_{j=0}^{k} \binom{k}{j} + \delta \\ &= \varepsilon 2^k + \delta \\ &= (1/2)\delta < 0 . \end{aligned}$$

Hence $f \notin B_n$.

Thus by Theorem 39.4 of Choquet [3], it follows that for any function f_0 in C_n there exists a nonnegative measure μ_0 on the closure of the extreme points of C_n such that $f_0(x)\int d\mu_0 = \int f(x)\,d\mu_0$. Since C_n meets every ray of the cone B_n and does not contain the origin, it follows that each function of B_n is a scalar multiple of such a representation.

If the set of extremal elements of C_n are dense in C_n, then the above result would be of no interest, but this is not the case. Consider $g_0(x) = (1/2) + 2^{n-2}f(1/2, n-1; x)$. Then g_0 belongs to B_n since it is the sum of two functions in B_n. Notice further that $g_0(1) = 1$ and hence $g_0 \in C_n$. The neighborhood of g_0,

$$\begin{aligned} V_0 &= V(0, 1; 1/8) + g_0 \\ &= \{f : f \in B_n - B_n, \, |f(i) - q_0(i)| < (1/8), \, i = 0, 1\}\,, \end{aligned}$$

does not meet any extreme point of C_n. Any positive constant function of C_n is $f(x) = 1$ for all x and hence $f(0) > 5/8$ at $x = 0$. Any function of the form $m\,f(\xi, k;)$ that belongs to zero at $x = 0$ and hence does not belong to V_0.

7. **Remarks.** Choquet [3] discusses convex cones of functions related to the cones discussed here. The main difference is that the differences, $\Delta_h^k f(x)$, alternate in sign as k takes on successive integral values in the cones that Choquet considered.

REFERENCES

1. F. F. Bonsall, *Semi-alqebras of continuous functions*, Proceedings of the International Symposium on Linear Spaces, (1961), 101–114.

2. N. Bourbaki, *Espaces vectoriels topologiques*, Act. Sci. Ind. no. 1189, Paris, 1953.

3. G. Choquet, *Theory of capacities*, Annales de l'Institut Fourier, **5** (1953 and 1954), 131–296.

4. E. K. McLachlan, *Extremal elements of the convex cone of semi-norms*, Pacifie J. Math., **13** (1963), 1335–1342.

5. D. V. Widder, *The Laplace transform*, Princeton Mathematics Series, **6** (1946).

OKLAHOMA STATE UNIVERSITY