MAXIMAL ALGEBRAS AND A THEOREM OF RADÓ

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1. A theorem of Radó [1, 4, 6, 9] asserts that a function f, continuous on the closed disc $D = \{z : |z| \leqq 1\}$, and analytic at all points of the interior of D where f doesn't vanish, is analytic on all the interior. One can of course take this as a statement about the uniformly closed algebra A_1—the disc algebra—formed by those f in $C(D)$ analytic on the interior of D, and in fact it is easy to restate the result in a form which makes sense for any function algebra. For let $T^1 = \{z : |z| = 1\}$, and call f locally approximable at z if f can be uniformly approximated by elements of A_1 on some neighborhood of z. Then it is clear that the result asserts that any f in $C(D)$, locally approximable at all z in $D\backslash(T^1 \cup f^{-1}(0))$, is in A_1.

Now since D can be viewed as the maximal ideal space of A_1, and T^1 as the Šilov boundary, we can formulate such an assertion for any uniformly closed algebra of functions—and, needless to say, it will fail in general.[1] But under appropriate maximality conditions the result does hold; in particular we shall show it holds for any uniformly closed function algebra A maximal on its Šilov boundary, provided the boundary is not all the maximal ideal space of A, and for intersections of such algebras.

This result holds as a consequence of two facts: Rossi's local maximum modulus principle [11], and a quite elementary lemma (2.1) which allows one to eliminate certain points as candidates for elements of the Šilov boundary of an algebra. In the original setting, where the elementary local maximum modulus principle for analytic functions can be used, our proof requires (beyond this lemma) only the fact that the disc algebra A_1 is a maximal subalgebra of $C(T^1)$ [7, 12]; no doubt it is no simpler than the proof given in [6]. However our arguments do establish some nontrivial variants of the result in the general setting (3.5, 3.6, 4.9), and, in particular, for functions analytic on polycylinders in C^n; deflated to the disc algebra almost all of these follow rather easily from Radó's result due to the topological simplicity of the one (complex) dimensional situation and the fact that there Radó's result can be applied locally.

One consequence of Radó's theorem is the fact that A_1 is *integrally closed* in $C(D)$, i.e., any f in $C(D)$ satisfying a polynomial equation

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[1] For example, for the subalgebra of A_1 of those f with $f'(0) = 0$; $f(z) \equiv z$ is locally approximable off $f^{-1}(0)$, but not in the subalgebra.

$$f^n + a_{n-1}f^{n-1} + \cdots + a_0 = 0$$

with coefficients in A_1 must lie in A_1. This extends to our maximal algebras (§ 5), and, as a consequence, for every uniformly closed subalgebra A of $C(\mathscr{M})$, where $\mathscr{M}$, the maximal ideal space of A, properly contains the Šilov boundary of A, we have a larger subalgebra a $C(\mathscr{M})$ with the same Šilov boundary which is integrally closed in $C(\mathscr{M})$.

Another consequence of one of our variants of Radó's theorem is the analogue, for intersections of maximal algebras, of the elementary removably singularity theorem for analytic functions (§ 6); from this one also has an analogue of the elementary facts on the behavior of analytic functions near isolated singularities, valid for functions locally approximable on $\mathscr{M}$ less a point.

Finally, the main portion of our argument can be applied to yield an abstract version of Schwarz's lemma: for any algebra A, if $f, g \in A$ and f/g is bounded on $\mathscr{M}\backslash g^{-1}(0)$ then it is bounded by its supremum over the Šilov boundary. Various consequences of this are given in § 4.

The author is indebted to Kenneth Hoffman and John Wermer for many helpful comments; in particular it was Wermer who observed that the author's original version of 2.2 could be used to prove Radó's theorem, and suggested its use to obtain integral closure.

We shall use $\boldsymbol{C}$ for the complex numbers, $\boldsymbol{R}$ for the reals, and F^0 for the interior of a set F.

2. In all that follows $C(X)$ will denote the Banach space of all bounded complex continuous functions on the space X, and A will denote a closed separating subalgebra of some $C(X)$, containing the constants. In general we shall view any such algebra A as a closed subalgebra of $C(\mathscr{M})$, where $\mathscr{M}$ is the maximal ideal space of A; when there is any necessity we may write $\mathscr{M}_A$ for $\mathscr{M}$. A *closed* subset X of $\mathscr{M}$ is a *boundary* for A if every f in A assumes its maximum modulus over $\mathscr{M}$ on X; any boundary is just a superset of the Šilov boundary ∂ of A.

Let X be a boundary for A, and let F be a closed *non-void* subset of X. An f in A will be said to *peak within* X *on* F if $f(F) = 1$ while $|f| < 1$ on $X\backslash F$. As is easily seen a point m of X lies in the Šilov boundary ∂ of A if and only if for every open neighborhood V of m in X there is an f in A which peaks within X on a nonvoid subset of V. The following lemma is fundamental to our considerations.

LEMMA 2.1. *Let $X \subset \mathscr{M}$ be a boundary for A, and V a (relatively) open subset of X. Suppose $g \in A$ peaks within X on a nonvoid subset of V, and let $\alpha = \sup |g(X\backslash V)|$ (which is necessarily <1).*

Then any $f \in A$ vanishing on V also vanishes on the nonvoid open subset $U = \{m \in \mathscr{M} : |g(m)| > \alpha\}$ of $\mathscr{M}$.

Proof. Suppose $|g(m)| > \alpha$ and $f(m) \neq 0$. Let μ be a (normalized, nonnegative, regular Borel) measure on X representing m, so $h(m) = \int h d\mu$ for all h in A [7, p. 181]. Let ν be the complex measure $(1/f(m))f\mu$ (the ordinary product of function and measure), which again represents m since

$$\int h d\nu = \frac{1}{f(m)} \int h f d\mu = \frac{1}{f(m)} \cdot h(m) f(m) , \qquad h \in A.$$

Now set $h = (1/g(m)) \cdot g \in A$; since $|g(m)| > \alpha = \sup |g(X \backslash V)|$ we have $h(m) = 1 > \sup |h(X \backslash V)|$. Replacing h by a sufficiently high power of itself we can suppose $\sup |h(X \backslash V)| < 1/(2\|\nu\|)$, where $\|\nu\|$ is the total variation norm of ν, while $h(m)$ is still 1.

Since $f(V) = 0$ the measure $\nu = (1/f(m))f\mu$ is carried by $X \backslash V$, so

$$1 = h(m) = \int h d\nu = \int_{X \backslash V} h d\nu < \frac{1}{2\|\nu\|} \cdot \|\nu\| = \frac{1}{2} ,$$

the desired contradiction.

Our main applications of 2.1 will be made via the following corollary, and usually with the set $\mathscr{F}$ a singleton.

COROLLARY 2.2. *Let $X \subset Y \subset \mathscr{M}$ be boundaries for A, V a relatively open subset of X, and $\mathscr{F}$ any subset of A. If V is contained in the topological boundary in Y of $\bigcap_{f \in \mathscr{F}} f^{-1}(0)$, then $V \cap \partial = \phi$.*

Suppose $V \cap \partial \neq \phi$, so some g in A peaks within X on a nonvoid subset F of V. Then each f in $\mathscr{F}$ must vanish on the open subset U of $\mathscr{M}$ given in 2.1, and $F \subset U$, so F lies in the interior of $\bigcap_{f \in \mathscr{F}} f^{-1}(0)$ in Y, not in its boundary.

For a boundary X for A, A is called *analytic on X* if every f in A vanishing on a nonvoid relatively open subset of X vanishes identically (on X, hence on ∂, hence on all of $\mathscr{M}$). In [5] an example was given of an algebra A analytic on ∂ but not on $\mathscr{M}$; the original purpose of 2.1 was to prove

COROLLARY 2.3. *If A is analytic on $\mathscr{M}$, A is analytic on ∂.*

Indeed if $f \in A$ vanishes on a relatively open subset V of ∂ then some g in A must peak within ∂ on a nonvoid subset of V, so that f vanishes on a nonvoid open subset of $\mathscr{M}$ by 2.1. Thus we have the more general assertion of

COROLLARY 2.4. *For any algebra A, an f in A vanishing on a nonvoid relatively open subset of ∂ vanishes on a nonvoid open subset of $\mathscr{M}$.*

In particular if $f^{-1}(0)$ is nowhere dense in $\mathscr{M}$ then $f^{-1}(0) \cap \partial$ is nowhere dense in ∂. Both 2.3 and 2.4 remain valid if $\mathscr{M}$ is replaced by any boundary for A, but neither need hold if ∂ is enlarged to an arbitrary boundary; for example both fail for the disc algebra A_1, with ∂ replaced by $X = T^1 \cup \{0\}$, and $\{0\}$ the relatively open subset of X.

As we shall see later (4.2), 2.1 yields some further information on zero sets of elements of algebras with $\mathscr{M} \neq \partial$.

Some simple variants of 2.1 are of interest, but will not be needed in what follows. For example

COROLLARY 2.5. *Let U and V be as in 2.1. Then any bounded sequence $\{f_n\}$ in A converging pointwise to zero on V converges pointwise to zero on U.*

For $\theta > 1$ let $U_\theta = \{m \in \mathscr{M} : |g(m)| \geqq \theta\alpha\}$. Then any bounded sequence $\{f_n\}$ in A converging uniformly to zero on V converges uniformly to zero on U_θ.

Proof. For the first part, suppose $f_n(m) \not\to 0$ for some m in U; replacing $\{f_n\}$ by a subsequence we can assume $f_n(m) \to c \neq 0$. Let μ again represent m, and let f be any weak* cluster point of $\{f_n\}$ in $L_\infty(\mu)$. Since a subnet of $\{f_n\}$ converges weak* to f we have

$$h(m)\int f d\mu = \int h f d\mu \, , \qquad h \in A \, ,$$

while $c = \lim f_n(m) = \lim \int f_n d\mu = \int f d\mu$. So for $h = g/g(m)$ we have, for all n,

$$c = h(m)^n \int f d\mu = \int h^n f d\mu \, .$$

But by dominated convergence, for any f' in $L_1(\mu)$ vanishing off V we have $\int f' f d\mu = \lim \int f_n f' d\mu = 0$, and thus $f = 0$ a.e. μ on V. So, since $\sup |h(X \setminus V)| < 1$,

$$c = \lim \int_{X \setminus V} h^n f d\mu = 0 \, ,$$

our contradiction.

The second assertion is entirely elementary. With $m \in U_\theta$, and μ and h as before, we have $|h| \leqq 1/|g(m)| \leqq 1/\theta\alpha$ on X, and $\leqq \alpha/\theta\alpha =$

$1/\theta$ on $X\backslash V$. Thus

$$|f_n(m)| = \left|\int h^k f_n d\mu\right| \leqq \left|\int_V\right| + \left|\int_{X\backslash V}\right|$$
$$\leqq \left(\frac{1}{\theta\alpha}\right)^k \sup|f_n(V)| + \frac{1}{\theta^k}\|f_n\| ;$$

since $\theta > 1$ the last term will be $<\varepsilon/2$ for some large k, and choosing $n \geqq N$ will then force the sum below ε.

We might note that there are trivial variants of this second assertion which allow $\{f_n\}$ to be unbounded, provided the sequence $\{\sup|f_n(V)|\}$ approaches zero rapidly enough. For example, if

$$\sup|f_n(V)| = o(\|f_n\|^{-\log\alpha\theta/\log\theta}) ,$$

as is easily verified.

3. Let X be a boundary for A. We shall call a function f, defined on part of X, *locally approximable* (*within*[2] X *by* A) *at* $x \in X$ if, for some neighborhood U *of* x *in* X, f is defined on U and is uniformly approximable there by elements of A; alternatively[3] $f|U \in (A|U)^-$, the closure in $C(U)$ of $A|U$. We shall say f is *locally approximable on* $Y \subset X$ if f is locally approximable at each point of Y; note that by definition the set of points of X at which a given function is locally approximable is open in X.

We have $\partial_A \subset X \subset \mathscr{M}_A$. Call A *relatively maximal in* $C(X)$ if $A|X \neq C(X)$ and no closed proper subalgebra B of $C(X)$ containing $A|X$ has $\partial_B = \partial_A$. (Since $\partial_A \subset \partial_B$ necessarily, we are requiring properly larger subalgebras of $C(X)$ to have properly larger Šilov boundaries.) Note that A is relatively maximal in $C(\partial_A)$ if and only if $A|\partial_A$ is a maximal closed subalgebra of $C(\partial_A)$; on the other hand if $X \neq \partial_A$ it follows quite simply from Zorn's lemma that there is a (necessarily proper) closed subalgebra $B \supset A$ of $C(X)$ with the same Šilov boundary which is relatively maximal in $C(X)$. (As we shall see later, an example of an algebra which is relatively maximal in $C(\mathscr{M})$ but not maximal is the algebra of functions in $C(D^n)$, analytic on the interior of D^n, the unit polycylinder in $\boldsymbol{C}^n$.)

The following simple observation will extend the range of our results.

LEMMA 3.1. *If* A *is relatively maximal in* $C(X)$ *and* $X \subset Y \subset \mathscr{M}_A$ *then* A *is relatively maximal in* $C(Y)$.

[2] We shall omit these terms when the algebra and boundary are clear.

[3] $f|U$ is the restriction of f to U, $A|U = \{g|U : g \in A\}$. Trivially the uniform closure $(A|U)^-$ of $A|U$ in $C(U)$ is isometrically isomorphic to the closure of $A|U^-$ in $C(U^-)$, and at times we may write $(A|U)^-$ where $(A|U^-)^-$ might also be used.

Suppose B is a larger subalgebra of $C(Y)$ with $\partial_B \subset \partial_A$, so that $\partial_B = \partial_A \subset X$. Then $B \mid X$ is closed in $C(X)$, and since we can assume $X \neq Y$, $B \mid X \neq C(X)$ since each point of $Y \backslash X$ provides a multiplicative linear functional on $B \mid X$. But A is relatively maximal in $C(X)$, so $A \mid X = B \mid X$, and each f in B coincides on $\partial_B = \partial_A$ with a g in A; since Y can clearly be viewed as a subset of $\mathscr{M}_B$, and $f - g \in B$ must vanish on all of $\mathscr{M}_B$ since it vanishes on ∂_B, $f = g$ on Y, and $B = A$.

The following is our direct extension of Radó's theorem.

THEOREM 3.2. *Suppose A is relatively maximal in $C(\mathscr{M}_A)$ with $\mathscr{M}_A \neq \partial_A$, or, more generally, is an intersection of closed subalgebras of $C(\mathscr{M}_A)$ each having a Šilov boundary which is a proper subset of $\mathscr{M}_A$ and each relatively maximal in $C(\mathscr{M}_A)$.*

Then any f in $C(\mathscr{M}_A)$ which is locally approximable on $\mathscr{M}_A \backslash (\partial_A \cup f^{-1}(0))$ is in A.

Proof. Consider first the special case in which A is relatively maximal in $C(\mathscr{M}_A)$, and let us write $\mathscr{M}$, ∂ for $\mathscr{M}_A$, ∂_A. Let B be the closed subalgebra of $C(\mathscr{M})$ generated by A and f.

For each m in $U = \mathscr{M} \backslash (\partial \cup f^{-1}(0))$ we have an open neighborhood U_m of m contained in U for which $f \mid U_m \in (A \mid U_m)^-$, so clearly $h \mid U_m \in (A \mid U_m)^-$ for any h in B. As a consequence $m \notin \partial_B$; for otherwise some h in B must peak within $\mathscr{M}$ on a subset of the open set U_m, so for some m' in U_m

$$| h(m') | > \sup | h(U_m^- \backslash U_m) | .$$

Since $h \mid U_m \in (A \mid U_m)^-$ this contradicts Rossi's local maximum modulus principle [11] (which asserts that $\partial_{(A \mid U)^-} \subset U^- \backslash U$ for any open $U \subset \mathscr{M} \backslash \partial$).

Similarly for any m in $f^{-1}(0)^0$, the interior of $f^{-1}(0)$, $m \notin \partial$, we have a neighborhood $U_m \subset \mathscr{M} \backslash \partial$ on which $f \mid U_m = 0 \in (A \mid U_m)^-$, and we again conclude that $m \notin \partial_B$. So $\partial_B \subset \partial \cup F$, where F is the topological boundary of $f^{-1}(0)$ in $\mathscr{M}$.

Now $F \backslash \partial$ is a relatively open subset of the boundary $X = F \cup \partial$ for B, and $F \backslash \partial$ lies in the topological boundary F of $f^{-1}(0)$ in the subspace $Y = \mathscr{M}$ of $\mathscr{M}_B$; so 2.2 applies, showing $(F \backslash \partial) \cap \partial_B = \phi$, whence $\partial_B \subset \partial$. Since $\partial \subsetneqq \mathscr{M}$, B is proper in $C(\mathscr{M})$, and since A is relatively maximal in $C(\mathscr{M})$, $B = A$. Thus $f \in A$ as desired.[5]

For the more general case[6] let $A = \bigcap A_\alpha$, where $\partial_{A_\alpha} \subsetneqq \mathscr{M}$, and each A_α is relatively maximal in $C(\mathscr{M})$. Clearly $\partial \subset \partial_{A_\alpha}$, and $\mathscr{M}$ is

[4] Actually $A \mid X \neq C(X)$ is redundant if $X \neq \partial_A$, as will usually be the case.

[5] Radó's theorem for A_1 now follows from Wermer's maximality theorem [**7**, 12].

[6] Our discussion here (and in later sections) would be considerably simplified if one had a positive answer to the following open question, raised some time ago by Kenneth Hoffman: if $A \subset B \subset C(\mathscr{M}_A)$ and $\partial_B = \partial_A$, must $\mathscr{M}_B = \mathscr{M}_A$?

a subspace of $\mathscr{M}_{A_\alpha}$ properly containing ∂_{A_α}, so $\partial_{A_\alpha} \neq \mathscr{M}_{A_\alpha}$. Let $\rho_\alpha : \mathscr{M}_{A_\alpha} \to \mathscr{M}$ be the map dual to the injection of A into A_α, which we can of course view as a retraction of $\mathscr{M}_{A_\alpha}$ onto its subspace $\mathscr{M}$; finally let $h \to \hat{h}$ denote the Gelfand representation of A_α— for h in A, in particular, $\hat{h} = h \circ \rho_\alpha$.

Now trivially $f \circ \rho_\alpha \in C(\mathscr{M}_{A_\alpha})$ is locally approximable (by $A \circ \rho_\alpha$, hence) by $\hat{A_\alpha}$ on $\mathscr{M}_{A_\alpha}\backslash(\partial \cup (f \circ \rho_\alpha)^{-1}(0))$, so certainly on $\mathscr{M}_{A_\alpha}\backslash(\partial_{A_\alpha} \cup (f \circ \rho_\alpha)^{-1}(0))$. Since $\hat{A_\alpha}$ is relatively maximal in $C(\mathscr{M}_{A_\alpha})$ by 3.1, $f \circ \rho_\alpha \in \hat{A_\alpha}$ by our special case, whence $f = (f \circ \rho_\alpha) \mid \mathscr{M}$ is in $\hat{A_\alpha} \mid \mathscr{M} = A_\alpha$: since this holds for every α, $f \in A$, completing the proof of 3.2.

The argument of the special case of 3.2 is central to all that follows (and will be needed again). There, in distinction to the more general case, the only property of $\mathscr{M}$ that is used is the local maximum modulus principle; $\mathscr{M}$ could just as well be any boundary X fo which

$$(3.1) \qquad \partial_{(A \cdot U)^-} \subset U^- \backslash U, \text{ for all relatively open } U \text{ in } X\backslash(\partial \cup F)\,,$$

where F is the boundary of $f^{-1}(0)$ in X. Moreover 3.2 evidently yields a positive assertion about any algebra A with $\mathscr{M} \neq \partial$; it will be worthwhile later to combine these observations in the following corollary to our proof, in which $\mathscr{M}$ can be taken as X.

COROLLARY 3.3. *Let $f \in C(X)$, where X is a boundary for A for which* (3.1) *holds. Let f be locally approximable (within X) on $X\backslash(\partial \cup f^{-1}(0))$ and let B be the closed subalgebra of $C(X)$ generated by A and f. Then*

(a) $\partial_B = \partial$ *(so that $B = A$ if A is relatively maximal in $C(X)$), and*

(b) *local maximum modulus applies to B on X, i.e., for an open $U \subset X\backslash\partial$,*

$$\partial_{(B|U)^-} \subset U^- \backslash U\,.$$

If $X = \partial$ the assertions of 3.3 are of course vacuous. (a) is of course proved in 3.2, and also follows from (b), whose proof is simply a modification of that of 3.2. For if $x \in U$ is not in F, the boundary in X of $f^{-1}(0)$, then x has a neighborhood U_x with $U_x^- \subset U\backslash F$ for which $f \mid U_x \in (A \mid U_x)^-$, so $h \mid U_x \in (A \mid U_x)^-$ for any h in $(B \mid U)^-$; thus $x \notin \partial_{(B\, U)^-}$ as in 3.2. On the other hand if x is in F then $x \notin \partial_{(B\, U)^-}$ by 2.2, so (b) follows.

3.3 has the following consequences.

THEOREM 3.4. *Let $\mathscr{M}_A \neq \partial_A$. Then there is a closed subalgebra B of $C(\mathscr{M}_A)$ containing A, with $\partial_B = \partial_A$, for which any f in $C(\mathscr{M}_A)$, locally approximable by B on $\mathscr{M}_A\backslash(\partial_A \cup f^{-1}(0))$, must lie in B.*

Consider any chain of closed subalgebras B of $C(\mathscr{M}_A)$ which have the same Šilov boundary as A and to which local maximum modulus applies on $\mathscr{M}_A : \partial_{(B|U)^-} \subset U^- \backslash U$, for U open in $\mathscr{M}_A \backslash \partial_A$. By just the argument used in 3.2, if B_0 denotes the closure of the union of the elements of the chain then $\partial_{(B_0|U)^-} \subset U^- \backslash U$ for any open $U \subset \mathscr{M}_A \backslash \partial_A$, so $\partial_{B_0} = \partial_A$ and local maximum modulus applies to B_0 on $\mathscr{M}_A$. By Zorn's lemma then we have a closed subalgebra B of $C(\mathscr{M}_A)$ maximal with respect to these properties, with $A \subset B$. But now for an f in $C(\mathscr{M}_A)$ which is locally approximable (within $\mathscr{M}_A$) by B on $\mathscr{M}_A \backslash (\partial_A \cup f^{-1}(0))$ we have by 3.3 precisely the same properties for the algebra generated by B and f; thus the latter coincided with B, and $f \in B$.

The following extension of 3.2, which allows us to replace 0 by a countable subset of C, merely adds a category argument to that of 3.2. In the original setting of Radó's theorem it can be obtained by a local application of that result (and category).

THEOREM 3.5. *Let A be relatively maximal in $C(\mathscr{M})$ with $\mathscr{M} \neq \partial$. Let E be a countable subset of C, and $\{F_n\}$ a sequence of nowhere dense hull-kernel closed*[7] *subsets of $\mathscr{M}$. If $f \in C(\mathscr{M})$ is locally approximable on*

$$\mathscr{M} \backslash (\partial \cup f^{-1}(E) \cup (\textstyle\bigcup F_n)) \tag{3.2}$$

then $f \in A$.

If A is only an intersection of relatively maximal subalgebras of $C(\mathscr{M})$, each having its Šilov boundary proper in $\mathscr{M}$, then the same assertion holds if $\bigcup F_n$ is closed, in particular if $\{F_n\}$ is finite.

Proof. Suppose first that A is relatively maximal, and let B be the closed subalgebra of $C(\mathscr{M})$ generated by A and f. Actually f is locally approximable on an open subset W of $\mathscr{M} \backslash \partial$ which contains (3.2), and also contains the open sets $f^{-1}(e)^0 \backslash \partial$, $e \in E$, as well; and so for each $m \in W$ we have a neighborhood U_m of m, $U_m \subset \mathscr{M} \backslash \partial$, for which $f \mid U_m \in (A \mid U_m)^-$, whence $h \mid U_m \in (A \mid U_m)^-$ for all h in B. Since W is open we can conclude from the local maximum modulus principle that $\partial_B \cap W = \phi$ as before.

Suppose $m \in \partial_B \backslash \partial$, so $m \in \partial_B \backslash (\partial \cup W)$. Since W contains (3.2) and each set $f^{-1}(e)^0 \backslash \partial$, such an m must lie in $\bigcup F_n$, or in $f^{-1}(E) \backslash \bigcup_{e \in E} f^{-1}(e)^0$, which is contained in the union of the boundaries of the sets $f^{-1}(e)$. Thus $\partial_B \backslash \partial$ is contained in a countable union of closed subsets of $\mathscr{M}$, and, by category, if $\partial_B \backslash \partial \neq \phi$ one of the sets $F_n \cap (\partial_B \backslash \partial)$ or

[7] Recall that a subset of $\mathscr{M} - \mathscr{M}_A$ is hull-kernel closed if and only if it is of the form $\bigcap_{g \in \mathscr{G}} g^{-1}(0)$, where $\mathscr{G} \subset A$.

(boundary $f^{-1}(e)) \cap (\partial_B \backslash \partial)$ has nonvoid interior V in the locally compact space $\partial_B \backslash \partial$, hence in ∂_B.

Now if $V \subset$ (boundary $f^{-1}(e)) \cap (\partial_B \backslash \partial)$ then $e - f$ is an element of B which vanishes on the relatively open subset V of ∂_B, while V lies in the boundary in $\mathscr{M}$ of $(e-f)^{-1}(0)$, so that 2.2 implies $V \cap \partial_B = \phi$, our contradiction. Similarly if $V \subset F_n \cap (\partial_B \backslash \partial)$, then since F_n is hull-kernel closed it has just the form of the intersection in 2.2; since F_n is nowhere dense in $\mathscr{M}$, V lies in boundary $F_n = F_n$, so 2.2 again yields a contradiction, and we conclude that $\partial_B \subset \partial$, whence $B = A$ and $f \in A$.

For the final assertion of 3.5 we consider $\mathscr{M}$ as a subspace of $\mathscr{M}_{A_\alpha}$ as in 3.2, with ρ_α our retraction of $\mathscr{M}_{A_\alpha}$ onto $\mathscr{M}$. Since now $\rho_\alpha^{-1}F_n$ need not be nowhere dense in $\mathscr{M}_{A_\alpha}$, we let $Y = \mathscr{M}_{A_\alpha} \backslash \bigcup \rho_\alpha^{-1}(F_n)^0$, and let B_α be the closed subalgebra of $C(Y)$ generated by (the restrictions of) $A_\alpha^\wedge$ and $f \circ \rho_\alpha$.

Our hypothesis that F_n is nowhere dense implies $\mathscr{M} \subset Y$ since $\rho_\alpha^{-1}(F_n)^0 \cap \mathscr{M} \subset F_n^0 = \phi$. And our hypothesis that $\bigcup F_n$ is closed implies $K = \bigcup \rho_\alpha^{-1} F_n = \rho_\alpha^{-1}(\bigcup F_n)$ is closed so that

$$U = \mathscr{M}_{A_\alpha} \backslash (\partial_{A_\alpha} \cup K)$$

is an open subset of $\mathscr{M}_{A_\alpha} \backslash \partial_{A_\alpha}$ contained in the subspace Y of $\mathscr{M}_{A_\alpha}$. Trivially $f \circ \rho_\alpha$ is locally approximable by $A_\alpha^\wedge$ on an open subset of Y which contains all points of U except (possibly) those lying in the boundaries of $(f \circ \rho_\alpha)^{-1}(e)$, $e \in E$. But now any m in U at which $f \circ \rho_\alpha$ is locally approximable has an open neighborhood U_m in $\mathscr{M}_{A_\alpha} \backslash \partial_{A_\alpha}$ with $U_m^- \subset U$ for which $h \mid U_m \in (A_\alpha \mid U_m)^-$, $h \in B$; since U_m is open in $\mathscr{M}_{A_\alpha}$, we know $m \notin \partial_B$ by just the argument of 3.2.

Thus $m \in \partial_B \backslash \partial_{A_\alpha}$ implies m lies in the boundary in Y of some $(f \circ \rho_\alpha)^{-1}(e)$, or in $K \backslash \bigcup \rho_\alpha^{-1}(F_n)^0 \subset \bigcup \{\rho_\alpha^{-1}(F_n) \backslash \rho_\alpha^{-1}(F_n)^0\}$, i.e., in the boundary of one of the sets $\rho_\alpha^{-1}(F_n) \cap Y$ in Y; and now the argument of the special case shows $\partial_B \subset \partial_{A_\alpha}$. By 3.1 $A_\alpha^\wedge$ is relatively maximal in $C(Y)$, so $B_\alpha = A_\alpha^\wedge \mid Y$, and since $\mathscr{M} \subset Y$, $B_\alpha \mid \mathscr{M} = A_\alpha^\wedge \mid \mathscr{M} = A_\alpha$, and $f \in A_\alpha$. Hence $f \in A$, completing the proof of 3.5.

As noted, the only point in the proof of the special case of 3.2 in which $\mathscr{M}$ had to be the full maximal ideal space of A, rather than a subset properly larger than the Šilov boundary, was in the application of local maximum modulus. In some special situations classical local maximum modulus can be applied, and we can then avoid using all of the maximal ideal space. For example, for $x \in X$, a boundary for A, call a *non-constant* map ρ_x of the open disc D^0 onto a subset of X containing x an *analytic disc through* x if $g \circ \rho_x$ is analytic for each g in A. Then

THEOREM 3.6. *Let $X \neq \partial$ be a boundary for A, and suppose A is relatively maximal in $C(X)$. Let $f \in C(X)$ and let F be the topological boundary of $f^{-1}(0)$ in X.*

Suppose that for every x in $U = X \backslash (\partial \cup F)$ there is an analytic disc ρ_x through x for which $f \circ \rho_x$ is analytic on D^0. Then $f \in A$.

As before, define B to be the closed subalgebra of $C(X)$ generated by A and f; for every disc ρ_x in our hypothesis we have $h \circ \rho_x$ analytic on D^0 for $h \in B$ as a uniform limit of analytic functions.

Now if $\partial_B \cap U$ is nonvoid then [10, p. 138] the open set U must contain a strong boundary point x of B, and since ρ_x is non-constant some g in B must peak within $\rho_x(D^0)$ on a proper subset containing x. So $g \circ \rho_x$ assumes its maximum modulus at a point of D^0, yet is non-constant and analytic on D^0; we conclude that $U \cap \partial_B$ is void, and $\partial_B \subset \partial \cup F$. Now the remainder of the proof of 3.2 applies.

Other variations of this sort can be obtained. We have pointed out 3.6 mainly to note an apparently nontrivial variation of Radó's theorem which it yields for the polycylinder algebra—the algebra A_n of all functions continuous on the polycylinder D^n in C^n and analytic on its interior. Recall that for A_n, $\mathscr{M} = D^n$ and $\partial = T^n$; moreover if X is any closed subset of D^n containing the topological boundary of D^n in C^n (and thus a boundary for A_n) then[8] $A_n \mid X$ is relatively maximal in $C(X)$.

COROLLARY 3.7. *Suppose X is a closed subset of D^n containing the topological boundary of D^n in C^n. Suppose $f \in C(X)$, and through each point of $X \backslash T^n$ where f does not vanish we have an analytic disc in X on which f is analytic.*

Then f is an element of the polycylinder algebra A_n restricted to X.

(Note that we of course have analytic discs on which f is analytic through points of $f^{-1}(0)^0$. Here an analytic disc is simply an analytic map of D^0 into X, which need not be $(1-1)$, let alone bianalytic.)

Finally we should note that something slightly weaker than local

[8] This is no doubt well known; the proof for $n = 2$ is as follows, with $A = A_2 \mid X$. Suppose $A \subset B \subset C(X)$, and $\partial_B = \partial_A = T^2$. Each disc $D_0 = \{(z, w_0) : |z| \leqq 1\}$ with $|w_0| = 1$ lies in X and is a peak set of A (hence of B), since $(z, w) \to (1/2)(1 + \bar{w}_0 w)$ peaks there. Consequently [8, p. 227] $B \mid D_0$ is closed in $C(D_0)$ and $\partial_{B|D_0} \subset \partial_B \cap D_0 = T^2 \cap D_0 = \partial_{A|D_0}$. Since $A \mid D_0$ is the relatively maximal disc algebra and we now have $\partial_{B|D_0} = \partial_{A|D_0}$, we conclude that $A \mid D_0 = B \mid D_0$; thus $z \to b(z, w_0)$ is analytic on $|z| < 1$ for $b \in B$, $|w_0| = 1$. Similarly $w \to b(z_0, \omega)$ is analytic for $|z_0| = 1$. But now

$$\iint e^{i(m\theta + n\phi)} b(e^{i\theta}, e^{i\phi}) d\theta d\phi = 0$$

for n or $m > 0$, so $b = g \in A$ on $\partial_B = T^2$, whence $b - g$ must vanish on $X \subset \mathscr{M}_B$.

approximability can be used in its stead in 3.2–3.5: rather than insisting that f be uniformly approximable by elements of A on U_m (hence necessarily on U_m^-) as we have done, we need only insist on uniform approximation on

$$K = \{m\} \cup (U_m^- \backslash U_m) = \{m\} \cup \text{boundary } U_m. \tag{3.3}$$

For example, in 3.2, $f \mid U_m \in (A \mid U_m)^-$ was used only to show $(\mathscr{M} \backslash (\partial \cup f^{-1}(0))) \cap \partial_B = \phi$, and since [10, p. 138] strong boundary points are dense in ∂_B while $\mathscr{M} \backslash (\partial \cup f^{-1}(0))$ is open, it suffices to show $m \in \mathscr{M} \backslash (\partial \cup f^{-1}(0))$ cannot be a strong boundary point. But if m is a strong boundary point and f is uniformly approximable on (3.3) then $h \mid K \in (A \mid K)^-$ for all h in B while some h in B must have

$$| h(m) | > \sup | h(U_m^- \backslash U_m) | \tag{3.4}$$

since m is a strong boundary point not in $U_m^- \backslash U_m$. Now some h' in A satisfies (3.4) (since $h \mid K \in (A \mid K)^-$), contradicting local maximum modulus again.

It may be worthwhile to note what this yields for the disc algebra A_1: if $f \in C(D)$, and for each z in $D \backslash (T^1 \cup f^{-1}(0))$ there is an r_z, $0 < r_z \leqq$ dist $(z, T^1 \cup f^{-1}(0))$ for which f can be approximated uniformly by polynomials on $\{z\} \cup \{z' : | z' - z | = r_z\}$, then $f \in A_1$. (Deleting $f^{-1}(0)$ everywhere, we have here simply a corollary to Wermer's maximality theorem for A_1 and the density of strong boundary points in ∂; from this the more general statement follows by Radó's theorem. Actually we can limit our z's to a dense countable set in $D \backslash (T^1 \cup f^{-1}(0))$ if we also assume that $r_z > k$ dist $(z, T^1 \cup f^{-1}(0))$ for some fixed $k > 0$.)

4. Schwarz's lemma. Our argument can also be applied to certain functions defined and continuous only on part of $\mathscr{M}$, for any algebra A. In particular, we have the following generalization of Schwarz's lemma (for $A = A_1$, take $g(z) \equiv z$), which has several consequences.

THEOREM 4.1. *Let f and g be in A and suppose f/g is bounded on $\mathscr{M} \backslash g^{-1}(0)$. Then*

$$\sup \left| \frac{f}{g} (\mathscr{M} \backslash g^{-1}(0)) \right| = \sup \left| \frac{f}{g} (\partial \backslash g^{-1}(0)) \right| \tag{4.1}$$

(In fact the assertion applies to the Gelfand representatives of any commutative Banach algebra.)

Proof. For each m in $U = \mathscr{M} \backslash (\partial \cup g^{-1}(0))$ let U_m be an open neighborhood of m with compact closure contained in U, chosen so small that $0 \in \boldsymbol{C}$ does not lie in the closed convex hull of $g(U_m^-)$. Then we have polynomials P_n for which $P_n(z) \to z^{-1}$ uniformly on $g(U_m)$, so that $(P_n \circ g) \mid U_m \to (1/g) \mid U_m$ in $C(U_m)$, and thus $(f/g) \mid U_m \in (A \mid U_m)^-$.

Letting B_0 be the uniformly closed subalgebra of $C(\mathscr{M}\backslash g^{-1}(0))$ generated by $A_0 = A \mid (\mathscr{M}\backslash g^{-1}(0))$ and f/g, we have

$$h \mid U_m \in (A \mid U_m)^- \tag{4.2}$$

for all h in B_0 and m in U.

Now let X be the closure of $\mathscr{M}\backslash g^{-1}(0)$ in $\mathscr{M}_{B_0}$, so that X is a boundary for the algebra $B_0^\wedge$ of Gelfand representatives of B_0. Set $B = B_0^\wedge | X$; B of course contains a continuous extension to X of each h in A_0, and of f/g, and we shall let h^* denote the extension of $h \mid (\mathscr{M}\backslash g^{-1}(0))$, for $h \in A$.

g^* cannot vanish on $\mathscr{M}\backslash g^{-1}(0)$. On the other hand g^* must vanish on $X\backslash\mathscr{M}$: for since $\mathscr{M}\backslash g^{-1}(0)$ is dense in X, $|g^*(x)| \geqq \varepsilon > 0$ implies x is in the closure of $\{m \in \mathscr{M} : |g(m)| \geqq \varepsilon\}$, which is already compact, so $x \in \mathscr{M}$. Consequently $g^{*-1}(0) = X\backslash\mathscr{M}$. Again since $\mathscr{M}\backslash g^{-1}(0)$ is dense in X, $g^{*-1}(0) = X\backslash\mathscr{M}$ must coincide with its boundary in X.

Now $\mathscr{M}\backslash g^{-1}(0) = X\backslash g^{*-1}(0)$ is open in X; on the other hand the imbedding of $\mathscr{M}\backslash g^{-1}(0)$ into X is a homeomorphism,[9] so that the relatively open subset $U = \mathscr{M}\backslash(\partial \cup g^{-1}(0))$ of $\mathscr{M}\backslash g^{-1}(0)$ is in fact relatively open (hence open) in X. Consequently each U_m is open in X and (4.2) suffices to show no m in U is in ∂_B, as in 3.2. So

$$\partial_B \subset X\backslash U \subset (X\backslash\mathscr{M}) \cup (\partial\backslash g^{-1}(0)) \subset (X\backslash\mathscr{M}) \cup F\,,$$

where F is the closure in X of $\partial\backslash g^{-1}(0)$, and $\partial_B \subset g^{*-1}(0) \cup F$.

If the relatively open subset $\partial_B\backslash F$ of ∂_B were nonvoid, then, since it lies in $g^{*-1}(0)$, hence in the boundary in X of this set, 2.2 would imply $\partial_B \cap (\partial_B\backslash F) = \phi = \partial_B\backslash F$; so $\partial_B\backslash F = \phi$, $\partial_B \subset F$ and trivially (4.1) follows. (Since the result applies to $A^\wedge$—with 1 adjoined if necessary—for any commutative Banach algebra A, the final assertion follows easily.)

Our first corollary to 4.1 gives some information about zero-sets which is quite familiar for the disc algebra: a (non-void) zero set $g^{-1}(0)$ $(g \in A)$ disjoint from the Šilov boundary has a smallest neighborhood on which elements of A can vanish, while no f in A vanishing on $g^{-1}(0)$ can tend to zero faster than every power of g unless f vanishes on a neighborhood of $g^{-1}(0)$.

We first observe that (4.1) can be trivially improved to have $\partial\backslash(f^{-1}(0) \cup g^{-1}(0))$ in place of $\partial\backslash g^{-1}(0)$ on the right side of (4.1) (since f/g vanishes on $f^{-1}(0)\backslash g^{-1}(0)$).

[9] Trivially $B_0 \subset C(\mathscr{M}\backslash g^{-1}(0))$ implies the map of $\mathscr{M}\backslash g^{-1}(0)$ into $\mathscr{M}_{B_0}$ is continuous, while the map of $\mathscr{M}_{B_0}$ into $\mathscr{M}$ dual to the injection of A into B_0-restricted to the image of $\mathscr{M}\backslash g^{-1}(0)$ in $\mathscr{M}_{B_0}$-provides a continuous inverse.

COROLLARY 4.2. *Let f and g be in A, with $\phi \neq g^{-1}(0) \subset f^{-1}(0)$, and suppose* $\inf |g(\partial \backslash f^{-1}(0))| = \delta > 0$ *(which will of course be the case if $g^{-1}(0) \cap \partial = \phi$). If f/g^n is bounded on $\mathscr{M} \backslash g^{-1}(0)$ for every $n \geqq 1$ then f vanishes on*

$$g^{-1}(D_\delta^0)\,, \tag{4.3}$$

where D_δ^0 is the open disc about 0 of radius δ.

In particular, if $g^{-1}(0)$ is nonvoid and disjoint from the Šilov boundary, then any f in A vanishing on a neighborhood of $g^{-1}(0)$ vanishes on (4.3) with $\delta = \inf |g(\partial)|$.

By (4.1), modified as indicated,

$$\left|\frac{f}{g^n}\right| \leqq \sup\left|\frac{f}{g^n}(\partial \backslash (f^{-1}(0) \cup g^{-1}(0)))\right| = \sup\left|\frac{f}{g^n}(\partial \backslash f^{-1}(0))\right| \leqq \frac{\|f\|}{\delta^n}$$

on $\mathscr{M} \backslash g^{-1}(0)$ so that

$$|f(m)| \leqq \lim \|f\| \cdot \left|\frac{g(m)}{\delta}\right|^n = 0$$

if $0 \neq |g(m)| < \delta$. By hypothesis $f(m) = 0$ if $g(m) = 0$, so f vanishes on all of (4.3).

For convenience let us say $g \in A$ divides $f \in A$ if $f = gh$, $h \in A$.

COROLLARY 4.3. *Suppose A is analytic on $\mathscr{M}$* (§ 2), *and $g \in A$ has $g^{-1}(0)$ nonvoid and disjoint from the Šilov boundary. Then if g divides a nonzero element f of A there is a largest integer n for which g^n divides f.*

Otherwise f/g^n is bounded for every n, and f must vanish on (4.3), hence on all of $\mathscr{M}$.

COROLLARY 4.4. *Suppose $A|\partial$ is an intersection of maximal closed subalgebras of $C(\partial)$, f and g are in A, and*[10] $\mathscr{M} \neq \partial \cup g^{-1}(0)$. *If f/g is bounded on $\mathscr{M} \backslash g^{-1}(0)$, and on ∂ has an extension in $C(\partial)$, then $f = gh$ for some h in A.*

---

[10] This hypothesis is superfluous if g does not vanish anywhere on ∂, but in general is essential to the result. For let $\mathscr{M}$ be the subset $(\{0\} \times D) \cup \{(r, z) : 0 \leqq r \leqq 1, |z| = 1\}$ of $\boldsymbol{R} \times \boldsymbol{C}$, and A all functions continuous on $\mathscr{M}$ and analytic on $\{0\} \times D^0$. Then $\mathscr{M}_A = \mathscr{M}$, $\partial_A = \mathscr{M} \backslash (\{0\} \times D^0)$ and setting $f(r, z) = r\bar{z}$, $g(r, z) = r$ we have $f/g(r, z) = \bar{z}$ so $f/g \notin A$. (If $0 \notin g(\partial)$ and $\mathscr{M} = \partial \cup g^{-1}(0)$ then each of the complementary sets ∂ and $g^{-1}(0)$ is open and closed; by a result of Šilov, or in fact by 3.2, the characteristic function of $g^{-1}(0)$ is an element of A. Since it vanishes on ∂ we conclude that $g^{-1}(0) = \phi$ and the assertion of 4.4 is vacuous.)

We shall only sketch the proof, which is quite similar to that of 4.1. Suppose first that $A\,|\,\partial$ is actually maximal. Let h_0 be an extension of f/g to $Y = \partial \cup (\mathscr{M}\backslash g^{-1}(0))$ with $h_0\,|\,\partial \in C(\partial)$; we now let B_0 be the uniformly closed algebra of *bounded* functions on Y generated by $A\,|\,Y$ and h_0. Of course we have $h\,|\,\partial$ and $h\,|\,(\mathscr{M}\backslash g^{-1}(0))$ continuous for $h \in B_0$, and this implies the $(1-1)$ map of Y into $\mathscr{M}_{B_0}$ is continuous when restricted to ∂ or $\mathscr{M}\backslash g^{-1}(0)$: we can view Y as a subset of $\mathscr{M}_{B_0}$, ∂ as a compact subspace.

As in 4.1 we let X be the closure of Y in $\mathscr{M}_{B_0}$, $B = \hat{B_0}\,|\,X$, and h^* the element of B corresponding to $h \in B_0$. If $g^*(x) \neq 0$ then for some $\varepsilon > 0$, x lies in the closure in X of $\{m \in \mathscr{M}: |\,g(m)\,| \geqq \varepsilon\}$, which is already compact in X as the continuous image of a compact subset of $\mathscr{M}\backslash g^{-1}(0)$. So $X\backslash g^{*-1}(0)$, an open subset of X, is contained in Y; thus $(X\backslash g^{*-1}(0))\backslash\partial$ is another subset of Y which is open in X. This last set is clearly the open subset $U = Y\backslash\partial = \mathscr{M}\backslash(\partial \cup g^{-1}(0))$ of Y, and U is open in X.

Now the imbedding of U into X is a homeomorphism (exactly as before; see footnote 9)), so any subset of U, open in $\mathscr{M}$, is open in X.

Consequently if we select, for $m \in U$, an open neighborhood U_m of m in $\mathscr{M}$ with $U_m^- \subset U$ (as in 4.1) for which $h^*\,|\,U_m = h\,|\,U_m \in (A\,|\,U_m)^-$, $h \in B_0$, then since U_m is in fact open in X the argument of 3.2 and local maximum modulus show $\partial_B \cap U_m = \phi$.

Thus $\partial_B \subset X\backslash U = X\backslash(Y\backslash\partial)$, so $\partial_B\backslash\partial \subset X\backslash Y$. But $X\backslash Y \subset g^{*-1}(0)$, as we have seen, so $X\backslash Y \subset g^{*-1}(0)\backslash\partial$, and $\partial_B\backslash\partial \subset g^{*-1}(0)\backslash\partial$. Since $Y\backslash\partial = \mathscr{M}\backslash(\partial \cup g^{-1}(0))$ is dense in $X\backslash\partial$ while g^* cannot vanish on this set, we clearly have $g^{*-1}(0)\backslash\partial$ contained in its boundary in X. So 2.2 applies to show $\partial_B\backslash\partial = \phi$, whence $\partial_B \subset \partial$ and $B\,|\,\partial$ is closed in $C(\partial)$.

By hypothesis $\mathscr{M}\backslash(\partial \cup g^{-1}(0)) \neq \phi$, so $\partial_B \subset \partial$ is proper in $\mathscr{M}_B$; thus $A\,|\,\partial \subset B\,|\,\partial \subsetneqq C(\partial)$, and by maximality $A\,|\,\partial = B\,|\,\partial$. Hence $h_0 = h$ on ∂ for some h in A, whence $f = gh$ on ∂, hence on all of $\mathscr{M}$.

Now if $A\,|\,\partial = \bigcap(A_\alpha\,|\,\partial)$ where each $A_\alpha\,|\,\partial$ is maximal in $C(\partial)$, then the preceding argument applied to A_α (with f and g taken in $C(\mathscr{M}_{A_\alpha})$) shows $h_0\,|\,\partial \in A_\alpha\,|\,\partial$; thus $h_0\,|\,\partial = h\,|\,\partial$ for some h in A, whence $f = gh$ on $\mathscr{M}$ as before.

COROLLARY 4.5. *Let $A\,|\,\partial$ be an intersection of maximal closed subalgebras of $C(\partial)$, and let g be an element of A with $\mathscr{M} \neq \partial \cup g^{-1}(0)$. Then any f in $C(\mathscr{M})$ with $fg \in A$ coincides on $\mathscr{M}\backslash g^{-1}(0)$ with an element of A.*

For fg/g is bounded on $\mathscr{M}\backslash g^{-1}(0)$ and on ∂ has the extension $f\,|\,\partial$ in $C(\partial)$, so that $fg = gh$ for some h in A by 4.4.

(If A is *analytic* on $\mathscr{M}$ (see § 2), $f \in A$; for then $g^{-1}(0)$ is nowhere dense in $\mathscr{M}$.)

Bishop [3, § 2, Lemma 3] has recently shown that (for any A) a point m in $\mathscr{M}\backslash\partial$ is represented by a (not necessarily unique) Jensen measure on ∂, i.e., there is a probability measure μ carried by ∂ for which Jensen's inequality holds:

$$\log|f(m)| \leqq \int \log|f|\,d\mu\,, \qquad f \in A\,.$$

(Applied to $f = e^{\pm g}$, $g \in A$, this yields $Re\,g(m) = \int Re\,g d\mu$ so that μ represents m on A.) As a consequence the argument of 4.4 yields

COROLLARY 4.5. *Suppose $f, g \in A$ and f/g is bounded on $\mathscr{M}\backslash g^{-1}(0)$ and on ∂ has an extension h_0 in $C(\partial)$. Then for each m in $\mathscr{M}\backslash(\partial \cup g^{-1}(0) \cup f^{-1}(0))$ there is a Jensen measure μ on ∂ representing m for which*

$$\int \log|g|\,d\mu - \log|g(m)| \leqq \int \log|f|\,d\mu - \log|f(m)|\,.$$

When (as in 4.4) f/g is actually the restriction of an element h of A, 4.5 follows trivially from Jensen's inequality for any Jensen measure μ representing m; for

$$\begin{aligned}\int \log|f|\,d\mu - \log|f(m)| &= \int \log|gh|\,d\mu - \log|gh(m)| \\ &= \left(\int \log|g|\,d\mu - \log|g(m)|\right) \\ &\quad + \left(\int \log|h|\,d\mu - \log|h(m)|\right)\end{aligned}$$

while the last term is nonnegative. In general we can construct the algebra B of 4.4, obtaining $\partial_B = \partial$ as there. Thus $m \in \mathscr{M}\backslash(\partial \cup g^{-1}(0))$, which provides an element of $\mathscr{M}_B\backslash\partial_B$, is represented on B by a Jensen measure μ on $\partial_B = \partial$ by Bishop's result. So

$$\log\left|\frac{f}{g}(m)\right| \leqq \int \log|h_0|\,d\mu$$

and

$$-\infty < \log|g(m)| \leqq \int \log|g|\,d\mu\,.$$

From the last we have $\partial \cap g^{-1}(0)$ a μ-null set so

$$\begin{aligned}\log|f(m)| - \log|g(m)| = \log\left|\frac{f}{g}(m)\right| &\leqq \int \log\left|\frac{f}{g}\right|d\mu \\ &\leqq \int \log|f|\,d\mu - \int \log|g|\,d\mu\end{aligned}$$

yielding 4.6.

If $A \mid \partial$ is not an intersection of maximal subalgebras of $C(\partial)$ and f, g are as in 4.4 one would not in general expect f/g to have an extension in $C(\mathscr{M})-$ or even an extension to $\mathscr{M}$ continuous at all points of ∂. However this is the case if A has unique representing measures.[11]

COROLLARY 4.7. *Suppose each $m \in \mathscr{M}$ is represented by a unique (probability) measure on ∂. Let $f, g \in A$, with f/g bounded on $\mathscr{M}\backslash g^{-1}(0)$, and suppose that, on ∂, f/g has an extension in $C(\partial)$. Finally, suppose $\mathscr{M} \neq \partial \cup g^{-1}(0)$.*

Then f/g has an extension in $C(\mathscr{M})$.

Exactly as in 4.4 we form the closed subalgebra B of $C(X)$; X contains (a homeomorph of) ∂ and a continuous $(1-1)$ image of $\mathscr{M}\backslash g^{-1}(0)$ as before. Again we obtain $\partial_B \subset \partial$, so that each $m \in \mathscr{M}_B$ is represented by a probability measure μ_m on ∂, which is necessarily multiplicative on $A \subset B$, hence represents an element m' of $\mathscr{M}$; the map $m \rightarrow m'$ is of course nothing but the continuous map on $\mathscr{M}_B$ into $\mathscr{M}$ dual to the injection of A into B. But since representing measures for A are unique $m \rightarrow m'$ is $1-1$: for if m_1, m_2 both map into m' then $\mu_{m_1} = \mu_{m_2}$, whence $m_1 = m_2$.

Thus $\mathscr{M}_B$ is homeomorphic to a compact subset of $\mathscr{M}$ which necessarily contains $(\mathscr{M}\backslash g^{-1}(0)) \cup \partial$, so that h_0 (see 4.4) has a continuous extension to the closure of this set, hence to $\mathscr{M}$.

Actually in 4.1, 4.4 and 4.7 various other combinations of f and g (e.g., $f \exp(1/g)$) could be used in place of f/g. More generally f/g could be replaced by any h in $C(\mathscr{M}\backslash g^{-1}(0))$ which is locally approximable on $\mathscr{M}\backslash(\partial \cup g^{-1}(0))$, as is clear from their proofs. Thus

THEOREM 4.8. *Let $g \in A$ and suppose $h \in C(\mathscr{M}\backslash g^{-1}(0))$ is locally approximable on $\mathscr{M}\backslash(\partial \cup g^{-1}(0))$. Then*

$$\sup | h(\mathscr{M}\backslash g^{-1}(0)) | = \sup | h(\partial\backslash g^{-1}(0)) | .$$

Suppose that $\mathscr{M} \neq \partial \cup g^{-1}(0)$, while $h \mid \partial$ has an extension in $C(\partial)$. Then

(i) *If $A \mid \partial$ is an intersection of maximal subalgebras of $C(\partial)$,*

[11] More generally we could insist on uniqueness of the Jensen measure for each m (see [**3**, §2, Lemma 3]). An example where the assertion of 4.4 fails is the following which was pointed out by Wermer. Let $X = \{(z, w) \in \boldsymbol{C}^2 : |z| = 1 = |w|\}$, and A the closed subalgebra of $C(X)$ generated by the coordinate functions z, w, and all the functions w^m/z^n with $m > n > 0$. Then $\mathscr{M} = \{(z, w) \in \boldsymbol{C}^2 : |w| \leqq |z| \leqq 1\}$, the coordinate function $g = z$ vanishes only at $(0, 0)$ in $\mathscr{M}$, and w/z is bounded on $\mathscr{M}\backslash g^{-1}(0)$, but has no continuous extension to $\mathscr{M}$.

h is the restriction of an element of A.

(ii) *If each m in $\mathscr{M}$ has a unique representing measure on ∂ then h has an extension in $C(\mathscr{M})$.*

With sufficiently strong hypotheses we can also obtain an analogue of Radó's theorem in which continuity need not be assumed everywhere.

THEOREM 4.9. *Suppose $A\,|\,\partial$ is maximal in $C(\partial)$, $\mathscr{M} \neq \partial$. Let F be a relatively closed subset of $\mathscr{M}\backslash\partial$, E a countable subset of $\boldsymbol{C}$, and K a countable union of hull-kernel closed sets (for example, points) contained in F. Suppose $f \in C(\mathscr{M}\backslash F)$, f is locally approximable on $\mathscr{M}\backslash(\partial \cup F)$, $f^{-1}(e)$ is nowhere dense in $\mathscr{M}\backslash F$ for each e in E, and for every m in the boundary F_0 of F in $\mathscr{M}$, $m \notin K$, the cluster values of f at m lie in E, i.e.,*

$$\bigcap f(V\backslash F)^- \subset E$$

where the intersection is taken over all neighborhoods of m.

Then f is the restriction of an element of A.

Proof. Again for each m in $U = \mathscr{M}\backslash(\partial \cup F)$ we choose an open neighborhood U_m with $f\,|\,U_m \in (A\,|\,U_m)^-$, and let B_0 be the closed subalgebra of $C(\mathscr{M}\backslash F)$ generated by $A\,|\,(\mathscr{M}\backslash F)$ and f; thus

$$h\,|\,U_m \in (A\,|\,U_m)^- \tag{4.4}$$

for all m in U and h in B_0. We can again view $Y = \mathscr{M}\backslash F = \partial \cup (\mathscr{M}\backslash F)$ as a subset of $\mathscr{M}_{B_0}$, and ∂ as a compact subset. Let B be the restriction of the Gelfand representatives $B_0^\wedge$ to the closure X of Y in $\mathscr{M}_{B_0}$, and ρ the restriction to X of the map $\mathscr{M}_{B_0} \rightarrow \mathscr{M}$ dual to $A \rightarrow B_0$. Trivially $\rho(X)$ is the closure, in $\mathscr{M}$, of Y.

Now ρ is $1-1$ on $\rho^{-1}Y$, so $\rho^{-1}Y = Y$; for each h in B_0 is continuous on Y while for each x in X, $\hat{h}(x)$ is a cluster value of h at $\rho(x)$. Thus $\rho(x) = y \in Y$ implies $\hat{h}(x) = h(y) = \hat{h}(y)$, and $x = y$. (Since each U_m is open in $\mathscr{M}$, hence in $\rho(X)$, this implies $U_m = \rho^{-1}U_m$ is open in X.)

Each m in $\rho(X)\backslash Y$ lies in the boundary F_0 of F, clearly. Since each U_m is open in X, by local maximum modulus and (4.4) we have $\partial_B \cap U = \phi$, so $\partial_B \subset X\backslash U$; since $\rho^{-1}Y = Y$ and $Y = \partial \cup U$ we have $\rho(\partial_B) \subset \rho(X)\backslash U \subset \partial \cup (\rho(X)\backslash Y) \subset \partial \cup F_0$, so $\partial_B \subset \partial \cup \rho^{-1}(F_0)$. For each x in $\rho^{-1}(F_0)$, $\hat{f}(x)$ is a cluster value of f at $\rho(x)$, so that either $\hat{f}(x) \in E$ or $\rho(x) \in K = \bigcup_{i=1}^{\infty} K_i$ (where K_i is hull-kernel closed). Thus the locally compact space $\partial_B\backslash\partial \subset \rho^{-1}(K) \cup (\hat{f}^{-1}(E) \cap X)$, a countable union of closed subsets of X. By category, one of the sets $\rho^{-1}(K_i) \cap (\partial_B\backslash\partial)$ or $\hat{f}^{-1}(e) \cap (\partial_B\backslash\partial)$, $e \in E$, has a nonvoid relative interior V in $\partial_B\backslash\partial$ if $\partial_B\backslash\partial \neq \phi$.

Suppose $\rho^{-1}(K_i) \cap (\partial_B \backslash \partial)$ has nonvoid relative interior V. Then if $S = \{h \in A : h(K_i) = 0\}$, $K_i = \bigcap_{h \in S} h^{-1}(0)$, and since $\hat{f} = h \circ \rho$ on X, we have $\rho^{-1}(K_i) = \bigcap_{h \in S} \hat{h}^{-1}(0)$. Trivially $\rho^{-1}(K_i)$ is all boundary in X (since $\rho^{-1}(K_i) \cap Y = \phi$ and Y is dense), so 2.2 applies to yield the contradiction $V = \partial_B \cap V = \phi$.

Again if $f^{-1}(e) \cap (\partial_B \backslash \partial)$ has nonvoid relative interior V in $\partial_B \backslash \partial$, then $(e - \hat{f})^{-1}(0)$ contains V, and coincides with its boundary in X since $(e - \hat{f})^{-1}(0) \cap Y$ is nowhere dense in $Y = \mathscr{M} \backslash F$ by hypothesis, hence has a dense complement. Since this again yields $V = V \cap \partial_B = \phi$ by 2.2, we conclude that $\partial_B \backslash \partial = \phi$.

The remainder of the proof is now clear.

COROLLARY 4.10. *Suppose that the hypotheses of* 4.9 *are satisfied except for the requirement that* $A | \partial$ *be maximal in* $C(\partial)$. *Then the closed subalgebra of* $C(\mathscr{M} \backslash F)$ *generated by* f *and* $A | (\mathscr{M} \backslash F)$ *has the same Šilov boundary as* A.

5. **Integral closure.** For a boundary X of A we shall call A *integrally closed* in $C(X)$ if, when $a_0, a_1, \cdots, a_{n-1}$ are in A and $f \in C(X)$ then

$$p(f) = f^n + a_{n-1} f^{n-1} + \cdots + a_0 = 0 \quad \text{on } X \tag{5.1}$$

implies $f \in A$. We shall see that algebras to which 3.2 applies have this property for[12] $X = \mathscr{M}$ as a consequence of 3.2 and the implicit function theorem for analytic functions on C^n.

Recall that if F is analytic near $(z^0, w^0) = (z_1^0, \cdots, z_n^0, w^0)$ in C^{n+1}, $F(z^0, w^0) = 0$ and $(\partial F / \partial w) = F_{n+1}(z^0, w^0) \neq 0$ then, for some $\delta > 0$ and neighborhood V of z^0 in C^n, there is a unique function φ on V for which

$$F(z, \varphi(z)) = 0 \quad \text{and} \quad |\varphi(z) - w^0| < \delta ;$$

and φ is analytic on V. Consequently if

$$F(a_1, \cdots, a_n, f) = 0 \tag{5.2}$$

on a neighborhood of $m \in \mathscr{M}$, where $a_1, \cdots, a_n \in A, f \in C(\mathscr{M})$, and $a_i(m) = z_i^0, f(m) = w^0$, then

$$f = \varphi(a_1, \cdots, a_n)$$

near m. Thus f can be uniformly approximated by a power series in $a_1, \cdots, a_n$ near m, and for some neighborhood U_m of m, $f | U_m \in (A | U_m)^-$. So we have

[12] The same argument, using 3.3, yields this for any boundary X for which local maximum modulus applies to A on X, if A is relatively maximal in $C(X)$.

LEMMA 5.1. *Let* $a_1, \cdots, a_n \in A, f \in C(\mathscr{M})$, *and suppose* F *is analytic on a neighborhood of* $(a_1(m), \cdots, a_n(m), f(m))$ *in* $\boldsymbol{C}^{n+1}$ *while* (5.2) *holds on a neighborhood of* m. *Then* f *is locally approximable at* m *if* $F_{n+1}(a_1(m), \cdots, a_n(m), f(m)) \neq 0$.

We can now easily obtain the integral closure of the algebras in 3.2. Slightly more generally we have

THEOREM 5.2. *Suppose* A *is an intersection of relatively maximal subalgebras of* $C(\mathscr{M})$ *with Šilov boundaries proper in* $\mathscr{M}$. *If* $f \in C(\mathscr{M})$ *is locally approximable on* $\mathscr{M}\backslash\partial$ *outside the set where* (5.1) *holds then* $f \in A$; *in particular* A *is integrally closed in* $C(\mathscr{M})$.

Proof. f, and so $p(f)$, is locally approximable on $\mathscr{M}\backslash\partial$ except where $p(f) = 0$, so that $p(f) = a \in A$ by (3.2). Changing a_0, we can thus assume

$$p(f) = f^n + a_{n-1}f^{n-1} + \cdots + a_0 = 0$$

everywhere on $\mathscr{M}$. But now f is locally approximable off the set where

$$p'(f) = nf^{n-1} + (n-1)a_{n-1}f^{n-2} + \cdots + a_1 = 0$$

by 5.1, so $p'(f)$ is locally approximable off $p'(f)^{-1}(0)$, and $p'(f) \in A$ by 3.2. Continuing we finally have $(n!)f + a \in A$, and $f \in A$.

COROLLARY 5.3. *Suppose* A *satisfies the hypothesis of* 5.2, *while* $f \in A$ *does not have an nth root in* A *for some* $n > 1$. *Then* $\mathscr{M}\backslash f^{-1}(0)$ *is not simply connected (and if* $\mathscr{M}$ *is locally connected, some component of* $\mathscr{M}\backslash f^{-1}(0)$ *is not simply connected.)*

Finally, if A *is also analytic on* $\mathscr{M}$, $\mathscr{M}\backslash g^{-1}(0)$ *is connected for each* g *in* A.

If $\mathscr{M}\backslash f^{-1}(0)$ were simply connected we could find an h in $C(\mathscr{M}\backslash f^{-1}(0))$ with $h^n = f$ on $\mathscr{M}\backslash f^{-1}(0)$; setting $h = 0$ on $f^{-1}(0)$ we obtain an nth root of f in $C(\mathscr{M})$, and $h \in A$ by 5.2. (Similarly if the components of $\mathscr{M}\backslash f^{-1}(0)$ were simply connected we could find such an h on each component, and, if the components are open, we can combine these to again obtain an nth root in $C(\mathscr{M})$.)

Finally if A is also analytic on $\mathscr{M}$, and $\mathscr{M}\backslash g^{-1}(0) = U \cup V \neq \phi$, with U, V open and disjoint, then

$$h = \begin{cases} g \text{ on } U \cup g^{-1}(0) \\ -g \text{ on } V \end{cases}$$

defines an h in $C(\mathscr{M})$ which lies in A by 3.2; since $h+g$ or $h-g$ vanishes on a nonvoid open set (V or U) one vanishes identically. But $h=g$ implies $V=\phi$, $h=-g$ implies $U=\phi$, so $\mathscr{M}\backslash g^{-1}(0)$ must be connected.

Actually if A satisfies the hypothesis of 5.2 and is analytic on $\mathscr{M}$ then A is algebraically closed in $C(\mathscr{M})$ in the obvious sense. More generally such an A is *analytically closed* in $C(\mathscr{M})$ in the following sense.

Let $a_1, \cdots, a_n \in A$, $f \in C(\mathscr{M})$, and let F be a function analytic on a neighborhood in C^{n+1} of the range of the map

$$\rho\colon m \longrightarrow (a_1(m), \cdots, a_n(m), f(m))\ ;$$

despite our earlier notation we now let $F_k = [(\partial/\partial z_{n+1})^k F]\circ\rho$, $k \geqq 0$. Clearly F is not "independent of z_{n+1} on $\rho(\mathscr{M})$" if and only if $F_k(m) \neq 0$ for some m and $k \geqq 1$, and so we shall call A *analytically closed in* $C(\mathscr{M})$ if, for such a_i, f and F, with $F_k(m) \neq 0$ for some $k \geqq 1$ and m,

$$F(a_1, \cdots, a_n, f) = 0 \tag{5.3}$$

implies $f \in A$.

THEOREM 5.4. *If A is an intersection of relatively maximal subalgebras A_α of $C(\mathscr{M})$ each having its Šilov boundary proper in $\mathscr{M}$, and A is analytic on $\mathscr{M}$, then A is analytically closed in $C(\mathscr{M})$.*

For f is locally approximable on $\mathscr{M}\backslash(\partial \cup F_1^{-1}(0))$ by 5.1, so that F_1 is also, and $F_1 \in A$ by 3.2. Of course we may have $F_1 = 0$, but even then we know f (and so F_2) is locally approximable on $\mathscr{M}\backslash(\partial \cup F_2^{-1}(0))$, so that $F_2 \in A$ by 3.1; since not every $F_k = 0$ we have some F_k a nonzero element of A, and choosing k least, f is locally approximable on $\mathscr{M}\backslash(\partial \cup F_k^{-1}(0))$.

But now the final portion of 3.5 applies, with E void and $F_k^{-1}(0)$ our (single) hull-kernel closed subset of $\mathscr{M}$ (which is necessarily nowhere dense since $F_k \neq 0$ and A is analytic on $\mathscr{M}$).

For an algebra to which Radó's theorem applies the preceding argument shows (5.3) implies $F_k \in A$ for all k, and clearly we can replace $F_k^{-1}(0)$ in the proof by $K = \bigcap_k F_k^{-1}(0)$, with f locally approximable off this set; thus the hypothesis that K is nowhere dense is an adequate replacement for the analyticity of A on $\mathscr{M}$, yielding the first half of

THEOREM 5.5. *Suppose* (5.3) *holds with F appropriately analytic and $f \in C(\mathscr{M})$. Let $K = \bigcap F_k^{-1}(0)$.*

(1) *If A satisfies the hypotheses of 5.2 and K is nowhere dense, $f \in A$,*

(2) *If* $A|\partial$ *is maximal in* $C(\partial)$, *and* $\mathscr{M} \neq \partial \cup K$, *then* f *coincides with an element of* A *off the interior of* K.

Since $F_k \in A$ for all k, and f is locally approximable off K, (2) follows from 4.9 (with E void, and K the K of 4.9). Of course we could assume, as in 5.2, that 5.3 holds wherever f is not known to be locally approximable.

Actually any algebra A with $\mathscr{M} \neq \partial$ is contained in a subalgebra B of $C(\mathscr{M})$ given by 3.4 to which (1) applies, as is easily seen. In particular, B provides an integral closure of A in $C(\mathscr{M})$.

THEOREM 5.6. *Suppose* $\mathscr{M}_A \neq \partial_A$. *Then* A *is contained in a subalgebra* B *of* $C(\mathscr{M}_A)$ *which is integrally closed in* $C(\mathscr{M}_A)$ *and has* $\partial_B = \partial_A$. *Thus, in particular, if* $f \in C(\mathscr{M}_A)$ *satisfies* (5.1) *for* a_i *in* A, *the subalgebra of* $C(\mathscr{M}_A)$ *generated by* A *and* f *has* ∂_A *as its Šilov boundary.*

With B given by 3.4, the proof is precisely that of 5.2, with B in place of A.

Finally we should note that something stronger than integral closure in $C(\mathscr{M})$ holds for our intersections of relatively maximal algebra—we could require only that (5.1) holds locally on $\mathscr{M}\backslash\partial$, i.e., that each m in the (non-compact) space $\mathscr{M}\backslash\partial$ has a neighborhood on which an equation of the form (5.1) holds. Then, rather than invoking 3.2, we could simply show that for B, the subalgebra of $C(\mathscr{M})$ generated by A and f, one has $\partial_B \subset \partial$. (Indeed if $m \in \partial_B\backslash\partial$ and we choose p as in (5.1) of least possible degree with $p(f) = 0$ on a neighborhood U_m of m, then f and $p'(f)$ are locally approximable on $U_m\backslash p'(f)^{-1}(0)$, so m cannot lie in this open set—nor in its boundary by 2.2. Thus m is interior to $p'(f)^{-1}(0)$, so $p'(f)$ vanishes near m, contradicting the assumption that p had least degree.)

6. Removable singularities.

We next note an analogue of the elementary removable singularities theorem for analytic functions.

THEOREM 6.1. *Suppose* $A|\partial$ *is an intersection of maximal subalgebras* A_α *of* $C(\partial)$, $m \in \mathscr{M}\backslash\partial$, *and* f *is a bounded continuous function on* $\mathscr{M}\backslash\{m\}$ *which is locally approximable by* A *on* $\mathscr{M}\backslash(\partial \cup \{m\})$. *Then* $f \in A|(\mathscr{M}\backslash\{m\})$.

For each $\alpha, f \circ \rho_\alpha$ is locally approximable by $\hat{A_\alpha}$ on $\mathscr{M}_{A_\alpha}\backslash(\rho_\alpha^{-1}(m) \cup \partial)$, while $\rho_\alpha^{-1}(m)$ is a hull-kernel closed set[13] contained in $\mathscr{M}_{A_\alpha}\backslash\partial_{A_\alpha} = \mathscr{M}_{A_\alpha}\backslash\partial$.

[13] For $\{m\}$ is hull-kernel closed in $\mathscr{M}$ and $\rho_\alpha : \mathscr{M}_{A_\alpha} \to \mathscr{M}$ is continuous even when hull-kernel topologies are used.

Thus by 4.9 (with $E = \phi$, $F = K = \rho_\alpha^{-1}(m)$), $f \circ \rho_\alpha \in A_\alpha^\wedge \mid (\mathscr{M}_A \backslash \rho_\alpha^{-1}(m))$, so $f \in A_\alpha \mid (\mathscr{M} \backslash \{m\})$ and $f \in A \mid (\mathscr{M} \backslash \{m\})$.

Trivially $\{m\}$ could just as well be any hull-kernel closed set in $\mathscr{M} \backslash \partial$. The result yields immediately an (imperfect) analogue of the elementary facts on behavior of analytic functions near isolated singularities.

COROLLARY 6.2. *Suppose $A \mid \partial$ is an intersection of maximal subalgebras of $C(\partial)$ and f is a continuous function on $\mathscr{M} \backslash \{m\}$, which is locally approximable on $\mathscr{M} \backslash (\partial \cup (m\})$. Then either*

(a) $f \in A \mid (\mathscr{M} \backslash \{m\})$

(b) $f = \text{const.} + 1/g$, $g \in A$, *and* $g^{-1}(0) = \{m\}$,

or

(c) *for each (deleted) open neighborhood V of m there is a compact K in $\boldsymbol{C}$ for which $f(V)$ is dense in $\boldsymbol{C} \backslash K$.*

Suppose (a) fails, so f cannot be bounded by 6.1. Let V be as in (c) and take $K = f(\mathscr{M} \backslash (V \cup \{m\}))$ which is compact. If (c) fails for this K then for some $z \in \boldsymbol{C} \backslash K$, $z - f$ is bounded away from zero on V; since it is also bounded away from zero on $\mathscr{M} \backslash (V \cup \{m\})$ (by the distance from z to K), $g = (1/(z - f)) \in C(\mathscr{M} \backslash \{m\})$, and is locally approximable on $\mathscr{M} \backslash (\partial \cup \{m\})$, so we can take g to be an element of A by 6.1. But now to obtain (b) we need only see that $g^{-1}(0) = \{m\}$; evidently g cannot vanish elsewhere, and if $g(m) \neq 0$ then g has a bounded inverse, whence f is bounded, contradicting our hypothesis that (a) fails.

Remark (Added in proof.) Wermer has pointed out the following completely elementary proof of 2.2, which actually applies if A is merely a multiplicative subsemigroup of $C(X)$. (For simplicity we shall suppose $\mathscr{F} = \{f\}$, a singleton):

For $x \in V$ we have a net $\{y_\delta\}$ in Y converging to x, with $f(y_\delta) \neq 0$ for each δ. Fixing δ, for $g \in A$ we have

$$\begin{aligned} |fg(y_\delta)| &\leqq \sup |fg(X)| = \sup |fg(X \backslash V)| \\ &\leqq \sup |f(X \backslash V)| \cdot \sup |g(X \backslash V)| \end{aligned}$$

so $|g(y_\delta)| \leqq c_\delta \sup |g(X \backslash V)|$, all g in A. Replacing g by its kth power and taking kth roots

$$|g(y_\delta)| \leqq c_\delta^{1/k} \sup |g(X \backslash V)|$$

whence $|g(y_\delta)| \leqq \sup |g(X \backslash V)|$. Since this holds for any δ,

$$|g(x)| \leqq \sup |g(X \backslash V)|, \; g \in A \,,$$

so $X \backslash V$ is a boundary, and $V \cap \partial = \phi$.

An even shorter (but nonelementary) proof can be obtained using Bishop's result on the existence of Jensen (representing) measures [3], as Bishop observes in his forthcoming paper "Conditions for analyticity of certain sets" (§ 3).

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