A PROOF OF THE NAKAOKA-TODA FORMULA

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If X_j $(1 \leqq j \leqq r)$ are objects we denote the corresponding r-tuple $(X_1, X_2, \cdots, X_r)$ by X and the $(r-1)$-tuple $(X_1, X_2, \cdots, X_{i-1}, X_{i+1}, \cdots, X_r)$ by $X(i)$. When X_j $(1 \leqq j \leqq r)$ are based topological spaces ΠX will denote their topological product and $\Pi^i X$ the subspace of ΠX whose points have at least i coordinates at base points (always denote by $*$).

Let $\alpha_j \in \pi_{n_j}(X_j)$ $(n_j \geqq 2, 1 \leqq j \leqq r, r \geqq 3)$ be elements of homotopy groups then we have

$$\star\alpha(\text{say}) = \alpha_1 \star \alpha_2 \star \cdots \star \alpha_r \in \pi_n(\Pi X, \Pi^1 X) ,$$

where $n = \Sigma n_j$ and $\star$ denotes the product of Blakers and Massey [1]. We thus also have

$$\star\alpha(i) \in \pi_{n-n_i}(\Pi X(i), \Pi^1 X(i)) .$$

There is a natural map $\Pi X(i), \Pi^1 X(i) \to \Pi^1 X, \Pi^2 X$ and we denote also by $\star\alpha(j)$ its image induced in $\pi_{n-n_i}(\Pi^1 X, \Pi^2 X)$. Let ∂ denote the homotopy boundary homomorphism in the exact sequence of the triple $(\Pi X, \Pi^1 X, \Pi^2 X)$. We shall prove the formula:

$$\partial\star\alpha = \Sigma(1 \leqq i \leqq r)(-1)^{\varepsilon(i)}[\alpha_i, \star\alpha(i)] \in \pi_{n-1}(\Pi^1 X, \Pi^2 X) , \qquad (0.1)$$

where $\varepsilon(1) = 0$, $\varepsilon(i) = n_i(n_1 + n_2 + \cdots + n_{i-1})$ $(i > 1)$ and where the brackets refer to the generalised Whitehead product of Blakers and Massey [1]. In the case of the universal example 0.1 becomes the formula of Nakaoka and Toda stated in [4] and proved there for $r = 3$. I. M. James[1] has raised the question of its validity for $r > 3$ and as the formula has applications (see [2], [3]) it would seem desirable to have a proof available in the literature. The present argument while inspired by [4] has a few novel features.

(1) DEFINITIONS AND LEMMAS. Let $x = (x_1, x_2, \cdots, x_n)$ denote a point of n-dimensional Euclidean space and let

$$\begin{aligned} V^n &= \{x; \Sigma x_i^2 \leqq 1\} , \\ S^{n-1} &= \{x; \Sigma x_i^2 = 1\} , \\ E_+^{n-1} &= \{x \in S^n; x_n \geqq 0\} , \\ E_-^{n-1} &= \{x \in S^n; x_n \leqq 0\} , \\ D_+^n &= \{x \in V^n; x_n \geqq 0\} , \end{aligned}$$

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$$D_-^n = \{x \in V^n; x_n \leqq 0\} ,$$
$$D_1^n = \{x \in V^n; x_1 \geqq 0\} ,$$
$$D_2^n = \{x \in V^n; x_1 \leqq 0\} .$$

We recall that if $Y \subseteqq X$ then X is a closed *n-cell* and Y is a *face* of X if there exists a homeomorphism $f: V^n \to X$ such that $f(E_+^{n-1}) = Y$. The subset $X^0 = f(S^{n-1})$ is the *boundary* of X. If X and Y are oriented cells we assign to $X \times Y$ the cross-product of the orientations of X and Y.

LEMMA 1.1. *Let X_1 be a face of the cell X and Y_1 a face of the cell Y. Then*

$$(X_1 \times Y) \cup (X \times Y_1) \text{ is a face of } X \times Y .$$

A proof of 1.1 may be found in [1] to which the reader may also refer for details concerning orientations. The proofs of the following two lemmas are standard exercises in homotopy theory and will be omitted.

LEMMA 1.2. *Suppose given a simplicial decomposition of a closed n-cell $F(n \geqq 3)$ and a subcomplex G which is a closed n-cell oriented coherently with F. If A is a simply-connected subset of a space Y and if $f: F \to Y$ is a map such that $f\{(F - G) \cup G^\circ\} \subseteqq A$ then $f: F, F^\circ \to Y, A$ and $f: G, G^\circ \to Y, A$ represent the same element of $\pi_n(Y, A)$.*

LEMMA 1.3. *Suppose given a simplicial decomposition of $V^{n+1}(n \geqq 3)$ and subcomplexes $F_i(i = 1, 2, \cdots, m)$ which are faces of V^{n+1} with disjoint interiors oriented coherently with S^n. Let A be a simply-connected subset of a simply-connected space Y, let $f: S^n \to Y$ be a map such that $f\{(S^n - \cup F_i) \cup (\cup F_i^\circ)\} \subseteqq A$, let $f: S^n \to Y$ represent $\alpha \in \pi_n(Y)$ and let $f: F_i, F_i^\circ \to Y, A$ represent $\alpha_i \in \pi_n(Y, A)$ $(i = 1, 2, \cdots, m)$. Then $j\alpha = \Sigma\alpha_i$ where $j: \pi_n(Y) \to \pi_n(Y, A)$ is the injection homomorphism.*

Let A be a simply-connected subset of a space Y. Let $f: V^p, S^{p-1} \to A, *$ and $g: V^q, S^{q-1}, E_+^{q-1} \to Y, A, *$ be representatives of $\alpha \in \pi_p(A)$ and $\beta \in \pi_q(Y, A)$. Let

$$h: S^{p-1} \times V^q \cup V^p \times E_+^{q-1}, S^{p-1} \times E_-^{q-1} \cup V^p \times S^{q-2} \to Y, A$$

be the map such that

$$h(x, y) = \begin{cases} f(x) & \text{if } (x, y) \in V^p \times E_+^{q-1} , \\ g(y) & \text{if } (x, y) \in S^{p-1} \times V^q . \end{cases}$$

Then if $S^{p-1} \times V^p \cup V^q \times E_+^{q-1}$ is oriented coherently with $V^p \times V^q$ we recall 3.1 of [1]:

DEFINITION 1.4. h represents $[\alpha, \beta] \in \pi_{p+q-1}(Y, A)$.

(2) *Proof of* 0.1. Let α_i be represented by a map

$$\psi_i : V^{n_i}, S^{n_i-1} \to X_i, *$$

with the property that

$$\psi_i(D_+^{n_i} \cup D_2^{n_i}) = * \,. \tag{2.1}$$

If we denote $V^{n_1} \times V^{n_2} \times \cdots \times V^{n_r}$ by V and $V^{n_1} \times V^{n_{i-1}} \times V^{n_{i+1}} \times \cdots \times V^{n_r}$ by $V(i)$ then $\star\alpha$ and $\star\alpha(i)$ are represented by maps

$$\psi : V, V^\circ \to \Pi X, \Pi^1 X \,,$$
$$\psi(i) : V(i), V(i)^\circ \to \Pi^1 X, \Pi^2 X$$

such that

$$\psi(x_1, \cdots, x_r) = (\psi_1(x_1), \cdots, \psi_r(x_r)) \,,$$
$$\psi(i)(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_r) \tag{2.2}$$
$$= (\psi_1(x_1), \cdots, \psi_{i-1}(x_{i-1}), *, \psi_{i+1}(x_{i+1}), \cdots, \psi_r(x_r)) \; (x_i \in V^{n_i}) \,.$$

Let $\rho_i : V^{n_i} \times V(i) \to V$ be the map such that

$$\rho_i(x_i, (x_1, \cdots, x_{i-1}, x_{i-1}, \cdots, x_r)) = (x_1, x_2, \cdots, x_r) \,.$$

As an easy consequence of our orientation convention we obtain:

LEMMA 2.3. *The degree of* ρ_i is $(-1)^{\varepsilon(i)}$.

$$\begin{array}{cccccccc}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 & 1 & - \\
1 & 1 & 0 & 1 & 1 & 1 & - & - \\
1 & 1 & 1 & 0 & 1 & - & - & - \\
1 & 1 & - & 1 & 0 & - & - & - \\
1 & 1 & - & - & - & 0 & - & - \\
1 & - & - & - & - & - & 0 & - \\
- & - & - & - & - & - & - & 0
\end{array}$$

r even

$$\begin{array}{ccccccc}
0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 & - \\
1 & 1 & 0 & 1 & 1 & - & - \\
1 & 1 & 1 & 0 & - & - & - \\
1 & 1 & - & - & 0 & - & - \\
1 & - & - & - & - & 0 & - \\
- & - & - & - & - & - & 0
\end{array}$$

r odd

The proof of 0.1 depends on the construction of certain closed cells $G_i \subsetneqq V(i)$ $(1 \leqq i \leqq r)$. Consider the two infinite arrays illustrated in the diagram. They contain between them exactly one centrally situated $r \times r$ matrix. Let $\eta(i, k, r)$ denote the symbol in the (i, k) position of this matrix. We define

$$G_i = \Pi D^{n_k}_{\eta(i,k,r)} ,$$

where topological product Π is taken over all values of k (in ascending order) except those for which $\eta(i, k, r) = 0$.

EXAMPLES If $r = 5$ then $G_2 = D_1^{n_1} \times D_1^{n_3} \times D_1^{n_4} \times D_-^{n_5}$.
If $r = 6$ then $G_4 = D_1^{n_1} \times D_-^{n_2} \times D_1^{n_3} \times D_-^{n_5} \times D_-^{n_6}$.

Certainly $G_i \subseteqq V(i)$. We shall refer later to the following property of the G_i which is obvious from the diagram.

LEMMA 2.4. *If $i < j \leqq r$ then there is an integer k with $i \neq k \neq j$ such that G_i has a factor $D_1^{n_k}$ and G_j a factor $D_-^{n_k}$.*

The proof of the following lemma we postpone.

LEMMA 2.5. *For each $i = 1, 2, \cdots, r$, there exists a face τ_i of G_i and of $V(i)$ such that if G_i has a factor $D_1^{n_k}$ then the projection of τ_i on $D_1^{n_k}$ does not intersect $D_-^{n_k}$ and such that if G_i has a factor $D_-^{n_k}$ then the projection of τ_i on $D_-^{n_k}$ does not intersect $D_1^{n_k}$.*

In view of 2.1 and 2.5 we have $\psi(i)(\tau_i) = *$. Moreover 2.1 and 2.2 imply that

$$\psi(i)\{(V(i) - G_i) \cup G_i^\circ\} \subseteqq \Pi^2 X .$$

Thus applying 1.2 (we may assume $\Pi^2 X$ simply-connected for this is certainly so in the case of the universal example) we obtain that

$$(\psi(i) \mid G_i) : G_i, G_i^\circ, \tau_i \to \Pi^1 X, \Pi^2 X, * \quad \text{represents } \star\alpha(i) . \tag{2.6}$$

We now define

$$F_i = \rho_i(S^{n_i - 1} \times G_i \cup V^{n_i} \times \tau_i) \qquad (1 \leqq i \leqq r)$$

and prove later:

LEMMA 2.7. *The F_i are faces of V with disjoint interiors. The map $(\psi\rho_i \mid \rho_i^{-1} F_i)$ has the property that*

$$(\psi\rho_i \mid \rho_i^{-1} F_i)(x, y) = \begin{cases} \psi_i(x) & \text{if } (x, y) \in V^{n_i} \times \tau_i , \\ \psi(i)(y) & \text{if } (x, y) \in S^{n_i - 1} \times G_i . \end{cases}$$

If we orient F_i coherently with V and $\rho_i^{-1} F_i$ coherently with $V^{n_i} \times V(i)$, 1.4 implies that $(\psi\rho_i \mid \rho_i^{-1}F_i)$ represents $[\alpha_i, \star\alpha(i)]$.

Since ρ_i is of degree $(-1)^{\varepsilon(i)}$, $(\psi \mid F_i)$ represents $(-1)^{\varepsilon(i)}[\alpha_i, \star\alpha(i)]$ and hence applying 1.3 the formula 0.1 follows in view of the commutativity in the diagram

$$\begin{tikzcd} \pi_n(\Pi X, \Pi^1 X) \arrow[r, "\partial"] \arrow[d, "d"] & \pi_{n-1}(\Pi^1 X, \Pi^2 X) \\ \pi_{n-1}(\Pi^1 X) \arrow[ur, "j"'] & \end{tikzcd}$$

where d denotes the boundary homomorphism in the homotopy sequence of the pair $(\Pi X, \Pi^1 X)$.

Proof of 2.5. Let D_0^n and $D_\times^n$ denote the subsets

$$D_0^n = \left\{x \in V^n;\, x_1 \geqq \frac{1}{2} \text{ and } x_n \geqq \frac{1}{2}\right\},$$

$$D_\times^n = \left\{x \in V^n;\, x_1 \leqq \frac{1}{2} \text{ and } x_n \leqq \frac{1}{2}\right\}.$$

Let $D \subseteqq G_i$ have a factor D_0^{nk} for every factor D_1^{nk} of G_i and a factor $D_\times^{nk}$ for every factor D_-^{nk} of G_i. Then certainly $\tau_i = D \cap V(i)^\circ$ has the desired property.

Proof of 2.7. If σ_i is the face of G_i complementary to τ_i then it may be observed that F_i is the face of $\rho_i(V^{ni} \times G_i)$ complementary to $\rho_i(V^{ni} \times \sigma_i)$. Thus

$$F_i^\circ = \rho_i(S^{ni-1} \times \sigma_i \cup V^{ni} \times \tau_i^\circ).$$

Suppose $i < j$ and let

$$\begin{aligned} H &= \rho_i(S^{ni-1} \times G_i) \cap \rho_j(S^{nj-1} \times G_j), \\ H' &= \rho_i(S^{ni-1} \times G_i) \cap \rho_j(V^{nj} \times \tau_j), \\ H'' &= \rho_i(V^{ni} \times \tau_j) \cap \rho_j(S^{nj-1} \times G_j). \end{aligned}$$

2.7 will follow when we have proved that $H \subseteqq F_i^\circ \cap F_j^\circ$, $H' = \emptyset$ and $H'' = \emptyset$. Since the images of H under the projections into V^{ni} and V^{nj} are contained in S^{ni-1} and S^{nj-1} respectively we have

$$H \subseteqq \rho_i(S^{ni-1} \times G_i^\circ) \cap \rho_i(S^{nj-1} \times G_j^\circ).$$

2.4 asserts the existence of an integer k with $i \neq k \neq j$ such that G_i has a factor D_1^{nk} and G_j a factor D_-^{nk}. Hence 2.5 implies that

$$H \cap \rho_i(S^{n_i-1} \times \tau_i) = H \cap \rho_j(S^{n_j-1} \times \tau_j) = \emptyset$$

and hence that

$$H \subseteqq \rho_i(S^{n_i-1} \times (G_i^\circ - \tau_i)) \cap \rho_j(S^{n_j-1} \times (G_j^\circ - \tau_j)) \subseteqq F_i^\circ \cap F_j^\circ \ .$$

2.5 also implies that $H' = H'' = \emptyset$ which completes the proof of 2.6.

References

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