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MAPS WHICH INDUCE THE ZERO MAP ON HOMOTOPY

DONALD W. KAHN

The work of Eilenberg and MacLane shows that a map from one space to another may induce the zero map on homotopy groups, yet be essential. The purpose of this paper is to give a characterization of such maps in terms of Postnikov decompositions of the spaces. As applications, we consider what additional information is needed to make such a map null-homotopic, and we prove a proposition concerning Chern classes.

In his paper [5], J.H.C. Whitehead characterized those maps which induce isomorphisms on homotopy groups by showing that (for reasonable spaces) they are exactly the homotopy equivalences. The more modest goal of this paper is to give a characterization of the maps which induce the zero homomorphism on homotopy groups. Here the result is not so simple, and the answer is given in terms of the Postnikov systems of the spaces involved. These maps occur in various cases, but for the purpose of illustration we consider the following two:

(1) The question of when such a map is null-homotopic and

(2) the example when the image space is B_U, the classifying space for the infinite unitary group.

Throughout this note, all spaces have the homotopy-type of a 1-connected CW-complex. All spaces have base points, which are preserved by maps and homotopies.

1. We shall use the following definition of a Postnikov system.

DEFINITION. A Postnikov system for X is a family $\{X_n, P_n, \pi_n\}$, $n > 0$, where X_n is a space and $P_n : X \to X_n$, $\pi_n : X_n \to X_{n-1}$, such that

(1) if X is $(m-1)$-connected, $X_i =$ point, $i < m$,

(2) P_n is an n-equivalence,

(3) π_n defines principal fibre space[1], with fibre $K(\pi_n(X), n)$, and

(4) $\pi_n P_n \simeq P_{n-1}$.

It is well-known that any X having the homotopy type of a 1-connected complex has a Postnikov system.[2] Furthermore, if $f: X \to X'$, where X and X' have as Postnikov systems $\{X_n, P_n, \pi_n\}$ and $\{X'_n, P'_n, \pi'_n\}$, then there are maps $f_n : X_n \to X'_n$ such that

(1) $\pi'_n \cdot f_n = f_{n-1} \cdot \pi_n$ and

(2) $f_n \cdot P_n \simeq P'_n \cdot f$. (See [2]).

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[1] With no loss of generality, we may take it to be a principal fibre bundle.

[2] If X was not 1-connected, but $\pi_1(X)$ was abelian, this would still hold.

It is easy to see that any two such f_n are necessarily homotopic, and we refer to any such map as an induced map for f.

THEOREM 1. *Let X and X' have Postnikov systems $\{X_n, P_n, \pi_n\}$ and $\{X'_n, P'_n, \pi'_n\}$. Let $f: X \to X'$ have induced maps $f_n: X_n \to X'_n$. If for $i > N > 1, f_i: \pi_i(X) \to \pi_i(X')$ is the zero map, then for each $i > N$, there is a map $h_i: X_{i-1} \to X'_i$ such that the diagram*

$$\begin{tikzcd} X_i \arrow[r, "f_i"] \arrow[d, "\pi_i"'] & X'_i \arrow[d, "\pi'_i"] \\ X_{i-1} \arrow[r, "f_{i-1}"] \arrow[ur, dashed, "h_i"] & X'_{i-1} \end{tikzcd}$$

is homotopy commutative.

Proof. Let k'^{i+1} denote the k-invariant for the fibration $\pi'_i: X'_i \to X'_{i-1}$. Up to sign, k'^{i+1} is the obstruction to forming a cross-section to this fibration, or an equivalent one over a base space which is actually a complex. The obstruction to lifting f_{i-1} to X'_i is then $f^*_{i-1}k'^{i+1}$. But as $f^*_{i-1}k'^{i+1} = f^c_i k^{i+1}$, where f^c_i is the coefficient homomorphism (see [2]), we have $f^*_{i-1}k'^{i+1} = 0$, for $i > N$. Hence, when $i > N$, there is a map $\bar{h}_i: X_{i-1} \to X'_i$ with $\pi'_i\bar{h}_i = f_{i-1}$.

Consider the maps f_i and $\bar{h}_i\pi_i$. Because $\pi'_i f_i = f_{i-1}\pi_i = \pi'_i\bar{h}_i\pi_i$, there is a map: $d: X_i \to K(\pi_i(X'_i), i)$ such that

$$\mu\cdot(d \times \bar{h}_i\pi_i) = f_i$$

where $\mu: K(\pi_i(X_i), i) \times X_i \to X_i$ is the usual action of the fibre. If $x \in X_i, d(x)$ is the unique element of $K(\pi_i(X_i), i)$ such that $f_i(x) = d(x)\cdot(\bar{h}_i\pi_i(x))$.

Now $\bar{h}_i$ maps the base point in the space X_{i-1} into the base point in the fibre over the base point in X'_{i-1}. Then $\bar{h}_i\cdot\pi_i$ maps the fibre in $\pi_i: X_i \to X_{i-1}$ into the identity, so that $d \mid K(\pi_i(X_i), i) = f_i$. Since f_i induces the zero map on homotopy, the composition

$$K(\pi_i(X_i), i) \xrightarrow{i} X_i \xrightarrow{d} K(\pi_i(X'_i), i)$$

is null-homotopic. Using the Serre sequence (see [4]) for the fibre space $\pi_i: X_i \to X_{i-1}$, we see that there is a map $\bar{d}$ so that

$$\begin{tikzcd} X_i \arrow[r, "d"] \arrow[d, "\pi_i"'] & K(\pi_i(X'_i), i) \\ X_{i-1} \arrow[ur, dashed, "\bar{d}"] & \end{tikzcd}$$

is homotopy commutative. Hence,

$$(d \times \bar{h}_i\pi_i) \simeq (\bar{d}\pi_i \times \bar{h}_i\pi_i) \simeq (\bar{d} \times \bar{h}_i)\pi_i \ ,$$

and then

$$\mu(\bar{d} \times \bar{h}_i)\cdot\pi_i \simeq f_i$$

while

$$\pi'_i\mu(\bar{d} \times \bar{h}_i) = \pi'_i\bar{h}_i = f_{i-1} \ .$$

Therefore, we set $h_i = \mu(\bar{d} \times \bar{h}_i)$, and the proof is complete.

THEOREM 2. *Let X and X' have Postnikov systems $\{X_n, P_n, \pi_n\}$ and $\{X'_n, P'_n, \pi'_n\}$. Let $f: X \to X'$ with induced maps f_n.*

f induces the zero map in homotopy in all positive dimensions, if and only if for each $i > 1$, there is a map h_i: $X_{i-1} \to X'_i$ *such that*

$$\begin{array}{ccc} X_i & \xrightarrow{f_i} & X'_i \\ {\scriptstyle \pi_i}\downarrow & \nearrow{\scriptstyle h_i} & \downarrow{\scriptstyle \pi'_i} \\ X_{i-1} & \xrightarrow[f_{i-1}]{} & X'_{i-1} \end{array}$$

is homotopy commutative.

Proof. If f induces the zero map, apply Theorem 1 with $N = 1$.

For the converse, if $i \leqq n$, we identify $\pi_i(X)$ with $\pi_i(X_i)$ and the same for X'. Then, we may identify f_i with $(f_n)_i : \pi_i(X_n) \to \pi_i(X'_n)$. But $f_{n\#} = (h_n \cdot \pi_n)_\#$, which is always zero in dimension n because $\pi_n(X_{n-1}) = 0$.

2. In general, if a map $f: X \to X'$ induces the zero map on homotopy groups, it need not be null homotopic. (Eilenberg-MacLane spaces give many examples, and with slight modifications one may take $X = X'$.) I will first consider conditions which insure that such maps are null-homotopic.

PROPOSITION 1. Assume X has the homotopy-type of a finite-dimensional complex. Let $f: X \to X'$. Suppose either

(1) $H^n(X; \pi_n(X')) = 0$, all n. (This for comparison)

or (2) $H^n(X_{n-1}; \pi_n(X')) = 0$, all n, and f induces the zero map on homotopy.

Then f is null-homotopic.

Proof. It is sufficient to show that $P'_n \cdot f \simeq 0$ for all n. The case of condition (1) is trivial. In case (2), $P'_n \cdot f \simeq f_n \cdot P_n \simeq h_n \cdot \pi_n \cdot P_n \simeq h_n \cdot P_{n-1}$. Assuming that $P'_{n-1} \cdot f \simeq 0$, $f_{n-1} \cdot P_{n-1} \simeq 0$, then $\pi'_n \cdot h_n \cdot P_{n-1} \simeq 0$. But then, by the Puppe sequence, h_n comes from a map into the fibre of $\pi'_n : X'_n \to X'_{n-1}$, which is null-homotopic by assumption.

REMARK. If there is other structure present, one can often say more. For example, let $f: X \to X'$ be a homomorphism of H-spaces. Then if X is $(m-1)$-connected, $f_{\#m} = 0$, and $H^n(X; \pi_n(X'))$ contains no primitive elements for $n > m$, then f is null-homotopic.

For the next case, we need some notation. Suppose X has finite homotopy groups. Denote by $\mathfrak{C}(X, n)$ the class of finite abelian groups (see [3]) generated by $\pi_i(X)$, $i \leqq n$. In other words, $\mathfrak{C}(X, n)$ is the class of torsion groups whose p-components are zero for all primes which do not figure in the homotopy of X_n.

PROPOSITION 2. Let X have finite homotopy groups, and let ξ be a U-bundle over X. Let $c_i(\xi) \in H^{2i}(X; Z)$ be the ith. Chern class (see [1]). Then $c_i(\xi)$ is contained in a subgroup of $H^{2i}(X; Z)$ which belongs to $\mathfrak{C}(X; 2i-1)$.

Proof. Let ξ be given as a map

$$f_\xi : X \to B_U .$$

f_ξ must induce the zero map on homotopy groups. By Theorem 2, $(f_\xi)_n : X_n \to (B_U)_n$ factors through X_{n-1}. Now, the ith universal Chern class lies in $H^{2i}((B_U)_{2i}; Z)$ so that $c_i(\xi) \in \pi^*_{2i}(H^{2i}(X_{2i-1}; Z))$. But by (3), we know that

$$H^{2i}(X_{2i-1}; Z) \in \mathfrak{C}(X, 2i-1) .$$

REMARKS. This proposition clearly holds for spaces whose even dimensional homotopy groups are finite. Furthermore, under suitable hypotheses, remarks of this sort may be made about other characteristic classes. Details are left to the reader.

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COLUMBIA UNIVERSITY
UNIVERSITY OF MINNESOTA