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A COUNTER-EXAMPLE TO A LEMMA OF SKORNJAKOV

B. L. OSOFSKY

In his paper, Rings with injective cyclic modules, translated in Soviet Mathematics 4 (1963), p. 36–39, **L. A. Skornjakov states the following lemma: If a cyclic R-module M and all its cyclic submodules are injective, then the partially ordered set of cyclic submodules of M is a complete, complemented lattice.**

An example is constructed to show that this lemma is false, thus invalidating Skornjakov's proof of the theorem: Let R be a ring all of whose cyclic modules are injective. Then R is semi-simple Artin. The theorem, however, is true. (See Osofsky [4].)

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In this paper, all rings have identity and all modules are unital left modules. $_R\mathfrak{M}$ will denote the category of R-modules, and $_RM$ will signify $M \in {}_R\mathfrak{M}$.

Let Q be a commutative, left self injective, regular, non-Artin ring, and let I be a maximal ideal of Q which is not a direct summand of $_QQ$. (For example, let Q be a direct product of fields, and I a maximal ideal containing their direct sum.) Let $N = Q \oplus Q/I$. We observe the following:

1. $_QN$ is injective. Q is injective by hypothesis, and Q/I is a simple module over the commutative regular ring Q; hence injective by a theorem of Kaplansky. (See [5].)

2. $_QM \subseteqq {}_QN$ is a direct summand of $_QN$ if and only if $_QM$ is finitely generated. If $_QM$ is a direct summand of $_QN$, $_QM$ is generated by the projections of $(1, 0 + I)$ and $(0, 1 + I)$. If $_QM$ is finitely generated, and π is the projection of N onto $(Q, 0 + I)$, then $\pi(_QM)$ is finitely generated. Hence $\pi(_QM)$ is a direct summand of $_QQ$. (See von Neumann [6].) Say $Q = \pi(_QM) \oplus K$. Since $\pi(_QM)$ is projective (it is a direct summand of Q), $_QM = (\pi(_QM))' \oplus (\operatorname{Ker} \pi \cap {}_QM)$. Since Q/I is simple, $Q/I = (\operatorname{Ker} \pi \cap {}_QM) \oplus K_2$ where $K_2 = 0$ or Q/I. Then $N = M \oplus K \oplus K_2$.

3. The direct summands of N do not form a lattice. In particular, $Q(1, 0 + I) \cap Q(1, 1 + I) = (I, 0 + I)$ is not a direct summand

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of $(Q, 0 + I)$, hence not of N.

N is not a counter-example to Skornjakov's lemma, since N is not cyclic. However, properties 1, 2 and 3 are preserved under category isomorphisms. For we have:

PROPOSITION. ${}_RM$ is finitely generated $\Leftrightarrow$ the union of a linearly ordered chain of proper submodules is proper.

Proof. $\Rightarrow$ Let $M = \sum_{i=1}^n Rx_i$, and let $\{N_\mu\}$ be a linearly ordered chain of submodules whose union is M. If $x_i \in N_{\mu_i}$, then $\{x_i \mid i = 1, \cdots, n\} \subseteqq N_\nu$, where $\nu = \max\{\mu_i \mid 1 \leqq i \leqq n\}$. Then $M = N_\nu$.

$\Leftarrow$ Given ${}_RM$, let $\aleph$ be the smallest cardinal such that M has a generating set of cardinality $\aleph$. Index such a generating set $\{x_\mu\}$ by $\{\mu \mid \mu < \Omega\}$, where Ω is the first ordinal of cardinality $\aleph$. Then $\{\sum_{\nu \leqq \mu} Rx_\nu\}$ is a linearly ordered chain of submodules whose union is M. If Ω is a limit ordinal (that is, if $\aleph$ is infinite), then each $\sum_{\nu \leqq \mu} Rx_\nu$ is generated by less than $\aleph$ elements; hence proper.

Thus M finitely generated corresponds to the categorical property that the collection of nonepimorphic monomorphisms into M is inductive under the ordering: $f \leqq g$ if and only if there is an h with $f = gh$.[1]

Let $R = \mathrm{Hom}_Q(Q \oplus Q, Q \oplus Q)$. By Morita [3], Theorem 3.4, the functor $\mathrm{Hom}_Q(Q \oplus Q, \ \ ) \colon {}_Q\mathfrak{M} \to {}_R\mathfrak{M}$ is a category isomorphism. Hence ${}_RM = \mathrm{Hom}_Q(Q \oplus Q, N)$ has properties 1, 2, 3. Moreover, if $K = \{\lambda \in R \mid (Q \oplus Q)\lambda \subseteqq (0, I)\}$, then M is isomorphic to R/K since ${}_Q(Q \oplus Q)$ projective implies the natural map from $R = \mathrm{Hom}_Q(Q \oplus Q, Q \oplus Q) \to \mathrm{Hom}_Q(Q \oplus Q, Q \oplus Q/I) = M$ is an epimorphism. Hence M is cyclic, and as in 2, every direct summand of M is cyclic. Thus M is the required counter-example.

We conclude with the observation that the technique used in 2 gives us a categorical equivalence to regular rings which is closer to the usual definition than Auslander's theorem that R is regular if and only if the global flat dimension of R is 0. (See Auslander [1].)

$P \in {}_R\mathfrak{M}$ is a progenerator if it is finitely generated, projective, and every $M \in {}_R\mathfrak{M}$ is an epimorphic image of a direct sum of copies of P.

PROPOSITION. The following are equivalent:

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[1] Although the categorical definition of finitely generated appears in H. Bass, *The Morita theorems*, University of Oregon (mimeographed notes), the author found no proof in the literature that this is equivalent to the module definition, and so is including this proof for completeness.

(a) R is regular.

(b) Every finitely generated submodule of a projective module is a direct summand.

(c) There is a progenerator $P \in {}_R\mathfrak{M}$ such that every finitely generated submodule of P is a direct summand.

Proof. (b) $\Rightarrow$ (a) (See von Neumann [6].)

(a) $\Rightarrow$ (c) R is a progenerator with the required properties.

(c) $\Rightarrow$ (b) Let N be a projective module, M a finitely generated submodule.

Let P be the progenerator of condition (c). Then there is an epimorphism $f: \Sigma \oplus P_i \rightarrow N$. Since N is projective, this splits and $\Sigma \oplus P_i = N' \oplus \ker f$, where $N' \approx N$. Thus M is a finitely generated submodule of $\Sigma \oplus P_i$, and if it is a direct summand of $\Sigma \oplus P_i$, it is a direct summand of N.

Since M is finitely generated, M is contained in a finite direct sum $\sum_{j=1}^{n} P_j$. If $n = 1$, M is a direct summand of P by hypothesis, and hence a direct summand of $\Sigma \oplus P_i$. Now assume any finitely generated submodule of $\sum_{j=1}^{n-1} P_j$ is a direct summand. Let π_n be the projection of $\sum_{j=1}^{n} P_j$ onto P_n. Then $\pi_n(M)$ is a direct summand of P_n, say $P_n = \pi_n(M) \oplus K_1$. Ker $\pi_n \cap M$ is a direct summand of M, hence finitely generated. Then by the induction hypothesis, $\sum_{j=1}^{n-1} P_j =$ (Ker $\pi_n \cap M) \oplus K_2$. Then $\sum_{j=1}^{n} P_j = K_1 \oplus K_2 \oplus M$, so M is a direct summand of $\Sigma \oplus P_i$, and hence of N.

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RUTGERS, THE STATE UNIVERSITY