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GENERALIZATION OF A THEOREM OF MARCINKIEWICZ

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Let $P(z)$ be a polynomial of degree $m > 2$ and $g(z)$ an entire function of order less than m. According to a result of Marcinkiewicz the function $g(z)\exp\{P(z)\}$ cannot be the characteristic function of a probability distribution. The special case, that $\exp\{P(z)\}$ cannot be a characteristic function, is generally known as Marcinkiewicz's theorem. In the present paper it is shown that if $f(z)$ is any nonconstant entire function then neither $g(z)f[\exp\{P(z)\}]$ nor $f\{P(z)\}$ can be characteristic functions. Also, necessary and sufficient conditions are discussed for functions of the form $f[\exp\{P(z)\}]$ to be characteristic functions.

1. Marcinkiewicz's theorem and its extensions. Let $F(x)$ be a distribution function, that is a nondecreasing, right-continuous function satisfying $F(-\infty) = 0$, $F(\infty) = 1$. The Fourier-Stieltjes transform

$$\phi(z) = \int_{-\infty}^{\infty} e^{izx} dF(x) , \tag{1.1}$$

which always exists for real z, is the characteristic function of $F(x)$. We shall be interested in cases where $\phi(z)$ exists for all complex z and under such circumstances $\phi(z)$ is an entire function of z (Lukacs [4], p. 132). One of the problems connected with characteristic functions is that of characterizing them, i.e., given a function, can we say whether or not it is a characteristic function. Necessary and sufficient conditions are given by Bochner's theorem (see e.g. Lukacs, [4], p. 62) but these are difficult to apply in individual cases and so it seems worthwhile to seek characterizations of a more particular kind.

If $\phi(z)$ is an entire function, then the moment generating function (m.g.f.),

$$M(t) = \int_{-\infty}^{\infty} e^{tx} dF(x) , \tag{1.2}$$

is an entire function of t. We prefer to work with the m.g.f. rather than the characteristic function since this avoids frequent and slightly inconvenient multiplications by i.

In connection with the characterization of entire m.g.f.'s, Marcinkiewicz [5] proved a strong necessary condition, namely that an entire function of finite order $\rho > 2$, the exponent of convergence of whose zeros is less than ρ, can not be a m.g.f. In particular this result implies that if $P(t)$ is a polynomial then $\exp\{P(t)\}$ is a m.g.f. if and only if $P(t) = a_2t^2 + a_1t$ with $a_2 \geqq 0$ and a_1 real. This latter result is usually known as the theorem of Marcinkiewicz. Lukacs ([4], p. 146) has extended this result to functions of the form $c_ke_k\{P(t)\}$ where $e_k(z)$ is the k^{th} iterated exponential function defined by $e_1(z) = e^z$, $e_k(z) = \exp\{e_{k-1}(z)\}(k = 2, 3, \cdots)$ and c_k is a normalizing constant. Lukacs [3] has also shown that the function

$$\exp[\lambda_1(e^t - 1) + \lambda_1(e^{-t} - 1) + P(t)] \tag{1.3}$$

is a m.g.f. if and only if $\lambda_1 \geqq 0$, $\lambda_2 \geqq 0$ and $P(t) = a_2t^2 + a_1t$ with $a_2 \geqq 0$ and a_1 real. Some further extensions of Marcinkiewicz's theorem have been given by Christensen [2], who shows, in particular, that for certain specified m.g.f.'s $g(t)$, a function of the form

$$c_kg(t)e_k\{P(t)\}$$

cannot be a m.g.f. if the degree of $P(t)$ exceeds 2. Some of the results of Ostrovskiĭ [6] partially overlap those of the present paper; see Section 7.

Some further generalizations are stated in the following section and proved in subsequent sections. We rely on certain elementary properties of m.g.f.'s. Firstly the function $M(t)$ defined by (1.2) is obviously real and positive when t is real. Further, $M(t)$ is a strictly convex function of t when t is real unless $M(t) \equiv 1$ (Lukacs, [4], p. 136). Further if $t = u + iv(u, v$ real) then

$$|M(u + iv)| \leqq M(u) \tag{1.4}$$

or, writing $t = re^{i\theta}$,

$$|M(re^{i\theta})| \leqq M(r\cos\theta)\,. \tag{1.5}$$

In establishing that certain functions are not m.g.f.'s we shall, in common with previous authors, show that these functions contradict the elementary inequality (1.4) or (1.5).

2. Statement of results. Essentially, Marcinkiewicz's results can be stated as follows: if $P(t)$ is a polynomial of degree $m > 2$ and if $g(t)$ is an entire function of order $\rho < m$, then $g(t)\exp\{P(t)\}$ cannot be a m.g.f. More generally, we prove the following.

THEOREM 1. *Let $f(t)$ be a nonconstant entire function, $P(t)$ a polynomial of degree $m > 2$ and $g(t)$ an entire function of order $\rho < m$. Then $g(t)f[\exp\{P(t)\}]$ cannot be a m.g.f.*

COROLLARY. *If $f(t)$ is a nonconstant entire function and $P(t)$*

a polynomial of degree greater than 2, *then* $f[\exp\{P(t)\}]$ *cannot be a m.g.f.*

Necessary and sufficient conditions for $f[\exp\{P\{(t)\}]$ to be a m.g.f. are available if we restrict the class of entire functions as in the following theorem.

THEOREM 2. *If* $f(t) = \sum_{n=0}^{\infty} f_n t^n$ *is a nonconstant entire function satisfying* $f(1) = 1, f_n \geqq 0 (n = 0, 1, \cdots)$ *and if* $P(t) = a_1 t + \cdots + a_m t^m$, *then* $f[\exp\{P(t)\}]$ *is a m.g.f. if and only if* $P(t) = a_1 t + a_2 t^2$ *with* a_1, a_2 *real and* $a_2 \geqq 0$.

It may be thought that the condition of nonnegativity on the coefficients f_n is a necessary condition for $f[\exp\{P(t)\}]$ to be a m.g.f. when $P(t) = a_1 t + a_2 t^2 (a_1$ real, $a_2 > 0)$. (It clearly is necessary if $a_2 = 0$.) That is not necessary is shown by the simple example given by taking $f(t) = 2t^2 - t, P(t) = t^2/2$, so that

$$f[\exp\{P(t)\}] = 2e^{t^2} - e^{(t^2/2)} ,$$

which is the m.g.f. of

$$dF(x) = \left\{\frac{1}{\sqrt{\pi}} e^{-(x^2/4)} - \frac{1}{\sqrt{2\pi}} e^{-(x^2/2)}\right\} dx .$$

(It is easily verified that $(d/dx)F(x) \geqq 0$.)

However, we can, as in the following theorem, write down necessary and sufficient conditions for $f[\exp\{P(t)\}]$ to be a m.g.f. without restrictions on $f(t)$. But these conditions are rather obvious and at the same time difficult to apply to individual cases; i.e., it would be difficult to determine whether a given entire function $f(t) = \sum f_n t^n$ satisfies the condition (2.1) below.

THEOREM 3. *If* $f(t) = \sum_{n=0}^{\infty} f_n t^n$ *is a nonconstant entire function and if* $P(t) = a_1 t + \cdots + a_m t^m$, *then* $f[\exp\{P(t)\}]$ *is a m.g.f. if and only if* $P(t) = a_1 t + a_2 t^2$ *with* a_1 *real,* f_n *is real* $(n = 0, 1, \cdots)$, $f_0 \geqq 0$, $f(1) = 1$ *and*

(i) $a_2 > 0$ *and*

$$\sum_{n=1}^{\infty} f_n n^{-1/2} \left\{ \exp\left(-\frac{n a_1^2}{4 a_2}\right)\right\} y^{1/n} \geqq 0 \qquad (0 < y \leqq 1) \tag{2.1}$$

or

(ii) $a_2 = 0$ *and* $f_n \geqq 0 \quad (n = 1, 2, \cdots)$.

One may also ask what may be said of functions of the type

$f\{P(t)\}$ where f is an entire function. Clearly, even if $P(t)$ were of degree 2, f would have to be rather special for $f\{P(t)\}$ to be a m.g.f. However, in the following theorem we show that we may rule out all entire functions if the degree of $P(t)$ exceeds 2.

THEOREM 4. *Let $f(t)$ be a nonconstant entire function and $P(t)$ a polynomial of degree greater than 2. Then $f\{P(t)\}$ cannot be a m.g.f.*

Finally, the following result generalizes that of Christensen ([2], Theorem 3.1) and partially generalizes that of Lukacs ([3], p. 489 or [4], p. 158), in connection with (1.3).

THEOREM 5. *Let $g(t) = \sum_{n=-\infty}^{\infty} g_n t^n$, $g_n \geqq 0$ $(n = 0, \pm 1, \cdots)$, be regular and nonconstant for $0 < |t| < \infty$ and let $f(t) = \sum_{n=0}^{\infty} f_n t^n$, $f_n \geqq 0$ $(n = 0, 1, \cdots)$, be a nonconstant entire function. If $P(t) = a_1 t + \cdots + a_m t^m$ and if α is real, then $g(e^{\alpha t}) f[\exp\{P(t)\}]$ is a m.g.f. if and only if $g(1)f(1) = 1$ and $P(t) = a_1 t + a_2 t^2$ with a_1, a_2 real and $a_2 \geqq 0$.*

3. Proof of Theorem 1.

We require the following lemma.

LEMMA A. *Let R be a large positive number and let $\phi(R)$ be a bounded function of R. Let*

$$P(t) = a_m t^m + a_{m-1} t^{m-1} + \cdots + a_1 t + a_0 \qquad (m \geqq 1)$$

where $a_m = \alpha_m \exp(i\beta_m) \neq 0$. Then the roots $t_k(R)$ of the equation $P(t) = R + i\phi(R)$ satisfy

$$t_k(R) \sim \left(\frac{R}{\alpha_m}\right)^{1/m} \exp\left\{\frac{(2k\pi - \beta_m)i}{m}\right\} \qquad (R \to \infty;\ k = 1, \cdots, m)$$

Proof. Clearly, the result is exact if $P(t) = a_m t^m$ and $\phi(R) \equiv 0$. The result is also intuitively clear in general, since $P(t) \sim a_m t^m$ and $R + i\phi(R) \sim R$ for large $|t|$ and R respectively. However, a proof is easily obtained by means of Rouché's theorem. Without loss of generality we may take $a_m = 1$, for otherwise we make a change of variable $s = \{\alpha_m \exp(i\beta_m/m)\}t$. The result is clear if $m = 1$. Suppose $m > 1$ and let $R = C^m (C > 0)$. Define

$$\begin{aligned} A(t) &= t^m + a_{m-1} t^{m-1} + \cdots + a_1 t + a_0 - C^m - i\phi(R) \, , \\ B(t) &= t^m - A(t) - C^m \, . \end{aligned}$$

For given $\varepsilon, 0 < \varepsilon < \sin(\pi/m)$, consider a circle with centre $C \exp(2i\pi/m)$ and radius εC. For t on this circle and for C large, it is easily seen that

$$\left|\frac{B(t)}{A(t)}\right| = O(C^{-1}) ,$$

and hence for C sufficiently large $|B(t)/A(t)| < 1$, so that $A(t)$ and $A(t) + B(t)$ have the same number of zeros inside the circle. But $A(t) + B(t) = t^m - C^m$ has exactly one zero in this circle, namely $C \exp(2i\pi/m)$. The corresponding zero, $t(C)$ say, of $A(t)$ therefore satisfies $|t(C) - C\exp(2i\pi/m)| < \epsilon C$. Hence

$$t(C) \sim C \exp(2i\pi/m) \qquad (C \to \infty) .$$

The conclusion of the lemma therefore follows for $k = 1$ and similarly for $k = 2, \cdots, m$.

We proceed now to the proof of Theorem 1. Let $F(R)$ be the maximum modulus of $f(z)$ on the circle $|z| = e^R$ and suppose that this maximum is attained at a point $z = \exp\{R + i\phi(R)\}$ where $0 \leqq \phi(R) < 2\pi$. Let

$$P(t) = \sum_{j=0}^{m} a_j t^j$$

where $a_m = \alpha_m \exp(i\beta_m) \neq 0$ $(0 \leqq \beta_m < 2\pi)$. Let $t_R = t_1(R)$ be a root of the equation $P(t) = R + i\phi(R)$, so that by Lemma A,

$$t_R \sim \left(\frac{R}{\alpha_m}\right)^{1/m} \exp\left\{\frac{(2\pi - \beta_m)i}{m}\right\} \qquad (R \to \infty) . \tag{3.1}$$

If $t_R = u_R + iv_R$ (u_R, v_R real) then as $R \to \infty$,

$$u_R \begin{cases} \sim \left(\dfrac{R}{\alpha_m}\right)^{1/m} \cos\left(\dfrac{2\pi - \beta_m}{m}\right) & \left(\cos\left(\dfrac{2\pi - \beta_m}{m}\right) \neq 0\right) \\ = O(R^{1/m}) & \left(\cos\left(\dfrac{2\pi - \beta_m}{m}\right) = 0\right) . \end{cases}$$

Hence for large R it follows that

$$\mathscr{R}[P(u_R)] \begin{cases} \sim R \cos\beta_m \cos^m\left(\dfrac{2\pi - \beta_m}{m}\right) & \left(\cos\beta_m \neq 0, \cos\left(\dfrac{2\pi - \beta_m}{m}\right) \neq 0\right) , \\ = o(R) & (\text{otherwise}) . \end{cases}$$

Now for $m \geqq 3$ and for any θ satisfying $0 < \theta \leqq 2\pi$ we have $|\cos(\theta/m)| < 1$. It follows that $|\cos^m\{(2\pi - \beta_m)/m\}| < 1$ and hence

$$\mathscr{R}[P(t_R) - P(u_R)] = R - \mathscr{R}[P(u_R)] > KR \tag{3.2}$$

for all sufficiently large R and some fixed $K > 0$.

Now if $f(z)$ is not a linear function, i.e., $f(z) \neq f_0 + f_1 z$, then the function

$$\frac{1}{r} \max_{|z|=r} |f(z)| \tag{3.3}$$

is ultimately a steadily increasing function of r. This can be seen by applying the maximum modulus principle to the function $f(z)/z$ in the annulus $0 < r' < |z| < r$ for r' fixed and r increasing. If $f(z) = f_0 + f_1 z (f_1 \neq 0)$, then the function (3.3) tends to a finite limit, namely $|f_1|$, as $r \to \infty$. In all cases, however, it follows that if $R > R'$ and if R is sufficiently large, then

$$\frac{F(R)}{e^R} > c\,\frac{F(R')}{e^{R'}} ,$$

$$\text{i.e.,} \quad \frac{F(R)}{F(R')} > ce^{R-R'} , \tag{3.4}$$

for a fixed $c(0 < c \leqq 1)$. We may take $c = 1$ if $f(z)$ is nonlinear, but we must take $0 < c < 1$ if $f(z)$ is linear. It therefore follows that for all sufficiently large R,

$$\begin{aligned} \left|\frac{f[\exp\{P(t_R)\}]}{f[\exp\{P(u_R)\}]}\right| &= \frac{F(R)}{|f[\exp\{P(u_R)\}|} \\ &\geqq \frac{F(R)}{F[\mathscr{R}\{P(u_R)\}]} \\ &> c \exp[R - \mathscr{R}\{P(u_R)\}] \\ &> e^{K_1 R} \qquad (K_1 > 0) , \end{aligned} \tag{3.5}$$

the last inequality following from (3.2).

We now turn to the function $g(t)$. Suppose $g(t)$ has an infinity of zeros, $\tau_n = r_n e^{i\theta_n} (n = 1, 2, \cdots)$, where $r_1 \leqq r_2 \leqq \cdots$. If

$$\varepsilon(0 < \varepsilon < m - \rho)$$

is given then outside the circles with centre τ_n and radius r_n^{-2m} we have, according to a theorem of Borel (Cartwright, [1], p. 22), that

$$\log |g(t)| > - |t|^{\rho+\varepsilon} \qquad (|t| > T(\varepsilon)) . \tag{3.6}$$

Further, since $g(t)$ is of order ρ, we have

$$\log |g(t)| < |t|^{\rho+\varepsilon} \qquad (|t| > T_1(\varepsilon)) . \tag{3.7}$$

It $g(t)$ has no zeros, or a finite number of zeros, then (3.6) holds *a fortiori* for all $|t|$ sufficiently large and (3.7) also holds.

Now define

$$M(t) = g(t) f[\exp\{P(t)\}] .$$

Then

$$\log\left|\frac{M(t_R)}{M(u_R)}\right| = \log|g(t_R)| - \log|g(u_R)| + \log\left|\frac{f[\exp\{P(t_R)\}]}{f[\exp\{P(u_R)\}]}\right| . \tag{3.8}$$

Consider the sequence of values $P(\tau_n)(n = 1, 2, \cdots)$. If $\mathscr{R}\{P(\tau_n)\}$ is bounded above as $n \to \infty$, then for R sufficiently large, all the points t_R are outside the circles with centre τ_n and radius r_n^{-2m}. We may therefore apply the inequality (3.6) to (3.8). Using also (3.5) and remembering that $\rho + \varepsilon < m$ we obtain

$$\log\left|\frac{M(t_R)}{M(u_R)}\right| > -|t_R|^{\rho+\varepsilon} - |u_R|^{\rho+\varepsilon} + K_1 R \tag{3.9}$$

for R sufficiently large, in virtue of (3.6) and (3.1). If $\mathscr{R}\{P(\tau_n)\}$ is not bounded above, let $R_1 \leqq R_2 \leqq \cdots, R_n \to \infty$, denote all the positive values of $\mathscr{R}\{P(\tau_n)\}$ and let $\sigma_1, \sigma_2, \cdots$ denote the corresponding members of the sequence $\{\tau_n\}$. Let t', t'' be any two points in the circle with centre σ_n and radius $|\sigma_n|^{-2m}$. For all σ_n sufficiently large we have

$$|P(t') - P(t'')| = O(|\sigma_n|^{-m-1})$$

and we can therefore find a constant K_2 such that

$$|\mathscr{R}\{P(t')\} - \mathscr{R}\{P(t'')\}| < K_2 |\sigma_n|^{-m-1} \qquad (n = 1, 2, \cdots) .$$

Hence if $R > 0$ lies outside the intervals

$$R_n - K_2\sigma_n^{-m-1}, R_n + K_2\sigma_n^{-m-1} \qquad (n = 1, 2, \cdots) \tag{3.10}$$

then t_R lies outside the circles with centre τ_n and radius $|\tau_n|^{-2m}$. The sum of the lengths of the intervals (3.10) is $2K_2 \sum \sigma_n^{-m-1}$ which is finite since $m + 1$ exceeds the order ρ of $g(t)$. Hence we can let $R \to \infty$ outside the intervals (3.10) and so again we obtain the inequality (3.9). We have thus contradicted (1.4) and $M(t)$ cannot be a m.g.f.

4. Proof of Theorems 2 and 3. We need the following result which seems natural enough but a simple proof has eluded the author.

LEMMA B. *Let $f(z) = \sum_{n=0}^{\infty} f_n z^n$ be a nonconstant entire function and $P(t) = a_m t^m + a_{m-1}t^{m-1} + \cdots + a_1 t (a_m \neq 0)$ a polynomial of degree $m \geqq 1$. If $f[\exp\{P(t)\}]$ is real for all real t, then the coefficients $f_n(n = 0, 1, \cdots)$ and $a_n(n = 1, \cdots m)$ are all real.*

Let $a_k = b_k + ic_k$ where a_k, b_k are real $(k = 1, \cdots m)$ and define $B(t) = \sum_k b_k t^k$, $C(t) = \sum_k c_k t^k$. Let p and q be the degrees of $B(t)$,

$C(t)$ respectively. Then $m = \max(p, q)$ and

$$P(t) = B(t) + iC(t) .$$

Let

$$H(t) = f[\exp\{P(t)] = \sum_{n=0}^{\infty} f_n \exp\{nB(t) + inC(t)\} . \tag{4.1}$$

For real t, $H(t) = \overline{H(t)}$, where the bar denotes complex conjugate. Hence, for real t

$$\sum_{n=0}^{\infty} f_n \exp[n\{B(t) + iC(t)\}] = \sum_{n=0}^{\infty} \bar{f}_n \exp[n\{B(t) - iC(t)\}] , \tag{4.2}$$

but since both sides of (4.2) define entire functions of t, the relation (4.2) holds over the whole t-plane.

Suppose first that $B(t) \equiv 0$. Then putting $z = \exp\{iC(t)\}$ we obtain from (4.2) that for all $z \neq 0$,

$$\sum_{n=0}^{\infty} f_n z^n = \sum_{n=0}^{\infty} \bar{f}_n z^{-n} .$$

Since a Laurent expansion is unique, it follows that $f_0 = \bar{f}_0, f_n = 0$ $(n = 1, 2, \cdots)$ so that $f(z) =$ constant, contrary to our hypothesis. Hence $B(t) \not\equiv 0$.

Suppose we can find a path L extending to infinity in the t-plane such that as $t \to \infty$ along L,

$$\mathscr{R}\{B(t) + iC(t)\} \sim \mathscr{R}\{B(t) - iC(t)\} \tag{4.3}$$

with both sides of (4.3) tending to $-\infty$. The exponential terms on both sides of (4.2) tend to zero and we obtain $f_0 = \bar{f}_0$ so that f_0 is real, possibly zero. The relation (4.2) now holds with the summations starting at $n = 1$. Suppose f_k is the first nonvanishing coefficient after f_0. Dividing through by $\exp[k\{B(t) + iC(t)\}]$ we have,

$$\begin{aligned} f_k + \sum_{n=k+1}^{\infty} f_n \exp[(n-k)\{B(t) + iC(t)\}] \\ = \bar{f}_k \exp\{-2kiC(t)\} \\ + \sum_{n=k+1}^{\infty} \bar{f}_n \exp[n\{B(t) - iC(t)\} - k\{B(t) + iC(t)\}] . \end{aligned} \tag{4.4}$$

If we now let $t \to \infty$ along L all terms inside the summation signs in (4.4) tend to zero and we have

$$\lim_{t\to\infty} \exp\{-2kiC(t)\} = f_k/\bar{f}_k .$$

Since $C(t)$ is polynomial with zero constant term, if follows that $C(t) \equiv 0$. From (4.2), therefore, f_n is real for all n. It remains to show that the path L exists.

We choose L from among those curves in the t-plane on which $\mathscr{I}\{C(t)\} = 0$. We have, for $t = re^{i\theta}$,

$$C(t) = c_q t^q + \cdots + c_1 t = c_q r^q e^{iq\theta} + \cdots + c_1 re^{i\theta},$$

and

$$\mathscr{I}\{C(t)\} = c_q r^q \sin q\theta + \cdots + c_1 r \sin\theta .$$

Each of the rays $\theta = \theta_n = n(\pi/q)(n = 0, 1, \cdots)$ is an asymptote to a curve $\mathscr{I}\{C(t)\} = 0$. We choose n so that $b_p \cos p\theta_n < 0$ and then take L as the curve $\mathscr{I}\{C(t)\} = 0$ which is asymptotic to the ray $\theta = \theta_n$. Then, as $t \to \infty$ along L,

$$\left.\begin{aligned} \mathscr{R}\{B(t) + iC(t)\} &\sim b_p r^p \cos p\theta_n , \\ \mathscr{R}\{B(t) - iC(t)\} &\sim b_p r^p \cos p\theta_n , \end{aligned}\right\} \quad (r \to \infty)$$

and L therefore satisfies our requirements. We observe that since $q \geqq 1$ and $p \geqq 1$, we can always find an integer n_1 such that $\cos(pn_1\pi/q) < 0$ and an integer n_2 such that $\cos(pn_2\pi/q) > 0$. We choose L asymptotic to the ray $\theta = \theta_{n_1}$ or $\theta = \theta_{n_2}$ according as $b_p > 0$ or $b_p < 0$. This completes the proof of Lemma B.

Turning to Theorem 2, we see that the sufficiency of the condition is clear since if $P(t) = a_1 t$ (a_1 real), then $f[\exp\{P(t)\}]$ is the m.g.f. of a lattice distribution, while if $P(t) = a_1 t + a_2 t^2 (a_1, a_2$ real, $a_2 > 0)$, then $f[\exp\{P(t)\}]$ is the m.g.f. of an infinite mixture of normal distributions together with a discrete probability f_0 at the origin.

To prove the necessity, we observe from Theorem 1 that if $f[\exp\{P(t)\}]$ is to be a m.g.f. at all, then $P(t)$ can only be of the form $P(t) = a_1 t + a_2 t^2$, and from Lemma B, the coefficients a_1 and a_2 must be real. Further we cannot have $a_2 < 0$ for in this case $\exp\{P(t)\}$ and, therefore, $f[\exp\{P(t)\}]$ would be bounded as $t \to \pm\infty$, which is impossible for a convex function. The theorem is therefore proved.

In proving Theorem 3 we see from Theorem 1 and Lemma B that for $f[\exp\{P(t)\}]$ to be a m.g.f., it is necessary that $P(t) = a_1 t + a_2 t^2$ (a_1, a_2 real) and that f_n be real $(n = 0, 1, \cdots)$. By the argument at the end of the previous paragraph it is also necessary that $a_2 \geqq 0$. Now let

$$M(t) = f\{\exp(a_1 t + a_2 t^2)\} = \sum_{n=0}^{\infty} f_n \exp\{n(a_1 t + a_2 t^2)\} \tag{4.5}$$

where $f_n (n = 0, 1, \cdots)$, a_1 and a_2 are real and $a_2 > 0$. We clearly have

$$M(t) = \int_{-\infty}^{\infty} e^{tx} dF(x)$$

where

$$dF(x) = f_0 dH(x) + \left[\sum_{n=1}^{\infty} \frac{f_n}{\sqrt{(4\pi n a_2)}} \exp\left\{-\frac{(x-na_1)^2}{4na_2}\right\}\right] dx\,, \tag{4.6}$$

$H(x)$ being the unit step function with a jump at $x = 0$. We thus have

$$dF(x) = f_0 dH(x) + \left[\frac{\exp\left(\dfrac{a_1 x}{2a_2}\right)}{\sqrt{(4\pi a_2)}} \sum_{n=1}^{\infty} f_n n^{-\frac{1}{2}} \left\{\exp\left(-\frac{na_1^2}{4a_2}\right)\right\} y^{\frac{1}{n}}\right] dx \tag{4.7}$$

where $y = \exp(-x^2/4a_2)$. We now see that for $M(t)$ as defined by (4.5) to be a m.g.f. it is necessary and sufficient that $f_0 \geqq 0$, $f(1) = 1$ and that the sum on the right hand side of (4.7) be nonnegative for $0 < y \leqq 1$.

If $a_2 = 0$, the result is obvious and so Theorem 3 is proved.

5. **Proof of Theorem 4.** We may assume without loss of generality that the coefficient of the highest power of t in $P(t)$ is unity. For if $P(t) = a_m t^m + a_{m-1}t^{m-1} + \cdots + a_0 (a_m \neq 0)$ and if $f(z) = \sum f_n z^n$ then $f\{P(t)\} = \sum f_n a_m^n \{P_1(t)\}^n$ where $P_1(t) = t^m + (a_{m-1}/a_m)t^{m-1} + \cdots + (a_0/a_m)$, and then $f\{P(t)\} = f_1\{P_1(t)\}$, where $f_1(z) = \sum f_n a_m^n z^n$ is an entire function. Accordingly, let

$$P(t) = t^m + a_{m-1}t^{m-1} + \cdots + a_0 \qquad (m > 2).$$

Consider the complex number $Re^{i\phi}$, where the argument ϕ may depend on R but is always defined to be in the interval $\pi/2 \leqq \phi < 5\pi/2$. We consider the roots of the equation

$$P(t) = Re^{i\phi}\,.$$

We assert that for given ε, $0 < \varepsilon < \pi/2m$, there is always a root t_R of this equation which satisfies

$$\begin{aligned} &|t_R| \sim R^{1/m} \\ &0 < \frac{\pi}{2m} - \varepsilon \leqq \arg(t_R) \leqq \frac{5\pi}{2m} + \varepsilon\,. \end{aligned} \qquad (R \to \infty) \tag{5.1}$$

We observe that if $P(t) \equiv t^m$ and we take $|t_R| = R^{1/m}$ and $\arg t_R = \phi/m$, then t_R satisfies (5.1). In general, if we consider a circle C_R

with centre $R^{1/m}\exp(i\phi/m)$ and radius $R^{(1/m)-\delta}(0<\delta<1/m)$, then for all sufficiently large R, the circle C_R lies within the angle

$$\frac{\pi}{2m}-\varepsilon<\arg t<\frac{5\pi}{2m}+\varepsilon\,. \tag{5.2}$$

It is now easily verified that for t on C_R,

$$\left.\begin{aligned}|P(t)-Re^{i\phi}|&\sim mR^{1-\delta}\\ |P(t)-t^m|&=O(R^{1-1/m})\end{aligned}\right\}\quad (R\to\infty)\,.$$

Hence, since $1-\delta>1-(1/m)$, it follows from Rouché's theorem that for all sufficiently large R, $P(t)-Re^{i\phi}$ and $t^m-Re^{i\phi}$ have the same number of zeros inside C_R. Since $t^m-Re^{i\phi}$ has a zero at $t=R^{1/m}e^{i\phi/m}$, the centre of C_R, if follows that $P(t)-Re^{i\phi}$ has at least one zero, say $t=t_R$, inside C_R. It immediately follows that for all sufficiently large R, t_R lies in the angle (5.2) and that

$$|t_R-R^{1/m}e^{i\phi/m}|<R^{1/m-\delta}\,,$$

which gives the result (5.1).

Now consider the function

$$M(t)=f\{P(t)\}\,. \tag{5.3}$$

If $M(t)$ is to be a m.g.f. then clearly $f(t)$ cannot be a polynomial, for if $f(t)$ were a polynomial, then $M(t)$ would also be and a polynomial cannot satisfy the inequality (1.4). We suppose therefore that $f(t)$ has an essential singularity at infinity. If $F(R)$ is the maximum modulus of $f(t)$ on the circle $|z|=R$, then $F(R)/R$ is ultimately a strictly increasing function of R. Hence for all sufficiently large R_1, R_2 with $R_1<R_2$ we have

$$\frac{F(R_2)}{F(R_1)}>\frac{R_2}{R_1}\,. \tag{5.4}$$

Suppose that $|f(z)|$ attains its maximum on $|z|=R$ at a point $Re^{i\phi}$ where ϕ is defined to be in the interval $\pi/2\leqq\phi<5\pi/2$. Choose t_R so that $P(t_R)=Re^{i\phi}$ and so that t_R satisfies (5.1). Let $u_R=\mathscr{R}t_R$. Then

$$\begin{aligned}\left|\frac{M(t_R)}{M(u_R)}\right|&=\left|\frac{f\{P(t_R)\}}{f\{P(u_R)\}}\right|\\ &=\frac{F(R)}{|f\{P(u_R)\}|}\\ &\geqq\frac{F(R)}{F\{|P(u_R)|\}}\,.\end{aligned} \tag{5.5}$$

Now in virtue of (5.1) we have, for all R sufficiently large,

$$0 < |t_R| \cos\left(\frac{5\pi}{2m} + \varepsilon\right) \leqq u_R \leqq |t_R| \cos\left(\frac{\pi}{2m} - \varepsilon\right).$$

Hence there exists $R_0 > 0$ and $\eta (0 < \eta < 1)$ such that

$$|P(u_R)| < \eta |t_R|^m < R \qquad (R > R_0),$$

so that on applying (5.4) to (5.5) we obtain

$$\left|\frac{M(t_R)}{M(u_R)}\right| \geqq \frac{R}{|P(u_R)|} > 1,$$

for $R > R_0$. It follows from (1.4) that $M(t)$ cannot be a m.g.f. and Theorem 4 is therefore proved.

6. **Proof of Theorem 5.** The sufficiency part of theorem 5 is clear. For $g(e^{\alpha t})/g(1)$ is the m.g.f. of a lattice distribution and $f[\exp\{P(t)\}]/f(1)$ is the m.g.f. of a lattice distribution if $a_2 = 0$ or of a mixture of normal distributions if $a_2 > 0$, with possibly a discrete probability at the origin.

To prove the necessity part of the theorem, suppose that

$$M(t) = g(e^{\alpha t}) f[\exp\{P(t)\}] \tag{6.1}$$

is a m.g.f., where $P(t) = a_1 t + \cdots + a_m t^m$. Then $M(t)$ is real for real t and since $g(e^{\alpha t})$ is real for real t so also is $f[\exp\{P(t)\}]$. Hence by Lemma B, the coefficients $a_1, \cdots, a_m$ must be real.

Suppose $m \geqq 3$ and $a_m > 0$. If ξ is real and positive then $P(\xi)$ is a positive strictly increasing function of ξ for all sufficiently large ξ. For given ξ, consider the equation

$$P(t) = P(\xi).$$

By Lemma A, there is a root of this equation, say $t = t_\xi$, which satisfies

$$\begin{aligned} t_\xi &\sim \left\{\frac{P(\xi)}{a_m}\right\}^{1/m} \exp\left(\frac{2\pi i}{m}\right) \\ &\sim \xi \exp\left(\frac{2\pi i}{m}\right) \qquad (\xi \to \infty). \end{aligned} \tag{6.2}$$

Hence as $\xi \to \infty$, we have $\mathscr{R} t_\xi \sim \xi \cos (2\pi/m) (m \neq 4)$, $\mathscr{R} t_\xi = O(\xi)$ $(m = 4)$. Since $P(\xi) \sim a_m \xi^m$, it follows that

$$P(t_\xi) = P(\xi) > P(\mathscr{R} t_\xi) \tag{6.3}$$

for all sufficiently large ξ, say $\xi > \xi_0$. Now $\mathscr{I} t_\xi$ is a continuous function of ξ and $\mathscr{I} t_\xi \sim \xi \sin(2\pi/m)$; hence we may choose $\xi_1 > \xi_0$ in order that $\mathscr{I} t_{\xi_1}$ is an integral multiple of $2\pi/\alpha$. It then follows that

$$g\{\exp(\alpha t_{\xi_1})\} = g\{\exp(\alpha \mathscr{R} t_{\xi_1})\} . \tag{6.4}$$

Hence

$$\frac{M(t_{\xi_1})}{M(\mathscr{R} t_{\xi_1})} = \frac{f[\exp\{P(\xi_1)\}]}{f[\exp\{P(\mathscr{R} t_{\xi_1})\}]} .$$

Now since $f(x)$ is nonconstant and has nonnegative coefficients

$$f_n(n = 0, 1, \cdots),$$

we have $f(x') > f(x'')$ if $x' > x'' > 0$. If therefore follows from (6.3) that

$$\frac{M(t_{\xi_1})}{M(\mathscr{R} t_{\xi_1})} > 1 ,$$

which contradicts the inequality (1.4). A similar argument deals with the case $a_m < 0$. It follows that if $M(t)$ as defined by (6.1) is to be a m.g.f. then we must have $m \leqq 2$, i.e., $P(t) = a_1 t + a_2 t^2$ with a_1 and a_2 real.

Finally if $a_2 < 0$ then on letting $t \to \infty$ along the imaginary axis through integral multiples of $2\pi i/\alpha$ we find that $M(t) \to \infty$ on account of the periodicity of $g(e^{\alpha t})$ and the nonnegativity of the coefficients $f_n(n = 0, 1, \cdots)$. This again contradicts (1.4) and so we must have $P(t) = a_1 t + a_2 t^2$ with a_1, a_2 real and $a_2 \geqq 0$. This completes the proof of the theorem.

7. Remark on the results of Ostrovskiĭ. The author is indebted to the referee for drawing his attention to the paper of Ostrovskiĭ [6] which he had unfortunately overlooked while writing the present paper. Theorem 4 would follow from Ostrovskiĭ's Theorem 4 under the more restrictive hypothesis that $|f(t)| \leqq f(|t|)$ for all $|t|$ sufficiently large. Otherwise, the results of the present paper are independent of those of Ostrovskiĭ.

References

1. Mary L. Cartwright, *Integral functions*, Cambridge University Press, London, 1956.

2. Inge Futtrup Christensen, *Some further extensions of a theorem of Marcinkiewicz*, Pacific J. Math. **12** (1962), 59-67.

3. E. Lukacs, *Some extensions of a theorem of Marcinkiewicz*, Pacific J. Math. **8** (1958), 487-501.

4. E. Lukacs, *Characteristic functions*, Griffin, London, 1960.

5. J. Marcinkiewicz, *Sur une propriété de la loi de Gauss*, Math. Zeits. **44** (1938), 612-618.
6. I. V. Ostrovskii, *On the application of a rule of Wiman and Valiron to the study of characteristic functions of probability laws*, Dokl. Akad Nauk SSSR **143** (1962), 532-535

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