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LOWER BOUNDS FOR THE EIGENVALUES OF A VIBRATING STRING WHOSE DENSITY SATISFIES A LIPSCHITZ CONDITION

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If a string has a density given by a nonnegative integrable function ρ defined on the interval $[0, a]$ and is fixed at its end points under unit tension, then the natural frequencies of vibration of the string are determined by the eigenvalues of the differential system

$$u'' + \lambda\rho(x)u = 0\ , \ u(0) = u(a) = 0\ . \tag{1}$$

As is well known, the eigenvalues of (1) form a positive strictly increasing sequence of numbers which depend on the density $\rho(x)$. We denote them accordingly by

$$0 < \lambda_1[\rho] < \lambda_2[\rho] < \cdots < \lambda_n[\rho] < \cdots\ .$$

In this paper we find lower bounds for these eigenvalues when the density ρ satisfies a Lipschitz condition with Lipschitz constant H and $\int_0^a \rho dx = M$. The bounds will be in terms of M and H.

Specifically, if $E(H, M)$ is the family of functions

$$\left\{\rho : \rho \in L(H) \text{ and } \int_0^a \rho(x)dx = M\right\}\ , \tag{2}$$

where

$$L(H) = \{\rho : |\rho(x_1) - \rho(x_2)| \leqq H|x_1 - x_2|;\ x_1, x_2 \in [0, a]\}\ ,$$

we find a unique function $\rho_0 \in E(H, M)$ for each $\lambda_n[\rho]$ such that

$$\lambda_n[\rho_0] = \min \lambda_n[\rho]$$

where the minimum is taken over all functions $\rho \in E(H, M)$.

Our results will be expressed in terms of the fundamental pair of solutions U_1 and U_2 of the Airy equation

$$\frac{d^2U}{ds^2} + sU = 0 \tag{3}$$

where $U_1(0) = 1$, $U_1'(0) = 0$ and $U_2(0) = 0$, $U_2'(0) = 1$. These functions are tabulated in [7]. The main conclusion is that

$$\min\nolimits_E \lambda_n[\rho] = \left[nt_1\left(\frac{Ha^2}{nM}\right)\right]^3 \Big/ Ha^3$$

where $t_1(K)$ is the least positive root of

$$U_1\left(\left(\frac{1}{K}-\frac{1}{4}\right)t\right)U_2'\left(\left(\frac{1}{K}+\frac{1}{4}\right)t\right) - U_2\left(\left(\frac{1}{K}-\frac{1}{4}\right)t\right)U_1'\left(\left(\frac{1}{K}+\frac{1}{4}\right)t\right)=0 \tag{4}$$

if $K \leqq 4$ and of

$$U_2'(t/\sqrt{K}) = \left(\frac{1}{\sqrt{K}}-\frac{1}{2}\right)tU_1'(t/\sqrt{K}) \tag{5}$$

if $K \geqq 4$.

This result is similar in nature to those obtained by Krein in [4]. There he found that if $\rho(x) \leqq H$ then

$$\frac{4Hn^2}{M^2}\chi\left(\frac{M}{Ha}\right) \leqq \lambda_n[\rho] \leqq \frac{Hn^2\pi^2}{M^2}$$

where $\chi(t)$ is the least positive root of

$$\sqrt{\chi}\tan\chi = t/(1-t)\ .$$

Furthermore, his inequalities are sharp. See [1], [2] and [6] and the references given there for other results of this nature. The maximum value of $\lambda_n[\rho]$ over the family $E(H, M)$ is not presented here. This problem is being investigated and the results will be presented in a later paper.

The method used in this paper has some general interest since it can be used to derive the results of Krein as well as some of the results given in the other papers mentioned above. This will be discussed in the final section.

2. The lower bound for $\lambda_1[\rho]$. In this section, we find a sharp lower bound for the lowest eigenvalue $\lambda_1[\rho]$ within the class of functions $E(H, M)$ defined by (2). It will be convenient to first prove a result concerning the lowest eigenvalue of the system

$$u'' + \lambda\rho(x)u = 0,\ u(0) = u'(\alpha) = 0\ , \tag{6}$$

where $x \in [0, \alpha]$.

THEOREM 1. *Let $\mu_1[\rho]$ be the lowest eigenvalue of a vibrating with one end fixed and the other free, density $\rho(x)$, and under unit tension between the points $x = 0$ and $x = \alpha$. If the density function $\rho \in E(H, m)$, then*

$$\mu_1[\rho]\alpha^3 H \geqq \tau_1^3(H\alpha^2/m) \tag{7}$$

where $\tau_1(k)$ *is the least positive root of*

$$(8)\qquad U_1\left(\left(\frac{1}{k}-\frac{1}{2}\right)\tau\right)U_1'\left(\left(\frac{1}{k}+\frac{1}{2}\right)\tau\right) - U_2\left(\left(\frac{1}{k}-\frac{1}{2}\right)\tau\right)U_1'\left(\left(\frac{1}{k}+\frac{1}{2}\right)\tau\right)=0$$

when $k = H\alpha^2/m \leqq 2$ *and of*

$$(9)\qquad U_2'\left(\sqrt{\frac{2}{k}}\tau\right)=\left(\sqrt{\frac{2}{k}}-1\right)\tau U_1'\left(\sqrt{\frac{2}{k}}\tau\right)$$

when $k > 2$. *Moreover, equality holds if and only if*

$$\rho(x) = H(x-\alpha/2) + m/\alpha$$

when $H\alpha^2 \leqq 2m$ *and*

$$\rho(x) = \begin{cases} 0 & , \ 0 \leqq x \leqq \alpha - \sqrt{2m/H}\ , \\ H(x-\alpha)+\sqrt{2mH} & , \ \alpha-\sqrt{2m/H} \leqq x < \alpha\ , \end{cases}$$

when $H\alpha^2 > 2m$.

Proof. We consider the two cases $H\alpha^2 \leqq 2m$ and $H\alpha^2 > 2m$ simultaneously. We compare any density $\rho \in E(H, M)$ with the function q defined by

$$(10)\qquad q(x) = H(x-\alpha/2) + m/\alpha$$

in the first case and by

$$(11)\qquad q(x) = \begin{cases} 0 & , \ 0 \leqq x \leqq \alpha - \sqrt{2m/H}\ , \\ H(x-\alpha)+\sqrt{2mH} & , \ \alpha-\sqrt{2m/H} \leqq x \leqq \alpha\ , \end{cases}$$

in the second case. In both cases, we note that p and q are continuous and

$$\int_0^\alpha \rho(x)\,dx = \int_0^\alpha q(x)dx = m$$

so that ρ and q have at least one common value for $x\varepsilon[0,\alpha]$. If ρ and q have a common value $\rho(a) = q(a)$, the Lipschitz condition implies that for $x > a$

$$\rho(x) - \rho(a) \leqq H(x-a)\ .$$

Consequently, for $x > a$, we have

$$\rho(x) \leqq H(x-a) + q(a) = q(x)\ .$$

Similarly for $x < a$ we have $\rho(x) \geqq q(x)$. For $x > a$ it follows that

$$\int_x^\alpha \rho(t)dt \leqq \int_x^\alpha q(t)dt \ . \tag{12}$$

For $x < \alpha$ we consider the integrals of ρ and q from 0 to α minus the integrals from 0 to x and arrive at the same conclusion for all $x \varepsilon [0, \alpha]$.

We let $\mu_1[\rho]$ denote the lowest eigenvalue of the differential system (6) and $\mu_1[q]$ denote the lowest eigenvalue of the same system with ρ replaced by q. We now use the following comparison theorem due to Nehari [5].

THEOREM (*Nehari*). *Let ρ, q be nonnegative continuous functions defined on $[0, \alpha]$ such that* (12) *is satisfied. Then*

$$\mu_1[q] \leqq \mu_1[\rho] \tag{13}$$

with strict inequality unless ρ and q are identical.

The computation of $\mu_1[q]$ for each of the two cases $k \leqq 2$ and $k > 2$ in terms of the fundamental solutions, U_1 and U_2, of the Airy equation (3) with (13) yields the conclusion of the theorem.

We use this theorem to prove the following more difficult

THEOREM 2. *Let $\lambda_1[\rho]$ be the lowest eigenvalue of a string fixed under unit tension between $x = 0$ and $x = a$. If the density function $\rho \in E(H, M)$, then*

$$\lambda_1[\rho]a^3 H \geqq t_1^3(Ha^2/M) \tag{14}$$

where $t_1(K)$ is the least positive root of (4) *when $K = Ha^2/M \leqq 4$ and of* (5) *when $K > 4$. Moreover, equality holds in* (14) *if and only if ρ is the symmetric function defined by*

$$\rho(x) = \begin{cases} H(x - a/4) + M/a \ , & 0 \leqq x \leqq a/2 \ , \\ \rho(a - x) & , \ a/2 \leqq x \leqq a \ , \end{cases} \tag{15}$$

when $Ha^2 \leqq 4M$ and by

$$\rho(x) = \begin{cases} 0 & , \ 0 \leqq x \leqq a/2 - \sqrt{\dfrac{M}{H}} \ , \\ H(x - a/2) + \sqrt{HM} \ , & a/2 - \sqrt{\dfrac{M}{H}} \leqq x \leqq a/2 \ , \\ \rho(a - x) & , \ a/2 \leqq x \leqq a \ , \end{cases} \tag{16}$$

when $Ha^2 > 4M$.

Proof. Let u_1 be the eigenfunction of (1) associated with the lowest eigenvalue $\lambda_1[\rho]$. If we take $u_1(x) > 0$ in the interval $(0, a)$, then $u_1'' = -\lambda[\rho]\rho(x)u_1(x) \leqq 0$ so that u_1 is concave. Thus it has only one maximum value which must be attained at some interior point α of $[0, a]$. We consider the differential system

$$u'' + \lambda\rho(x)u = 0, \quad u(0) = u'(\alpha) = 0 \ , \quad x \in [0, \alpha] \ ,$$

obtain from (1) by restricting the domain of ρ to $[0, \alpha]$ and imposing the given boundary conditions. Then $\lambda_1[\rho]$ is also the lowest eigenvalue of this system and u_1 restricted to $[0, \alpha]$ is the corresponding eigenfunction. $\lambda_1[\rho]$ is also the lowest eigenvalue of the system obtained from (1) by restricting x to the interval $[\alpha, a]$ and imposing the boundary conditions $u'(\alpha) = u(a) = 0$. If we let $m = \int_0^\alpha \rho(x)dx$, $m' = \int_\alpha^a \rho(x)dx = M - m$ and $\beta = a - \alpha$, and apply Theorem 1 to each restricted system defined above, we obtain the inequalities

$$\lambda_1[\rho] \geqq \tau_1^3(H\alpha^2/m)/\alpha^3 H, \ \lambda_1[\rho] \geqq \tau_1^3(H\beta^2/m')/\beta^3 H \tag{17}$$

where $\tau_1(k)$ is the last positive root of (8) or (9), depending on the magnitude of k. The theorem would be proved if we could show that α, β, m, m' can be varied in such a way that the quantities on the right side of the inequalities (17) always remain less than $\lambda_1[\rho]$ and at least one of them is greater than or equal to

$$\tau_1^3(H(a/2)^2/(M/2))/(a/2)^3 H = [2\tau_1(Ha^2/2M)]^3/a^3 H \ .$$

If we let $t = 2\tau$ and $K = 2k$ in (8) and (9) then we get the equations (4) and (5) so that the above considerations would yield the inequality (14).

We carry out the process just outlined by considering the function defined by

$$\eta(\alpha, m) = \tau_1^3(k)/\alpha^3 H \ , \tag{18}$$

$k = \alpha^2 H/m$ for all $\alpha \in [0, a]$ and all $m \in [0, M]$. We define a function f on $[0, a]$ by the equation

$$\eta(\alpha, f(\alpha)) = \eta(a - \alpha, M - f(\alpha)) \ . \tag{19}$$

To show that this determines a well-defined function we note that, by the comparison theorem for eigenvalues, $\eta(\alpha, m)$ is a strictly monotone decreasing function of m for each α. Furthermore, $\eta(\alpha, m) \to +\infty$ as $m \to 0$. Consequently, $\eta(\alpha, m) - \eta(a - \alpha, M - m)$ is strictly decreasing between $-\infty$ and $+\infty$ as m varies between 0 and M, proving that f is uniquely determined.

It now follows that

$$\lambda_1[\rho] \geqq \eta(\alpha, f(\alpha)) = \eta(a - \alpha, M - f(\alpha)) \tag{20}$$

whenever α is a maximum point of the corresponding first eigenfunction u_1. For if $m = \int_0^\alpha \rho(x)dx > f(\alpha)$ the comparison theorem and (19) yield the inequality

$$\eta(\alpha, m) \leqq \eta(\alpha, f(\alpha)) = \eta(a - \alpha, M - f(\alpha)) \leqq \eta(a - \alpha, M - m)$$

and (20) then follows from (17). Similarly, if $m < f(\alpha)$, the comparison theorem and (19) imply

$$\eta(\alpha, m) > \eta(\alpha, f(\alpha)) = \eta(a - \alpha, M - f(\alpha)) > \eta(a - \alpha, M - m)$$

and (20) again follows.

We want to show that $\eta(\alpha, f(\alpha))$ has a minimum value at $\alpha = a/2$, i.e., that the minimum of $\eta(\alpha, m)$ on graph of f is located at the point $(a/2, M/2)$. We first show that the graph of f is centrally symmetric about the point $(a/2, M/2)$, i.e., that $f(a - \alpha) = M - f(\alpha)$. This is a consequence of the defining relation (19) of f since (x, y) is a point on the graph of f if and only if $(a - x, M - y)$ is. Thus for any $(\alpha, f(\alpha))$ on the graph, $(a - \alpha, M - f(\alpha))$ is also on the graph and hence $(a - \alpha, f(a - \alpha)) = (a - \alpha, M - f(\alpha))$ by the uniqueness of the definition of f. This implies that $f(a - \alpha) = M - f(\alpha)$, proving the central symmetry. In particular, $f(a/2) = M/2$. Also from this symmetry we have that $f'(\alpha) = f'(a - \alpha)$ whenever the derivative exists. We will show presently that this is always the case.

We now assume that η has a minimum value over the graph of f at a point $(\alpha, f(\alpha))$ where $\alpha \neq a/2$. Then from (19) we see that $(a - \alpha, f(a - \alpha))$ also is a minimum point. Suppose that the minimum value of η at these points is c_0 and consider the level set

$$\{\alpha, m) : \eta(\alpha, m) = c_0\} . \tag{21}$$

By the comparison theorem for eigenvalues this defines a function g of $\alpha \in [0, a]$. We will show presently that g' exists. It will then follow that $f'(\alpha) = g'(\alpha)$ and $f'(a - \alpha) = g'(a - \alpha)$ if α is the minimizing value of $\eta(\alpha, f(\alpha))$, for otherwise the derivative of η in the direction of the curve determined by f would be nonzero unless the directional derivative of $\eta(\alpha, m)$ vanishes in a direction other than that of a level cure. That the latter possibility is not the case will be concluded at the same time we show that g' exists. Thus we conclude that f and g are tangent at α as well as at $a - \alpha$. From the symmetry we concluded that $f'(\alpha) = f'(a - \alpha)$ so that the same must be true of g at the minimizing value of α. We will show, however, that g is a strictly convex function of α and thus reach a contradiction. We will then be able to conclude that the minimum value of η on the graph of f must occur at the point $(a/2, M/2)$.

To complete the proof we must show that f' and g' exist for all $\alpha \in [0, a]$, that the directional derivative of η in any direction save that of a level curve is nonzero and that g is a strictly convex function.

To prove the first two assertions, we recall that the system (6) with ρ replaced by the functions defined by (10) or (11) has the lowest eigenvalue $\eta(\alpha, m)$. We make a change of the independent variable to get the system

$$v'' + \lambda h(y)v = 0\ , \qquad v(0) = v'(1) = 0\ , \qquad y \in [0, 1] \tag{22}$$

where $h(y) = H\alpha^3(y - 1/2) + \alpha m$ if $\alpha^2 H \leqq 2m$ and

$$h(y) = \begin{cases} H\alpha^3(y-1) + \alpha^2\sqrt{2mH}\ , & 1 - \sqrt{\dfrac{2m}{\alpha^2 H}} \leqq y \leqq 1\ , \\ 0 \qquad , & 0 \leqq y \leqq 1 - \sqrt{\dfrac{2m}{\alpha^2 H}}\ , \end{cases}$$

if $\alpha^2 H > 2m$. We note that the lowest eigenvalue of this new system is still $\eta(\alpha, m)$. The derivatives f' and g' will exist and even be continuous if the partial derivatives η_α and η_m exist, are continuous, and $\eta_m \neq 0$, for then

$$f'(\alpha) = -\frac{\eta_\alpha(\alpha, f(\alpha)) - \eta_\alpha(1-\alpha, 1-f(\alpha))}{\eta_m(\alpha, f(\alpha)) - \eta_m(1-\alpha, 1-f(\alpha))}$$

and

$$g'(\alpha) = -\frac{\eta_\alpha(\alpha, g(\alpha))}{\eta_m(\alpha, g(\alpha))}\ .$$

Furthermore, it is evident that, if in addition $\eta_\alpha \neq 0$, the directional derivative of η will be nonzero except in the direction of a level curve.

To verify these properties of η_α and η_m, we use the formula

$$\eta_\alpha = -\eta \frac{\displaystyle\int_0^1 \frac{\partial h}{\partial \alpha} v_1^2(y)dy}{\displaystyle\int_0^1 h(y)v_1^2(y)dy}\ . \tag{23}$$

for η_α and the same formula for η_m with $\partial h/\partial\alpha$ replaced by $\partial h/\partial m$. It seems likely that this formula is known, but since we know of no reference we give an outline of the derivation. Let the change in h due to a change $\Delta\alpha$ in α be denoted by Δh and consider the system (22) with h replaced by $h + \Delta h$. We denote lowest eigenvalue of this new system by $\eta + \Delta\eta$ and the corresponding eigenfunction by $\bar{v}_1$. Then $\bar{v}_1$ satisfies the equation $\bar{v}_1'' + (\eta + \Delta\eta)(h + \Delta h)\bar{v}_1 = 0$. We multiply this by v_1, use the fact that $v_1'' + \eta h v_1 = 0$ and carry out an integration by parts to get

$$\Delta\eta = -\eta\frac{\int_0^1 \Delta h v_1\bar{v}_1 dy}{\int_0^1 (h+\Delta h)v_1\bar{v}_1 dy}\ .$$

Dividing by $\Delta\alpha$ and letting $\Delta\alpha \to 0$, we get the formula (23). The interchange of limit and integration is justified since $\bar{v}_1 \to v_1$, $\Delta h \to 0$ and $\Delta h/\Delta\alpha \to h/\alpha$ all uniformly (see [3] $p.$ 151 for $\bar{v}_1 \to v_1$). The same proof holds for η_m.

It is clear from (23) and the corresponding formula for η_m that η_α and η_m exist and are continuous whenever $\partial h/\partial\alpha$ and $\partial h/\partial\alpha$ exist and are continuous. This is always the case with the possible exception of those points (α, m) such that $H\alpha^2 = 2m$. But even for this case we note that since

$$\frac{\partial h}{\partial\alpha} = 3H\alpha^2(y - 1/2) + m$$

when $\alpha^2 H < 2m$ and

$$\frac{\partial h}{\partial\alpha} = \begin{cases} 0\ , & 0 \leqq y \leqq 1 - \sqrt{\dfrac{2m}{\alpha^2 H}}\ , \\ 3H\alpha^2(y-1) + 2\alpha\sqrt{2mH}\ , & 1 - \sqrt{\dfrac{2m}{\alpha^2 H}} \leqq y \leqq 1\ , \end{cases}$$

when $\alpha^2 H > 2m$ we have the same limiting value $6my - 2m$ as (α, m) approaches a point on the curve $\alpha^2 H = 2m$. Hence η_α exists and is continuous for all positive α and m. The same remark holds for η_m.

We can also use (23) to show that $\eta_\alpha < 0$ by showing that each of the factors there are positive. This is clearly the case for η and the denominator. To show that $\int_0^1 \partial h/\partial v_1^2(y)dy > 0$, we must consider the two cases $\alpha^2 H \leqq 2m$ and $\alpha^2 H > 2m$. In the first case, this inequality will be evident if we note that

$$\begin{aligned}\int_0^1 3K\alpha^2(y - 1/2)v_1^2(y)dy &= \int_0^{1/2}\cdots + \int_{1/2}^1\cdots \\ &= \int_{1/2}^1 3K\alpha^2[(y-1/2)v_1^2(y) + (1/2 - y)v_1^2(1-y)]dy\ .\end{aligned}$$

The expression in the square brackets is positive since v_1^2 is an increasing function. In the second case, we have

$$\int_{1-\sqrt{2m/\alpha^2 H}}^1 [3H\alpha^2(y-1) + 2\alpha\sqrt{2mH}\,]v_1^2(y)dy\ .$$

The expression in the square brackets is greater than $3H\alpha^2(y - 1) +$

$3/2\alpha\sqrt{2mH}$. It can be shown that the above integral with this substituted for the square brackets is positive by the same method used in the first case.

It can be shown in a similar way that $\eta_m < 0$.

To complete the proof of Theorem 2, we must show that the level curve defined by (21) is convex. Since the proof is rather long we put it in a separate section.

3. Proof of the convexity of level curve $\eta(\alpha, m) = c_0$. To prove the convexity of the level line

$$\{(\alpha, m) : \eta(\alpha, m) = c_0\} \tag{21}$$

we note that η is the least positive root of (8) or (9) depending on the magnitude of $H\alpha^2/m$ and that the function g satisfies the equation $\eta(\alpha, g(\alpha)) = c_0$. It follows that g is determined by the least positive root $m = g(\alpha)$ of the equation

$$\begin{aligned} U_1\left(\left(\frac{m}{H\alpha} - \frac{\alpha}{2}\right)\sqrt[3]{c_0 H}\right) U_2\left(\left(\frac{m}{H\alpha} + \frac{\alpha}{2}\right)\sqrt[3]{c_0 H}\right) \\ = U_2\left(\left(\frac{m}{H\alpha} - \frac{\alpha}{2}\right)\sqrt[3]{c_0 H}\right) U_1'\left(\left(\frac{m}{H\alpha} - \frac{\alpha}{2}\right)\sqrt[3]{c_0 H}\right) \end{aligned}$$

if $H\alpha^2 \leqq 2m$ and of

$$U_2'\left(\sqrt{\frac{2m}{H}}\sqrt[3]{Hc_0}\right) = \left(\sqrt{\frac{2m}{H}} - \alpha\right)\sqrt[3]{Hc_0}\, U_1\left(\sqrt{\frac{2m}{H}}\sqrt[3]{Hc_0}\right)$$

if $H\alpha^2 > 2m$.

It will be necessary for us to consider the level line determined by $\eta(\alpha, m) = c$ for any positive value of c. We show that if the level line corresponding to a given value of c and a given value of H is convex, then the level line corresponding to c_0 and H_0 is convex. For some value of c and H, we assume that the corresponding function defined by $m_c = g(\alpha)$ is convex so that, by (18), $\tau_1(H\alpha^2/m_c) = \sqrt[3]{Hc\alpha}$. But for any value of c_0 with $\eta(\alpha, m) = c_0$, we have $m_0 = g_0(\alpha)$ determined by $\tau_1(H\alpha^2/m_0) = \sqrt[3]{Hc_0}\alpha$ or

$$\tau_1(H(\sqrt[3]{c_0/c}\alpha)^2/(\sqrt[3]{c_0/c})^2 m_0) = \sqrt[3]{Hc}\sqrt[3]{c_0/c}\alpha\,.$$

This implies that $m_c = (c_0/c)^{2/3} m_0 = g(\sqrt[3]{c_0/c}\alpha)$ so that the convexity of g implies that of g_0. Similarly, if g is the function determined for an arbitrary value of H, it can be shown that $m_H = (\sqrt[3]{H_0/H})^{-1} m_0 = g(\sqrt[3]{H_0/H}\alpha)$, so that g_0 is convex if g is. Thus, to verify the convexity of the level curve corresponding to c_0, we may choose $H = 1$ and c may be any positive value which is convenient.

We first consider the case where $H\alpha^2 = \alpha^2 \geqq 2m$ and let $z = \sqrt[3]{c}(m/\alpha+\alpha/2)$ and $w = \sqrt[3]{c}(m/\alpha-\alpha/2)$ so that (8) becomes

$$F(z, w) = U_1(w)U_2'(z) - U_2(w)U_1'(z) = 0 .$$

It will be convenient to define

$$H(z, w) = U_1(z)U_2(w) - U_2(z)U_1(w) ,$$
$$I(z, w) = U_1'(w)U_2'(z) - U_2'(w)U_1'(z) ,$$

and

$$G(z, w) = U_1(z)U_2'(w) - U_1'(w)U_2(z) .$$

If we restrict z and w to the set $S = \{(z, w) : F(z, w) = 0\}$ then the identities $F_z = zH$, $F_w = I$, $H_z = 0$, $H_w = G$, $I_z = zG$ and $I_w = 0$ are all valid.

We then calculate

$$m' = \frac{dm}{d\alpha} = -\frac{F_\alpha}{F_m} = -\frac{F_z z_\alpha + F_w w_\alpha}{F_z z_m + F_w w_m} \tag{24}$$
$$= \frac{z}{\sqrt[3]{C}}\frac{I + wH}{I + zH} , \qquad (z, w) \in S ,$$

Note that $m' = -\eta_\alpha/\eta_m < 0$ since η_α and η_m are negative. We want to show

$$\frac{d^2m}{d\alpha^2} = \frac{\partial m'}{\partial z}\frac{dz}{d\alpha} + \frac{\partial m'}{\partial w}\frac{dw}{d\alpha} > 0 . \tag{25}$$

Since z and w are position for the case under consideration, we have from the definition of z and w that

$$\frac{dz}{d\alpha} = -\frac{w}{\alpha} + \frac{\sqrt[3]{c}}{\alpha}m' < 0$$

and

$$\frac{dw}{d\alpha} = -\frac{z}{\alpha} + \frac{\sqrt[3]{c}}{\alpha}m' < 0 .$$

Thus (25) will be verified if we show that $\partial m'/\partial z$ and $\partial m'/\partial w$ are negative.

Using the identities in F, G, H and I listed above, we find that

$$\frac{\partial m'}{\partial z} = m'\left[1 + \frac{zG}{I + wH} - \frac{zG + H}{I + zH}\right]$$

and

$$\frac{dm'}{dw} = m'\Big[\frac{H + wG}{I + wH} - \frac{zG}{I + zH}\Big] .$$

Since $m' < 0$, we need only show that the terms in the square brackets are positive on S. We do this by showing that

(i) $I + zH < 0$,

(ii) $I + wH > 0$,

(iii) $G > 0$,

(iv) $H + zG > H + wG > 0$

on S for c sufficiently large.

We first note that on S,

$$\begin{aligned} H(z, w) &= U_2(w)[U_1(z) - U_2(z)U_1(w)/U_2(w)] \\ &= U_2(w)[U_1(z) - U_2(z)U_1'(z)/U_2'(z)] \\ &= [U_1(z)U_2'(z) - U_2(z)U_1'(z)]U_2(w)/U_2'(z)] \\ &= U_2(w)/U_2'(z) \end{aligned}$$

since $F(z, w) = 0$ and the wronskian $W(U_1, U_2) = 1$. Similarly, it may be seen that $I(z, w) = -U_2'(z)/U_2(w)$ on S. Thus, $IH = -1$ and since H and I have finite values on S, they must be nonzero and of opposite sign there. At $w = 0$ we see that $F(z, 0) = U_2'(z) = 0$ gives the value $z = z_0 = 1.5 \cdots$ and hence $I(z_0, 0) = -U_1'(z_0) = .9 > 0$ (see [7] p. 30). We may thus conclude that $H(z, w) < 0$ and $I(z, w) > 0$ on S. We note that $z > w$ so that

$$I(z, w) + zH(z, w) < I(z, w) + wH(z, w) .$$

But, by (24), we know that the ratio of these two quantities is negative so that (i) and (ii) must be satisfied.

To prove (iii) we first show that $G(z, w)$ must be of one sign on S. We consider the lowest eigenvalue ν_1 of the system

$$U'' + \nu\rho(x)U = 0 , \qquad U'(0) = U(\alpha) = 0 , \qquad x \in [0, \alpha] \tag{26}$$

with $\rho(x) = x - \alpha/2 + m/\alpha$. Solving this system we find that $\nu_1 = \sigma_1^3(\alpha^2/m)/\alpha^3$ where $\sigma_1(K)$ is the least positive root of the equation $G((1/K + 1/2)\sigma, (1/K - 1/2)\sigma) = 0$. Now suppose that $G(z, w) = 0$ for some point in S. By the definition of z and w and the fact that $\eta(\alpha, m) = c$ we see that

$$G\Big(\eta^{1/3}\Big(\frac{m}{\alpha} + \frac{\alpha}{2}\Big), \eta^{1/3}\Big(\frac{m}{\alpha} - \frac{\alpha}{2}\Big)\Big) = 0 .$$

But then $\eta = (\sigma/\alpha)^3$ must be an eigenvalue of the system (26). By Nehari's comparison theorem the lowest eigenvalue of this system is strictly greater than η so that η can not be the lowest or any other eigenvalue of (26). Thus G does not vanish on S and computing G at the point corresponding to $w = 0$ we get $G(z_0, 0) = .48 \cdots > 0$. Hence (iii) is established.

The first inequality of (iv) now follows from (iii) and the inequality $z > w$. We show that the last inequality holds for $z = \sqrt[3]{c}\,(m/\alpha + \alpha/2)$ and $w = \sqrt[3]{c}\,(m/\alpha - \alpha/2)$ sufficiently large and that this is sufficient for our purpose. We will need an asymptotic expansion of H and G. These can be obtained from the asymptotic formulas for the fundamental solutions of the Airy equation which in turn can be obtained from the asymptotic formulas for the Bessel functions of order $\pm 1/3$, $\pm 2/3$ (see [7]). We thus find that

$$U_1(s) = \frac{\Gamma(2/3)3^{1/6}}{\sqrt{\pi}\, s^{1/4}} \cos\left(\frac{8s^{3/2} - \pi}{12}\right) + 0\left(\frac{1}{s^{7/4}}\right),$$

$$U_2(s) = \frac{\Gamma(4/3)3^{5/6}}{\sqrt{\pi}\, s^{1/4}} \sin\left(\frac{8s^{3/2} - \pi}{12}\right) + 0\left(\frac{1}{s^{7/4}}\right),$$

$$U_1'(s) = \frac{-\Gamma(2/3)3^{1/6}s^{1/4}}{\sqrt{\pi}} \sin\left(\frac{8s^{3/2} - \pi}{12}\right) + 0\left(\frac{1}{s^{5/4}}\right),$$

and

$$U_2'(s) = \frac{(4/3)3^{5/6}s^{1/4}}{\sqrt{\pi}} \cos\left(\frac{8s^{3/2} + \pi}{12}\right) + 0\left(\frac{1}{s^{5/4}}\right).$$

From these formulas it follows directly that

$$F(z, w) = \frac{2}{\sqrt{3}}\left(\frac{z}{w}\right)^{1/4} \cos\left(\frac{4(z^{3/2} - w^{3/2}) + \pi}{6}\right) + 0\left(\frac{1}{w^{3/2}}\right),$$

$$G(z, w) = \frac{2}{\sqrt{3}}\left(\frac{w}{z}\right)^{1/4} \cos\left(\frac{4(z^{3/2} - w^{3/2}) - \pi}{6}\right) + 0\left(\frac{1}{w^{3/2}}\right),$$

and

$$H(z, w) = \frac{2}{\sqrt{3}}\frac{1}{(zw)^{1/4}} \sin\left(\frac{4(w^{3/2} - z^{3/2}) + \pi}{6}\right) + 0\left(\frac{1}{w^{2}}\right),$$

where have used the relation $\Gamma(p)\Gamma(1-p) = \pi/\sin\pi p$ and the appropriate trigonometric formulas.

Since F vanishes on the set S it follows that the cosine term can be made arbitrarily small for z and w sufficiently large. It then follows that the expression for G is arbitrarily close to

$$\frac{2}{\sqrt{3}}\left(\frac{w}{z}\right)^{1/4} \cos\{[4(z^{3/2} - w^{3/2}) + \pi]/6 - \pi/3\}$$
$$\doteq \left(\frac{w}{z}\right)^{1/4} \sin[4(z^{3/2} - w^{3/2}) + \pi]/6 \doteq \left(\frac{w}{z}\right)^{1/4}$$

and in a similar manner we see that H is arbitrarily close to $-1/\sqrt{3}(wz)^{1/4}$. We finally conclude that $H + wG$ can be made arbitrarily close to

$$\frac{1}{z^{1/4}}\left[w^{5/4} - \frac{1}{\sqrt{3}\,w^{1/4}}\right]$$

which, from the definition of z and w, will be positive for c sufficiently large provided $m/\alpha - \alpha/2 > 0$. But this is just the case under consideration. The case where $\alpha^2 = 2m$ will be discussed at the end of this section.

We turn to the case where $H\alpha^2 > 2m$. Here we take $H = 2$ and consider the curve defined by $\eta(\alpha, m) = 1/2$. The $g(\alpha)$ is determined by the least positive root m_1 of the equation

$$U_2'(\sqrt{m}) = (\sqrt{m} - \alpha)\, U_1'(\sqrt{m})\ .$$

We let $y = \sqrt{m_1}$ so that

(27) $$\alpha = f(y) = y - U_2'(y)/U_1'(y)\ .$$

It has already been shown that η_m and η_α are negative so that $d\alpha/dm = -\eta_m/\eta_\alpha < 0$. This implies that $f'(y) < 0$. Now

$$\frac{d^2\alpha}{dm^2} = f''(y)\left(\frac{dy}{dm}\right)^2 + f'(y)\frac{d^2y}{dm^2}$$

where $f'(y) < 0$ and $d^2y/dm^2 < 0$. Hence $d^2\alpha/dm^2 > 0$ if $f''(y) > 0$.

To show that $f''(y) > 0$ we first investigate the range of y determined by the condition $\alpha^2 > m$ or $\alpha > y = \sqrt{m}$. We show that this condition requires that $y < y_0$ where $y_0 = 1.51 \cdots$ is the least positive root of $U_2'(s) = 0$. For any value of y we have that $-U_2'(y)/U_1'(y)$ has the derivative $-y/U_1'^2(y) < 0$ and hence is decreasing except at zeros of $U_1'(y)$ where it has asymptotic. In particular, it is decreasing for $y \in (0, s_0)$ where s_0 is the least positive zero of $U_1'(s) = 0$ and has a zero at $y_0 < s_0$. Comparing $-U_2'(y)/U_1'(y)$ with $\alpha - y$, we see that unless $y < y_0$, the value of y determined for a given α by (27) satisfies $y > \alpha$ which is a contradiction. In particular, when $y = y_0$, $\alpha = y_0$.

To show $f''(y) > 0$ for $y \in [0, y_0]$, we use the fact that U_1 decreases from the value one at $y = 0$ to a value $U_1(y_0) > .48$ at y_0, (see [7]). Calculating $f''(y)$, we have

$$f''(y) = -\,[U_1'(y) - 2y^2U_1(y)]/[U_1'(y)]^3\ .$$

Since $U_1'(y)$ is negative we want the expression in the square bracket in the numerator to be positive. But

$$-U_1'(y) = -\int_0^y U_1''(t)dt = \int_0^t tU_1(t)dt < y^2/2$$

since $0 < U_1(t) < 1$ for $t \in (0, y_0)$. Hence

$$\begin{aligned} U_1' - 2y^2U_1(y) &\geqq -y^2/2 + 2y^2U_1(y) \\ &= 2y^2[U_1(y) - 1/4] > 0 \,, \end{aligned}$$

which proves $f''(y) > 0$, $y \in (0, y_0]$.

Thus, in both cases $H\alpha^2 > 2m$ and $H\alpha^2 < 2m$, the level lines are convex. If $H\alpha^2 = 2m$, we use the fact that these lines have a continuously turning tangent so that the convexity is proved for all cases.

4. Bounds for the higher eigenvalues. While the proof of Theorem 2 was rather long, it leads to an immediate proof of the following:

THEOREM 3. *Let $\lambda_n[\rho]$ be the nth eigenvalue of a string fixed under unit tension between $x = 0$ and $x = a$. If the density function $\rho \in E(H, M)$, then*

$$\lambda_n[\rho]a^3H \geqq n^3t_1\left(\frac{a^2H}{nM}\right) \tag{28}$$

where $t_1(K)$ is the least positive root of (4) *when $K \leqq 4$ and of* (5) *when $K > 4$. Moreover, equality holds if and only if $\rho \equiv \rho_0$ where ρ_0 is defined by*

$$\rho_0(x) = \begin{cases} H\left(x - \dfrac{a}{4n}\right) + \dfrac{M}{a}\,, & 0 \leqq x \leqq \dfrac{a}{2n}\,, \\ \rho_0\left(\dfrac{a}{n} - x\right)\,, & \dfrac{a}{2n} \leqq x \leqq \dfrac{a}{n}\,, \\ \rho_0\left(x + \dfrac{ka}{n}\right)\,, & (k-1)\dfrac{a}{n} \leqq x \leqq \dfrac{ka}{n}\,, \end{cases} \tag{29}$$

$(k = 2, 3, \cdots, n)$ if $a^2H/nM \leqq 4$ and

$$\rho_0(x) = \begin{cases} 0\,, & 0 \leqq x \leqq \dfrac{a}{2n} - \sqrt{\dfrac{M}{nH}}\,, \\ H\left(x - \dfrac{a}{2n}\right) + \sqrt{\dfrac{HM}{n}}\,, & \dfrac{a}{2n} - \sqrt{\dfrac{M}{nH}} \leqq x \leqq \dfrac{a}{2n}\,, \\ \rho_0\left(\dfrac{a}{n} - x\right)\,, & \dfrac{a}{2n} \leqq x \leqq \dfrac{a}{n}\,, \\ \rho_0\left(x + \dfrac{ka}{n}\right)\,, & (k-1)\dfrac{a}{n} \leqq x \leqq \dfrac{ka}{n}\,, \end{cases} \tag{30}$$

$(k = 2, 3, \cdots, n)$ if $a^2H/nM > 4$.

Proof. We give a proof by induction. For $n = 1$, the theorem is a restatement of Theorem 2. Because of the nature of the induction, it will be necessary to prove the theorem for $n = 2$ before going to the general case. Thus, we start by considering the second eigenvalue $\lambda_2[\rho]$ and the associated eigenfunction u_2 of the system (1). This eigenfunction has exactly one nodal point in the open interval $(0, a)$ which we denote by α. $\lambda_2[\rho]$ is also then the lowest eigenvalue of each of the differential systems

$$u'' + \lambda\rho(x)u = 0 \,, \qquad u(0) = u(\alpha) = 0 \,, \qquad x \in [0, \alpha]$$

and

$$u'' + \lambda\rho(x)u = 0 \,, \qquad u(\alpha) = u(a) = 0 \,, \qquad x \in [\alpha, a] \,.$$

By Theorem 2, we conclude that

$$\lambda_2[\rho] \geqq t_1^3\Big(\frac{\alpha^2 H}{m}\Big)\Big/\alpha^3 H \text{ and } \lambda_2[\rho] \geqq t_1\Big(\frac{\beta^2 H}{m'}\Big)^3\Big/\beta^3 H \tag{31}$$

where $\alpha + \beta = a$, $m = \int_0^\alpha \rho(x)dx$ and $m' = M - m$.

We now use the same argument as that used in the proof of Theorem 2. We increase or decrease m so that $m + m'$ remains constant and the right hand quantities of the inequalities (31) become equal. For each α there is determined a unique value of m so that a function f is defined. The minimum of $t_1(\alpha^2 H/m)\alpha^3 H$ on the graph of f is then found to occur at $\alpha = a/2$, $m = M/2$ just as in the proof of Theorem 2. Hence we find that

$$\lambda_2[\rho]a^3 H \geqq t_1^3\Big(\frac{a^2 H}{2M}\Big) \,,$$ [1]

To complete the induction, we consider the nth eigenvalue $\lambda_n[\rho]$ and the corresponding eigenfunction u_n of (1). This function will have $n - 1$ distinct nodal points $x_k (k = 1, \cdots, n-1)$ in the open interval $(0, a)$. We assume that these points are ordered, i.e., $x_k < x_{k+1}$, and consider the differential systems

$$u'' + \lambda^{(1)}\rho(x)u = 0 \,, \qquad u(0) = u(x_1) = 0 \,, \qquad x \in [0, x_1] \tag{32}$$

and

$$u'' + \lambda^{(2)}\rho(x)u = 0, \qquad u(x_1) = u(a) = 0 \,, \qquad x \in [x_1, a] \tag{33}$$

where x_1 is the smallest nodal point. Then $\lambda_n[\rho]$ is equal to the

[1] We note at this point that it is easy to prove the theorem for $n = 2^q$, q a positive integer. One would hope to be able to carry out a reverse induction as in [6]. Unfortunately, the method used there cannot be directly applied here.

lowest eigenvalue $\lambda_1^{(1)}$ of the system (32) and it is equal to the $(n-1)$ st eigenvalue $\lambda_{n-1}^{(2)}$ of the system (33). Then corresponding eigenfunctions are just u_n with the domain restricted to $[0, x_1]$ for the system (32) and to $[x_1, a]$ for the system (33). From Theorem 2, we have

$$\lambda_n[\rho] = \lambda_1^{(1)} \geqq t_1^3\left(\frac{x_1^2 H}{m_1}\right)\Big/ x_1^3 H = \eta(x_1, m_1) \tag{34}$$

where $m_1 = \int_0^{x_1} \rho(x)dx$. By the induction hypothesis

$$\lambda_n[\rho] = \lambda_{n-1}^{(2)} \geqq t_1^3\left(\frac{\alpha^2 H}{m}\right)\Big/ \alpha^3 H = \eta(\alpha, m) \tag{35}$$

where $\alpha = (a - x_1)/(n-1)$ and $m = (1/(n-1))\int_{x_1}^{a} \rho(x)dx$. Equality holds in (34) if and only if ρ is defined by (15) or (16). Equality holds in (35) if and only if ρ is defined by (29) or (30) with n replaced by $(n-1)$, M/n by m, and a/n by α. Since the function defined by (29) or (30) is periodic of period $(a - x_1)/(n-1)$ the nodal points of the $(n-1)$st eigenfunction will occur at the points $x_1 + k\alpha (k = 1, 2, \cdots, n-2)$. By holding the string fixed at the last nodal point we get a string fixed between x_1 and $x_1 + (n-2)\alpha$ whose $(n-2)$nd eigenvalue is equal to $\eta(\alpha, m)$ as defined in (35). This is also the lowest eigenvalue of the piece of the string between $x_1 + (n-2)\alpha$ and a.

We want now to piece together the part of the string between 0 and x_1 with density defined (15) or (16) and that part between x_1 and $x_1 + (n-2)\alpha$ with density defined by (29) or (30) in such a way that the $(n-1)$st eigenvalue of the resulting string fixed between 0 and $x_1 + (n-2)\alpha$ is less than $\lambda_n[\rho]$. This can be done by increasing (or decreasing) the mass of the string between 0 and x_1 and decreasing (or increasing) the mass of the string between x_1 and $x_1 + (n-2)\alpha$ in such a way that the total mass between 0 and $x_1 + (n-2)\alpha$ remains constant and such that the equality

$$\eta(x_1, m_1 \pm \theta) = \eta(\alpha, m \mp \theta/(n-2)) \tag{36}$$

results. Here θ denotes the change in the mass m_1. This is essentially the same argument used in deriving (19) in the proof of Theorem 2. Equation (36) defines a function f. We assume as part of the induction hypothesis that the minimum value of η over the graph of this function occurs at the point $([x_1 + (n-2)\alpha]/n - 1, [m_1 + (n-2)m]/n - 1)$, and that this value of η is the $(n-1)$st eigenvalue of a string with density defined by (29) or (30) with n replaced by $n-1$, M/n by $[m_1 + (n-2)m]/(n-1)$, and a/n by $[x_1 + (n-2)\alpha]/(n-1)$. We now repeat this process, first fixing this new string at its first nodal point and adjoining the right-hand piece to that part of the string between $x_1 + (n - 1)\alpha$ and a. Continuing this process indefinitely, we define a sequence of numbers

$$\eta(x_1^{(\nu)}, m_1^{(\nu)})(\nu = 1, 2, \cdots)$$

all less than $\lambda_n[\rho]$. We show that this sequence converges to the value $\eta(a/n, M/n)$.

We see that the above process generates a sequence of first nodal points which satisfies the recurrence relation.

$$x_1^{(\nu+2)} = [x_1^{(\nu)} + (n - 2)x_1^{(\nu+1)}]/(n - 1)$$

with the initial conditions $x_1^{(1)} = x_1$ and $x_1^{(2)} = [x_1 + (n - 2)\alpha]/(n - 1)$. This may be solved by letting $x^{(\nu)} = r^\nu$ and determining r. We thus find that

$$x^{(\nu)} = c_1 + \left(\frac{-1}{n - 1}\right)^\nu c_2$$

where c_1 and c_2 are constants to be determined from the initial conditions. We see immediately that $x^{(\nu)} \rightarrow c_1$ as $\gamma \rightarrow \infty$ where $c_1 = a/n$. Similarly $m_1^{(\nu)} \rightarrow M/n$. By the construction of the sequence $\{\eta(x_1^{(\nu)}, m_1^{(\nu)})\}$ we have

$$\lambda_n[\rho] \geqq \eta(a/n, \ M/n) .$$

This proves Theorem 3.

5. **Remarks.** The methods used to prove Theorems 2 and 3 can also be used to find lower bounds for the eigenvalues of a vibrating string when the end points of the string are free, i.e., when $u'(0) = u'(a) = 0$ and when an end is fixed and the other is free, i.e., $u(0) = u'(a) = 0$. We do not state these theorems but merely note that for the free end point problem the lower bound of the nth eigenvalue $\mu_n[\rho]$ is the same as the lower bound for the $(n - 1)$st eigenvalue of the fixed end point problem. The same can be said for Krein's results quoted in the introduction.

For the fixed free problem the lower bound for the nth eigenvalue $\mu_n[\rho]$ is the same as the lower bound for the $(2n)$th eigenvalue of the fixed end point problem which is obtained from the fixed free problem by defining the density ρ to be symmetric about $x = a$ and considering the system

$$u'' + \lambda\rho(x)u = 0 , \qquad u(0) = u(2a) = 0$$

$x \in [0, 2a]$.

Finally, we note that the methods used in this paper can be used to obtain the lower bounds given by Krein for the nth eigenvalue of a string with density $|\rho(x)| \leqq H$. Our methods also yield bounds for the nth eigenvalue of a string with a continuous concave density, i.e., where

$$\rho\left(\frac{x_1 + x_2}{2}\right) \geqq \frac{1}{2}[\rho(x_1) + \rho(x_2)] .$$

These bounds will not be sharp except in the case of the lowest eigenvalue.

In general, it is to be expected that lower bounds will be obtained if the extreme eigenvalue, which corresponds to $\eta(\alpha, m)$ in this paper, yields convex level lines in the α, m plane whenever it is set equal to a constant. If we apply this idea to the concave case just mentioned, the extreme eigenvalue turns out to be $\mu_0/\alpha m$ where μ_0 is a fixed constant so the level lines are given by $m = \text{const.}/\alpha$ which is clearly convex.

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