PACIFIC JOURNAL OF MATHEMATICS
Vol. 21, No. 2, 1967

ON ASYMPTOTIC ESTIMATES FOR KERNELS OF CONVOLUTION TRANSFORMS

ZEEV DITZIAN

In this paper we shall try to answer two open questions posed by Dauns and Widder in their paper "Convolution transforms whose inversion functions have complex roots" (Pacific Journal of Mathematics, 1965, Volume 15(2), pp. 427–442) on page 441.

We shall be interested in the function $G_{2m}(t)$ defined by

$$G_{2m}(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{st}ds}{E_{2m}(s)} \tag{1.1}$$

where

$$E_{2m}(s) = \prod_{k=m+1}^{\infty}\left(1 - \frac{s^2}{a_k^2}\right) \tag{1.2}$$

where $\{a_k\}$ is a sequence of complex numbers such that

$$|\arg a_k| < \frac{\pi}{4} - \eta$$

for a fixed η, $0 < \eta < \pi/4$,

$$\sum_{k}^{\infty} |a_k|^{-2} < \infty\ , \qquad 0 < \operatorname{Re} a_i \leqq \operatorname{Re} a_{i+1} \text{ for all } i$$

and

$$\lim_{m\to\infty} |a_{m+1}|^2 \sum_{k=m+1}^{\infty} |a_k|^{-2} = \infty\ . \tag{1.3}$$

If a sequence $\{a_k\}$ satisfies all the above assumptions, we shall denote it by $\{a_k\} \in \text{class } C$. We obtain condition B, defined in [1, p. 436], if we replace (1.3) by (1.4)

$$\lim_{n\to\infty} |a_{n+1}|^{4/3} \sum_{n+1}^{\infty} |a_k|^{-2} = \infty\ . \tag{1.4}$$

If we take $a_k = k^\lambda\, 1/2 < \lambda < \infty$ then $\{a_k\} \in \text{class } C$, but of these sequences only those for which $1/2 < \lambda < 3/2$ satisfy condition B.

We define as in [1]

$$V_m = \sum_{k=m+1}^{\infty} a_k^{-2} \quad\text{and}\quad S_m = \sum_{k=m+1}^{\infty} |a_k|^{-2} \tag{1.5}$$

and whenever $\{a_k\} \in$ class C we prove

$$\lim_{m\to\infty} \int_{-\infty}^{\infty} | tG'_{2m}(t) | \, dt = (\cos^2 \varphi_m - \sin^2 \varphi_m)^{-3/2} \tag{1.6}$$

where

$$\varphi_m = \frac{1}{2} \arg V_m \left(-\frac{\pi}{2} < \arg V_m < \frac{\pi}{2} \right)$$

which answers the question posed in remark (3) [1, p. 441].

We shall also prove under the restriction $\{a_k\} \in$ class C Corollary 4.3 and an analogous theorem to Theorem 4.1.

As a by product we shall have

$$\lim_{m\to\infty} S_m^{1/2} \frac{d^n}{dt^n} G_{2m}(S_m^{1/2} t) = \frac{1}{\sqrt{4\pi}} \left(\frac{S_m}{V_m} \right)^{1/2} \frac{d^n}{dt^n} \exp\left(-t^2 \frac{S_m}{4V_m} \right) \tag{1.7}$$

which is more than necessary for proving other results and is an interesting estimate of $G_{2m}^{(n)}(t)$ by itself.

2. **Some lemmas.** In the author's thesis [2] and in a paper in collaboration with A. Jakimovski [3; Lemma 2.1.] the following lemma was proved:

LEMMA 2.1. *Suppose* $\sum_{k=1}^{\infty} | a_k |^{-2} < \infty$ *then the assumptions*

$$\sum_{k=m+1}^{\infty} | a_k |^{-(2+\alpha)} = o\left(\left(\sum_{k=m+1}^{\infty} | a_k |^{-2} \right)^{1+(\alpha/2)} \right) \qquad m \longrightarrow \infty \tag{2.1}$$

for some fixed $\alpha > 0$ *and*

$$\lim_{m\to\infty} \left(\max_{k>m} | a_k |^{-2} \right) \left(\sum_{k=m+1}^{\infty} | a_k |^{-2} \right)^{-1} = 0 \tag{2.2}$$

are equivalent, and therefore the assumptions (2.1) *for all positive* α *are equivalent.*

Proof. Let us assume (2.1) for some $\alpha > 0$. If (2.3) is not valid then a subsequence $\{m(r)\}$ of $m+1, m+2, \cdots$ exists such that for some $\beta > 0$

$$\left(\max_{k \geqq m(r)+1} | a_k |^{-2} \right) S_{m(r)}^{-1} \geqq \beta > 0$$

for all $r \geqq 1$. Therefore

$$\sum_{k=m(r)+1}^{\infty} | a_k |^{-2-\alpha} \geqq \left(\max_{k \geqq m(r)+1} | a_k |^{-2} \right)^{1+(\alpha/2)} \geqq \beta^{1+(\alpha/2)} S_{m(r)}^{1+(\alpha/2)}$$

which contradicts (2.1).

Assuming (2.2) then

$$\begin{aligned}\sum_{k=m+1}^{\infty} | a_k |^{-2-\alpha} &= \sum_{k=m+1}^{\infty} | a_k |^{-2} \leqq \Big(\max_{k\geqq m+1} | a_k |^{-\alpha}\Big) S_m \\ &= \Big(\Big(\max_{k\geqq m+1} | a_k |^{-2}\Big) S_m^{-1}\Big)^{\alpha/2} S_m^{1+(\alpha/2)} \\ &= o(S_m^{1+(\alpha/2)}) \qquad (m \longrightarrow \infty) .\end{aligned}$$

The following two lemmas are easy to verify.

LEMMA 2.2. *If* $\{a_k\} \in \text{class } C$ *then* $\{a_k\}$ *satisfies assumption* (2.2). *If* $|\arg a_k| < \pi/4 - \eta$, $0 < \operatorname{Re} a_i \leqq \operatorname{Re} a_{i+1}$ *and* $\{a_k\}$ *satisfies assumption* (2.2) *then* $\{a_k\} \in \text{class } C$.

LEMMA 2.3. *If* $|\arg a_k| < (\pi/4) - \eta$ *and* $\sum | a_k |^{-2} < \infty$, *then*

$$\cos\Big(\frac{\pi}{2} - 2\eta\Big) S_n \leqq | V_n | \leqq S_n . \tag{2.3}$$

We define now $F_m(z)$ by

$$F_m(z) = E_m(z \cdot S_m^{-1/2}) = \sum_{k=m+1}^{\infty} \Big(1 - \frac{z^2}{a_k^2 S_m}\Big) . \tag{2.4}$$

LEMMA 2.4. *Suppose* $\{a_k\} \in \text{class } C$ *then there exist constants* $k(p) > 0$ *independent of* m *so that for all real* y

$$| F_m(iy) | > 1 + k(p) y^{2p} \qquad \text{for } m > m_0(p) . \tag{2.6}$$

Proof. Define $a_k = | a_k | e^{i\beta_k}$, $-(\pi/4) + \eta < \beta_k < (\pi/4) - \eta$

$$\begin{aligned}| F_m(iy) | &= \Big| \prod_{k=m+1}^{\infty} \Big(1 - \frac{(iy)^2}{a_k^2 S_m}\Big)\Big| \\ &\geqq \prod_{k=m+1}^{\infty} \Big(1 + \frac{y^2}{| a_k |^2 S_m} \cos 2\beta_k\Big) \\ &\geqq \prod_{k=m+1}^{\infty} \left(1 + \frac{y^2 \cos\Big(\frac{\pi}{2} - 2\eta\Big)}{| a_k |^2 S_m}\right) \\ &= 1 + \sum_{p=1}^{\infty} \frac{y^{2p} \cos^p\Big(\frac{\pi}{2} - 2\eta\Big)}{S_m^p p!} \sum_{\substack{k(i)>m \\ i\neq j\ k(i)\neq k(j)}} | a_{k(1)} \cdots a_{k(p)} |^{-2} .\end{aligned}$$

Since we have $\lim_{m\to\infty} \max_{k>m} | a_k |^{-2} S_m^{-1} = 0$ we can find $m_0(p)$ so that for $m > m_0(p)$ $\max_{k>m} | a_k^{-2} | < (1/2p) S_m$. Therefore we have

$$\sum_{\substack{k(i)>m\\ k(i)\neq k(j),i\neq j}} |a_{k(1)}\cdots a_{k(p)}|^{-2}$$
$$= \sum_{\substack{k(i)>m\\ k(i)\neq k(j),i\neq j}} \left(S_m - \sum_{i=1}^{p-1}|a_{k(i)}|^{-2}\right)|a_{k(1)}\cdots a_{k(p-1)}|^{-2}$$
$$\geqq \frac{1}{2}S_m \sum_{\substack{k(i)>m\\ k(i)\neq k(j),j\neq i}} |a_{k(1)}\cdots a_{k(p-1)}|^{-2} \geqq \left(\frac{1}{2}\right)^p S_m^p .$$

Hence

$$|F_m(iy)| \geqq 1 + y^{2p}\frac{\cos^p\left(\frac{\pi}{2}-2\eta\right)S_m^p}{S_m^p p!\,2^p} = 1 + k(p)y^{2p} .$$

3. The asymptotic estimates for $G_{2m}^{(k)}(t)$.

THEOREM 3.1. *Let* $\{a_k\}\in$ clase C; *then for all* $n = 0, 1, \cdots$

$$(3.1)\qquad \lim_{m\to\infty} S_m^{1/2}\frac{d^n}{dt^n}G_{2m}(S_m^{1/2}t) = \frac{1}{\sqrt{4\pi}}\left(\frac{S_m}{V_m}\right)^{1/2}\frac{d^n}{dt^n}\exp\left(-t^2\frac{S_m}{4V_m}\right)$$

uniformly in $-\infty<t<\infty$ (*we choose* $\arg V_m^{1/2} = (1/2)\arg V_m$).

Proof. Following the proof of the special case $n=0$ and $\arg a_k = 0$ by Hirschman—Widder [4; pp. 140-1] we have

$$S_m^{1/2}G_{2m}(S_m^{1/2}t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^{zt}dz}{F_{2m}(s)} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{iyt}dy}{F_{2m}(iy)} .$$

By an estimate of [5; p. 246] we have for $|z|<R$ and

$$R\cdot[|a_k|S_m^{1/2}]^{-1} \leqq \frac{1}{2}$$

$$\left|\log\left\{\left(1-\frac{z^2}{a_k^2S_m}\right)\exp(z^2/a_k^2S_m)\right\}\right| \leqq 4R^3\frac{1}{|a_k|^3S_m^{3/2}} .$$

Recalling that $\sum_{k=m+1}^{\infty}1/a_k^2S_m = V_m/S_m$ and since by Lemma 2.1

$$\sum_{k=m+1}^{\infty}\frac{1}{|a_k|^3S_m^{3/2}} = o(1)\ \ m\to\infty ,$$

we have for $|z|<R$ and $m>m_0(R)$

$$\left|F_m(z) - \exp\left(-\frac{V_mz^2}{S_m}\right)\right| < \varepsilon_1 .$$

Since by Lemma 2.4 $R>R_0(\varepsilon_2,\eta)$ implies

$$\int_R^{\infty}\frac{|y|^n\,dy}{|F_{2m}(iy)|} < \varepsilon_2 \quad\text{and}\quad \int_{-\infty}^{-R}\frac{|y|^n\,dy}{|F_{2m}(iy)|} < \varepsilon_2$$

we have

$$\begin{aligned}
&S_m^{1/2}\frac{d^n}{dt^n}G_{2m}(S_m^{1/2}t)\\
&\quad=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{(iy)^n e^{iyt}}{F_m(iy)}dy\\
&\quad=\frac{1}{2\pi}\int_{-\infty}^{\infty}(iy)^n\exp\Big(-\frac{V_m}{S_m}y^2+iyt\Big)dy+o(1)\\
&\quad=\frac{d^n}{dt^n}\Big\{\exp\Big(-\frac{t^2S_m}{4V_m}\Big)\\
&\qquad\times\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\,[-(V_m^{1/2}S_m^{-1/2}y-itS_m^{1/2}(4V_m)^{-1/2})^2]dy\Big\}+o(1)\\
&\quad=\Big(\frac{S_m}{V_m}\Big)^{1/2}\frac{d^n}{dt^n}\Big\{\exp\Big(-\frac{t^2S_m}{4V_m}\Big)\frac{1}{2\pi}\int_\Gamma e^{-z^2}dz\Big\}+o(1)\\
&\quad=\frac{1}{\sqrt{4\pi}}\Big(\frac{S_m}{V_m}\Big)^{1/2}\frac{d^n}{dt^n}\exp\Big(\frac{-t^2S_m}{4V_m}\Big)+o(1)\qquad(m\uparrow\infty)
\end{aligned}$$

using the residue theorem, the fact that e^{-z^2} is entire and that

$$|\arg V_m^{1/2}|<\frac{\pi}{4}\qquad\text{for all } m\,.$$

As a corollary we derive

THEOREM 3.2. *If $\{a_k\}$ satisfies assumption C then*

$$\int_{-\infty}^{\infty}|G_{2m}(t)|\,dt=(\cos^2\varphi_m-\sin^2\varphi_m)^{-1/2}+o(1)\qquad m\to\infty \tag{3.2}$$

and

$$\int_{-\infty}^{\infty}|tG'_{2m}(t)|\,dt=(\cos^2\varphi_m-\sin^2\varphi_m)^{-3/2}+o(1)\qquad m\to\infty \tag{3.3}$$

where $2\varphi_m=\arg V_m$.

Proof. Since by Lemma 2.4 of [1].

$$|G_{2m}(t)|<MS_m^{-1/2}\exp\,(-KS_m^{-1/2}|t|) \tag{3.4}$$

we have

$$\int_{-\infty}^{\infty}|G_{2m}(t)|\,dt=\int_{-\infty}^{\infty}|S_m^{1/2}G_{2m}(S_m^{1/2}t)|\,dt=\int_{-R}^{R}|S_m^{1/2}G_{2m}(S_m^{1/2}t)|\,dt$$
$$+\,o(1)\qquad(R\uparrow\infty)\,.$$

This combined with (3.1) and a simple integration yield (3.2).

To prove (3.3) we use Lemma 3.2 case A (since for $\{a_k\}\in$ class C $S_m\geqq 4r_{m+1}^{-2}\equiv 4\,|a_{m+1}|^{-2}$ for $m>m_0$) which is

(3.5) $$| G'_{2m}(t) | \leqq M_1 S_m^{-1} \exp(-K_1 S_m^{-1/2} | t |) .$$

Therefore we have

$$\int_R^\infty | S_m t G'_{2m}(S_m^{1/2} | t |) | \, dt \leqq M_1 \frac{1}{(K_1)^2} e^{-K_1 R} = o(1) \qquad R \longrightarrow \infty .$$

This implies

$$\int_{-\infty}^\infty | tG'_{2m}(t) | \, dt = \int_{-\infty}^\infty | S_m t G'_{2m}(S_m^{1/2} t) | \, dt$$
$$= \frac{1}{2\sqrt{4\pi}} \left| \frac{S_m}{V_m} \right|^{3/2} \int_{-\infty}^\infty t^2 \exp\left(-\frac{t^2}{4} S_m \operatorname{Re} \frac{1}{V_m}\right) dt + o(1)$$
$$= \left(\frac{| V_m |^{-1}}{(\operatorname{Re}(V_m^{-1}))} \right)^{3/2} + o(1) = (\cos^2 \varphi_m - \sin^2 \varphi_m)^{-3/2} + o(1) \quad (m \uparrow \infty) .$$

4. **Remarks.** I. For the theorems and the lemmas proved in this paper $0 < \operatorname{Re} a_i \leqq \operatorname{Re} a_{i+1}$ is not essential and the condition (2.2) can replace it and (1.3).

II. Theorem 3.1 which replaces Theorem 4.1 yields for the case $n = 0$ only the following

(4.6) $$G_{2m}(t) = (4\pi V_m)^{-1/2} \exp(-t^2/4V_m) + o(S_m^{-1/2}) \qquad m \longrightarrow \infty$$

but if one follows the proof of Theorem 4.1 of [1] and Lemma 4.2 of [1] almost litarally one obtains for $\{a_k\} \in$ class C

(4.7) $$G_{2m}(t) = (4\pi V_m)^{-1/2} \exp(-t^2/4V_m) + o(| a_{m+1} |^{-2} S_m^{-3/2}) \qquad m \longrightarrow \infty$$

which is somewhat more general.

References

1. J. Dauns and D. V. Widder, *Convolution transforms whose inversion function have complex roots*, Pacific J. Math. (2) **15** (1965).
2. Z. Ditzian, *Inversion formulae and various properties of integral transforms* (Thesis, 1965, Hebrew University).
3. Z. Ditzian and A, Jakimovski, *Inversion and Jump formulae for convolution transform* (to appear).
4. I. I. Hirschman and D. V. Widder, *The Convolution Transform*, Princeton Press, 1955.
5. E. C. Titchmarsh, *The Theory of Functions*, Oxford University Press, 1932.

Received May 23, 1966.

WASHINGTON UNIVERSITY
ST. LOUIS, MISSOURI AND
MICHIGAN STATE UNIVERSITY
EAST LANSING, MICHIGAN