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A REFINEMENT OF SELBERG'S ASYMPTOTIC EQUATION

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The elementary proofs of the prime number theorem are essentially based on asymptotic equations of the form

$$\text{(A)} \qquad f(x)\log x + \int_1^x f\left(\frac{x}{t}\right) d\psi(t) = O(x)\,,$$

where $f(x)$ is some function concerning the primes, $\psi(x)$ is Tchebychev's function and the limits in the integral—as throughout in this paper—are taken from 1— to $x+$. This paper gives an elementary method for refining the right hand side of (A).

This method is based on the lemma of Tatuzawa and Iseki [2], and, assuming the prime number theorem, on an estimation of remainder integral which is more accurate than earlier ones.

Writing

$$\psi(x) = \sum_{n \leqq x} \Lambda(n) = \sum_{p^\nu \leqq x} \log p, \quad R(x) = \psi(x) - x$$

we have the two equivalent forms of Selberg's asymptotic equation

$$(1) \qquad R(x)\log x + \int_1^x R\left(\frac{x}{t}\right) d\psi(t) = O(x)\,,$$

$$(2) \qquad \psi(x)\log x + \int_1^x \psi\left(\frac{x}{t}\right) d\psi(t) = 2x\log x + O(x)\,,$$

each of which is known to imply the prime number theorem: $\psi(x) = x + o(x)$. In this paper we give refinements of (1) and (2), showing that

$$(1') \qquad R(x)\log x + \int_1^x R\left(\frac{x}{t}\right) d\psi(t) = -(\gamma + 1)x + o(x)\,,$$

$$(2') \qquad \psi(x)\log x + \int_1^x \psi\left(\frac{x}{t}\right) d\psi(t) = 2x\log x - (2\gamma + 1)x + o(x)\,,$$

where γ denotes Euler's constant. The prime number theorem, however, has then to be assumed. In addition, we give some similar results.

2. Using the idea of Tatuzawa and Iseki [2] we start from the following

LEMMA. *Let $f(x)$ be defined for all $x \geqq 0$ and let*

$$g(x) = \log x \sum_{n \leqq x} f\left(\frac{x}{n}\right) = \log x \int_1^x f\left(\frac{x}{t}\right) d[t] \; . \tag{3}$$

Then

$$f(x) \log x + \int_1^x f\left(\frac{x}{t}\right) d\psi(t) = \int_1^x g\left(\frac{x}{t}\right) dM(t) \; , \tag{4}$$

where

$$M(x) = \sum_{n \leqq x} \mu(n) \; .$$

The Lemma follows immediately by substituting the expression (3) for g in the right hand side of (4) and noting that

$$\Lambda(n) = \sum_{d/n} \mu(d) \log \frac{n}{d} = - \sum_{d/n} \mu(d) \log d \; .$$

Inserting $f(x) = R(x)$ in (4), we obtain the left hand side of (1′). On the other hand,

$$\sum_{n \leqq x} \psi\left(\frac{x}{n}\right) = \sum_{n \leqq x} \log n = x \log x - x + O(\log x)$$
$$= x \sum_{n \leqq x} \frac{1}{n} - (\gamma + 1)x + O(\log x) \; ,$$

and writing $x = [x] + O(1)$, we obtain

$$g(x) = \log x \left(\sum_{n \leqq x} \psi\left(\frac{x}{n}\right) - \sum_{n \leqq x} \frac{x}{n} \right) = \log x \left(-(\gamma + 1)[x] + u(x)\right) \; ,$$

where $u(x) = O(\log x)$. Thus we have

$$\int_1^x g\left(\frac{x}{t}\right) dM(t) = -(\gamma + 1) \int_1^x \left[\frac{x}{t}\right] \log \frac{x}{t} \, dM(t) + \int_1^x u\left(\frac{x}{t}\right) \log \frac{x}{t} \, dM(t) \; .$$

From the lemma with $f(x) \equiv 1$ we see that the first expression on the right can be written as

$$-(\gamma + 1)(\log x + \psi(x)) = -(\gamma + 1)x + o(x) \; ,$$

assuming the prime number in the form $\psi(x) = x + o(x)$. This yield

$$R(x) \log x + \int_1^x R\left(\frac{x}{t}\right) d\psi(t) = -(\gamma + 1)x + o(x) + \int_1^x u\left(\frac{x}{t}\right) \log \frac{x}{t} \, dM(t) \; ,$$

which proves (1′), as soon as we can show that the last expression is $o(x)$. We choose $0 < \delta < 1$ and write

$$I = \int_1^x u\left(\frac{x}{t}\right) \log \frac{x}{t}\, dM(t) = \int_1^{\delta x} + \int_{\delta x}^x = I_1 + I_2 \,.$$

The integral I_1 can, by virtue of $u(x) = O(\log x)$, be estimated as follows:

$$I_1 = O\left(\int_1^{\delta x} \log^2 \frac{x}{t}\, d[t]\right) = O\left(\int_0^{\delta x} \log^2 \frac{x}{t}\, dt\right) = \eta(\delta)x \,,$$

where $\eta(\delta)$ vanishes with δ. In order to estimate I_2 we choose an arbitrary $\varepsilon > 0$ and $x' > 0$ such that $|M(x)| < \varepsilon x$, if $\delta x > x'$, which can be done by the prime number theorem. Then, integrating by parts, we obtain

$$\begin{aligned} |I_2| &\leqq \left|\log \frac{1}{\delta}\, u\left(\frac{1}{\delta}\right) M(\delta x)\right| + \left|\int_{\delta x}^x M(t)\, d\left(\log \frac{x}{t}\, u\left(\frac{x}{t}\right)\right)\right| \\ &\leqq \log \frac{1}{\delta} \left|u\left(\frac{1}{\delta}\right)\right| \delta\varepsilon x + \left[\int_1^{1/\delta} \frac{|d(\log t\, u(t))|}{t}\right] \varepsilon x = K_\delta \varepsilon x \,, \end{aligned}$$

where K_δ is a constant depending only on δ. Thus we can choose the number δ small enough to make the expression $|I_1|/x$ arbitrary small and then for fixed δ choose the value of $x (> x')$ so that the same holds for I_2. Thus we have $I = I_1 + I_2 = o(x)$, and (1′) is proved. (2′) then follows immediately by inserting $R(x) = \psi(x) - x$ and observing the equation

$$\int_1^x \frac{d\psi(t)}{t} = \log x - \gamma + o(1) \,,$$

which also follows from the prime number theorem.

3. The method used above can be applied to several similar problems. Taking the prime number theorem for granted and using (3) and (4) with $f(x)$ equal to

$$M(x) = \sum_{n \leqq x} \mu(n)$$

or

$$xB(x) = x \sum_{n \leqq x} \frac{\mu(n)}{n} \,,$$

we obtain (see [1] p. 36)

$$M(x) \log x + \int_1^x M\left(\frac{x}{t}\right) d\psi(t) = o(x) \,,$$

$$B(x) \log x + \int_1^x B\left(\frac{x}{t}\right) \frac{d\psi(t)}{t} = 1 + o(1) \,.$$

Futhermore, writing

$$\varepsilon(x) = \int_1^x \frac{d\psi(t)}{t} - \log x + \gamma = h(x) - \log x + \gamma \ ,$$

and applying the same method to the function $x\varepsilon(x)$, we can show that (see [1] pp. 22–23)

$$\varepsilon(x) \log x + \int_1^x \varepsilon\left(\frac{x}{t}\right) dh(t) = (\kappa + \gamma^2) + o(1) \ ,$$

or

$$\int_1^x h\left(\frac{x}{t}\right) dh(t) + \int_1^x \log t \, dh(t) = \log^2 x - 2\gamma \log x + \kappa + 2\gamma^2 + o(1) \ ,$$

where

$$\kappa = \lim_{x \to \infty} \left(2 \sum_{n \leq x} \frac{\log n}{n} - \log^2 x\right) .$$

Wirsing ([3] p. 8) gives by stronger assumption and assertion a similar result for the function

$$r(x) = \sum_{n \leq x} \left(\frac{1}{n} - \frac{\Lambda(n)}{n}\right) - 2\gamma \ .$$

Corresponding equations concerning the prime number theorem for arithmetic progressions can also be deduced.

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