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ON THE WEAK LAW OF LARGE NUMBERS AND THE GENERALIZED ELEMENTARY RENEWAL THEOREM

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$\{X_n\}$ is a sequence of independent, nonnegative, random variables and $G_n(x) = P\{X_1 + \cdots + X_n \leqq x\}$. $\{a_n\}$ is a sequence of nonnegative constants such that, for some $\alpha > 0, \gamma > 0$, and function of slow growth $L(x)$,

$$\sum_1^N a_r \sim \frac{\alpha N^\gamma L(N)}{\Gamma(1+\gamma)}, \quad \text{as } N \to \infty .$$

A Generalized Elementary Renewal Theorem (GERT) gives conditions such that, for some $\mu > 0$,

$$(*) \qquad \Psi(x) = \Sigma a_r G_r(x) \sim \frac{\alpha L(x)}{\Gamma(1+\gamma)} \left(\frac{x}{\mu}\right)^\gamma, \quad \text{as } x \to \infty .$$

The Weak Low of Large Numbers (WLLN) states that $(X_1 + \cdots + X_n)/n \to \mu$, as $n \to \infty$, in probability. Theorem 1 proves that WLLN implies (*). Theorem 3 proves that (*) implies WLLN if, additionally, it is given that

(i) $\sum_1^n P\{X_j > n\varepsilon\} \to 0$ as $n \to \infty$, for every small $\varepsilon > 0$;

(ii) for some $\varepsilon > 0$, $n^{-1} \sum_1^n \int_0^{n\varepsilon} P\{X_j > x\} dx$ is a bounded function of n. Theorem 2 supposes the $\{X_n\}$ to have finite expectations and proves (*) implies WLLN if it is given that

$$\limsup_{n\to\infty} \frac{\mathscr{E}X_1 + \mathscr{E}X_2 + \cdots + \mathscr{E}X_n}{n} \leqq \mu ,$$

in which case $(\mathscr{E}X_1 + \cdots + \mathscr{E}X_n)/n$ necessarily tends to μ as $n \to \infty$. Finally, an example shows that (*) can hold while the WLLN fails to hold. Much use is made of the fact that a necessary and sufficient condition for the WLLN is that, for all small $\varepsilon > 0$,

$$\frac{1}{n}\int_0^{n\varepsilon} \sum_1^n P\{X_j > x\} dx \to \mu, \quad \text{as } n \to \infty .$$

Let $\{X_n\}$, $n = 1, 2, \cdots$, be a sequence of independent, *nonnegative*, random variables; write $F_n(x) \equiv P\{X_n \leqq x\}$; $S_n = X_1 + X_2 + \cdots + X_n$; $G_n(x) = P\{S_n \leqq x\}$; when the first moments exist, write $\mu_n = \mathscr{E}X_n$. Let $\{a_n\}$ be a sequence of nonnegative constants such that, for some constants $\alpha > 0$, $\gamma > 0$, and some function of slow growth $L(x)$,

$$(1.1) \qquad \sum_{n=1}^N a_n \sim \frac{\alpha N^\gamma L(N)}{\Gamma(1+\gamma)}, \quad \text{as } N \to \infty .^{1}$$

[1] We carry the factor $\Gamma(1+\gamma)$ to simplify comparisons with Smith (1964).

By a Generalized Elementary Renewal Theorem (GERT) we shall mean a theorem that establishes conditions such that

$$\sum_{n=1}^{\infty} a_n G_n(x) \sim \frac{\alpha L(x)}{\Gamma(1+\gamma)} \left(\frac{x}{\mu}\right)^{\gamma}, \text{ as } x \to \infty . \tag{1.2}$$

for some constant $\mu > 0$. The special case of (1.2) when $a_n = 1$ for all n is the Elementary Renewal Theorem (ERT).

By a Weak Law of Large Numbers (WLLN) we shall mean a theorem that establishes conditions such that

$$\frac{S_n}{n} \to \mu, \text{ as } n \to \infty . \tag{1.3}$$

in probability.

This paper extends and leans heavily upon an earlier one (Smith, 1964) which we shall henceforth refer to simply as S. It will be concerned with weakening conditions of S for a GERT to hold for nonnegative random variables (specifically, we drop the assumption of finite means) and with investigating to what extent a GERT implies the WLLN and vice versa.

Two conditions play an important role in our work. They are

(A) For every small $\varepsilon > 0$,

$$\sum_{j=1}^{n} \{1 - F_j(n\varepsilon)\} \to 0, \text{ as } n \to \infty ;$$

(B) For every small $\varepsilon > 0$,

$$\frac{1}{n} \int_0^{n\varepsilon} \sum_{j=1}^{n} \{1 - F_j(x)\} dx \to \mu, \text{ as } n \to \infty .$$

It is an easy exercise to show that (B) implies (A); all we can infer from (A), concerning (B), is that the upper and lower limits, as $n \to \infty$, of

$$\frac{1}{n} \int_0^{n\varepsilon} \sum_{j=1}^{n} \{1 - F_j(x)\} dx$$

are independent of the small $\varepsilon > 0$.

It is known from the work of Bobrov (1939), described by Gnedenko and Kolmogorov (1954; see especially page 141), that condition (B) is *necessary* and *sufficient* for the WLLN (1.3) to hold. Thus the WLLN implies (A) and (B). It is interesting, therefore, that we are able to prove (in §2):

THEOREM 1. *A sufficient condition for the Generalized Elementary Renewal Theorem* (1.2) *is that* (B) *shall hold. Thus the*

WLLN *implies the* GERT.[2]

The question naturally arises, does the holding of a GERT (1.2) imply the truth of the WLLN? After some efforts we have discovered that the answer to this question is in the negative. In §5 we shall present the necessary counter-example for which (1.2) is true and (1.3) false. However, it transpires that the validity of a GERT does imply the WLLN *if a weak supplemental condition is given to be satisfied.* The situation is perhaps comparable to that arising with Abelian and Tauberian theorems in analysis; the Abelian theorem usually holds "in general," the converse Tauberian theorem usually requires the satisfaction of an extra "Tauberian Condition." We shall prove, in §3 and §4 the following two theorems relevant to this paragraph.

THEOREM 2. *If the* GERT (1.2) *holds, if all the means* $\mu_n = \mathscr{E} X_n$ *are finite, and if*

$$\limsup_{n \leftarrow \infty} \frac{\mu_1 + \mu_2 + \cdots + \mu_n}{n} \leqq \mu ,$$

then the WLLN (1.3) *holds and*

$$\frac{\mu_1 + \mu_2 + \cdots + \mu_n}{n} \to \mu \text{ as } n \to \infty .$$

THEOREM 3. *Suppose the* GERT (1.2) *holds together with conditions* (A) *and the following*:

(C) *For some* $\varepsilon > 0$,

$$\limsup_{n \to \infty} \frac{1}{n} \int_0^{\varepsilon n} \textstyle\sum_1^n \{1 - F_j(x)\} dx < \infty .$$

Then the WLLN (1.3) *holds. Condition* (C) *must be satisfied if there exists no sequence* $\{n_k\}$ *such that* $n_k^{-1} \sum_1^{n_k} X_j$ *tends to infinity in probability.*

As we have said, this paper leans heavily on S; one result buried in S turns out to be especially important. The argument of §5 of S (pp. 689–698) essentially (if not ostensibly) proves the following:

FUNDAMENTAL LEMMA. *Suppose condition* (C) *holds and there exists a* $\delta > 0$, *such that, for every* $\varepsilon > 0$,

$$\liminf_{n \to \infty} \frac{1}{n} \int_0^{\varepsilon n} \textstyle\sum_1^n \{1 - F_j(x)\} dx \geqq \delta .$$

[2] Consequently it is impossible to have (1.2) and (1.3) holding simultaneously, but for different values of μ.

Then for any small $\eta > 0$ *we can find a large* $C(\eta)$ *such that*

$$\sum_{n \geqq Cx}^{\infty} a_n G_n(x) < \eta x^{\gamma} L(x)$$

for all large x.

One further comment is called for. If $\gamma = 0$ in (1.2) the constant μ disappears from the right-hand said and we have a simpler relation

$$\sum_{n=1}^{\infty} a_n G_n(x) \sim \alpha L(x), \text{ as } x \to \infty \ . \tag{1.4}$$

It seems that this special case needs special treatment, and that (1.4) will hold under *considerably* more general conditions than Theorem 1 suggests; we hope to study (1.4) elsewhere, and throughout this present paper take $\gamma > 0$.

2. Proof of Theorem 1. To begin with we shall establish (leaning heavily on arguments in S) the following.

LEMMA 2.1. *Under conditions* (A) *and* (B) *there is an unbounded nondecreasing function* $l(n)$ *such that*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} \int_0^{r/l(r)} \{1 - F_r(x)\} dx = \mu \ .$$

Proof. From (A) and (B) we can evidently find an unbounded nondecreasing $\lambda(n)$ such that, as $n \to \infty$,

$$\sum_{j=1}^{n} \left\{1 - F_j\left(\frac{n}{\lambda(n)}\right)\right\} \to 0 \ , \tag{2.1}$$

$$\frac{1}{n} \int_0^{n/\lambda(n)} \sum_1^n \{1 - F_j(x)\} dx \to \mu \ . \tag{2.2}$$

Lemma 9 of S then shows that we can find another unbounded nondecreasing $w(n) \leqq \lambda(n)$ and such that $w(n)/n$ decreases to zero as $n \to \infty$. But

$$\frac{1}{n} \int_{n/\lambda(n)}^{n/w(n)} \sum_1^n \{1 - F_j(x)\} dx \leqq \frac{1}{w(n)} \sum_1^n \left\{1 - F_j\left(\frac{n}{\lambda(n)}\right)\right\} \ .$$

Since the right member tends to zero as $n \to \infty$, we can infer from (2.1) and (2.2) that, as $n \to \infty$.

$$\sum_{j=1}^{n} \left\{1 - F_j\left(\frac{n}{w(n)}\right)\right\} \to 0 \ , \tag{2.3}$$

$$\frac{1}{n}\int_0^{n/w(n)} \sum_1^n \left\{1 - F_j(x)\right\}dx \to \mu\ . \tag{2.4}$$

Let $l(n)$ increase much more slowly than $w(n)$; we make the function $l(n)$ more precise later. Support $t(n)$ is the least integer such that $r/l(r) \geqq n/w(n)$ for all $r \geqq t(n)$. Then, for any $\varepsilon > 0$,

$$\frac{n}{w(n)t(n)} \leqq \frac{1}{l(t(n))} \leqq \varepsilon \tag{2.5}$$

for all large n.

Since $w(n)\uparrow$ we have, for large n,

$$w(n) \geqq w\left(\frac{n \log w(n)}{w(n)}\right)\ .$$

Thus, if

$$s(n) = \frac{n \log w(n)}{w(n)}\ .$$

we have

$$\frac{s(n)}{\log w(s(n))} \geqq \frac{n}{w(n)}\ .$$

But we may assume $n/w(n)\ \uparrow$ with n, and hence $n/\log w(n))\ \uparrow$ also. Thus, if $l(n) = \log w(n)$ we have $r/l(r) \geqq n/w(n)$ for all $r \geqq s(n)$. Hence $t(n) \leqq s(n)$. But $s(n)/n \to 0$; thus $t(n)/n \to 0$.

If we set

$$T_1(n) = \frac{1}{n}\sum_1^{t(n)}\int_0^{n/w(n)}\{1 - F_j(x)\}dx$$

then, by (2.5),

$$T_1(n) \leqq \left(\frac{t(n)}{n}\right)\left\{\frac{1}{t(n)}\sum_1^{t(n)}\int_0^{\varepsilon t(n)}\{1 - F_j(x)\}dx\right\}\ .$$

Thus $T_1(n) \to 0$ as $n \to \infty$.

If we set

$$T_2(n) = \frac{1}{n}\sum_{j=t(n)+1}^{n}\int_0^{j/l(j)}\{1 - F_j(x)\}dx$$

then it is clear that $(\mu - \varepsilon) \leqq T_1(n) + T_2(n)$ for large n, and hence that $T_2(n) > \mu - 2\varepsilon$ for large n. Therefore

$$\liminf_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\int_0^{j/l(j)}\{1 - F_j(x)\}dx \geqq \mu\ .$$

But

$$\frac{1}{n}\sum_{j=1}^{n}\int_0^{j/l(j)}\{1-F_j(x)\}dx \leqq \frac{1}{n}\sum_{j=1}^{n}\int_0^{n/l(n)}\{1-F_j(x)\}dx ,$$

since $j/l(j)\uparrow$, and this, in turn, is smaller than

$$\frac{1}{n}\sum_{j=1}^{n}\int_0^{\varepsilon n}\{1-F_j(x)\}dx ,$$

for large n, since $l(n)\rightarrow\infty$. Thus

$$\limsup_{n\rightarrow\infty}\frac{1}{n}\sum_{j=1}^{n}\int_0^{j/l(j)}\{1-F_j(x)\}dx \leqq \mu$$

and the lemma is proved.

To prove the theorem we begin by setting

$$\begin{aligned} Y_r &= X_r \qquad \text{if } X_r \leqq r/l(r) \\ &= r/l(r) \text{ otherwise.} \end{aligned}$$

For $\varepsilon > 0$ we write

$$L_n(\varepsilon) = \frac{1}{n}\int_0^{n\varepsilon}\sum\nolimits_1^n P\{Y_j > x\}dx .$$

But if $r \leqq n$, for all large n, $r/l(r) \leqq n/l(n) \leqq \varepsilon n$, so that

$$L_n(\varepsilon) = \frac{1}{n}\sum\nolimits_1^n\int_0^{r/l(r)}\{1-F_r(x)\}dx$$

and hence, by Lemma 2.1, $L_n(\varepsilon)\rightarrow\mu$ as $n\rightarrow\infty$.

The sequence $\{Y_n\}$ satisfies the conditions of Theorem 1 of S. Thus

$$\sum_1^\infty a_r P\{Y_r \leqq x\} \sim \frac{\alpha x^\gamma L(x)}{\Gamma(1+\gamma)^r\mu^r} , \tag{2.6}$$

as $x\rightarrow\infty$.

Furthermore both $\{Y_n\}$ and $\{X_n\}$ satisfy conditions (A) and (B). Thus, by the result of Bobrov already quoted, as $n\rightarrow\infty$,

$$\frac{Y_1+\cdots+Y_n}{n}\rightarrow\mu , \quad \text{in probability,}$$

$$\frac{X_1+\cdots+X_n}{n}\rightarrow\mu , \quad \text{in probability.}$$

Hence, if $Z_n = X_n - Y_n$,

$$\frac{Z_1+\cdots+Z_n}{n}\rightarrow 0 , \quad \text{in probability.}$$

Plainly $G_n(x) \equiv P\{X_1 + \cdots + X_n \leqq x\} \leqq P\{Y_1 + \cdots + Y_n \leqq x\}$.

In all that follows let us write

$$\Psi(x) = \sum_1^\infty a_n G_n(x) .$$

Thus to prove our theorem we need only show

$$\liminf_{x\to\infty} \frac{\Psi(x)}{x^\gamma L(x)} \geqq \frac{\alpha}{\mu^\gamma \Gamma(1+\gamma)} . \tag{2.7}$$

It is not hard to show that $\{Y_n + \varepsilon\}$ also satisfies Theorem 1 of S, with μ increased to $\mu + \varepsilon$. Thus

$$\frac{1}{x^\gamma L(x)} \sum_{n=1}^\infty a_n P\{Y_1 + Y_2 + \cdots + Y_n + n\varepsilon \leqq x\} \sim \frac{\alpha}{(\mu+\varepsilon)^\gamma \Gamma(1+\gamma)} .$$

Also, by the "fundamental lemma" of §1, given any $\eta > 0$ we can find $C \equiv C(\eta)$ such that

$$\frac{1}{x^\gamma L(x)} \sum_{n=Cx}^\infty a_n P\{Y_1 + \cdots + Y_n + n\varepsilon \leqq x\} < \eta$$

for all large x. Therefore

$$\begin{aligned} \liminf_{x\to\infty} \frac{1}{x^\gamma L(x)} &\sum_1^{Cx} a_n P\{Y_1 + \cdots + Y_n + n\varepsilon \leqq x\} \\ &\geqq \frac{\alpha}{(\mu+\varepsilon)^\gamma \Gamma(1+\gamma)} - \eta . \end{aligned} \tag{2.8}$$

Now

$$\begin{aligned} P\{X_1 + \cdots + X_n \leqq x\} &\geqq P\{Y_1 + \cdots + Y_n + n\varepsilon \leqq x \ \&\ Z_1 + \cdots + Z_n \leqq n\varepsilon\} \\ &\geqq P\{Y_1 + \cdots + Y_n + n\varepsilon \leqq x\} \\ &\quad - P\{Z_1 + \cdots + Z_n > n\varepsilon\} , \end{aligned}$$

and so

$$\begin{aligned} \sum_{n=1}^{Cx} a_n P\{X_1 + \cdots + X_n \leqq x\} &\geqq \sum_{n=1}^{Cx} a_n P\{Y_1 + \cdots + Y_n + n\varepsilon \leqq x\} \\ &\quad - \sum_{n=1}^{Cx} a_n P\{Z_1 + \cdots + Z_n > n\varepsilon\} . \end{aligned} \tag{2.9}$$

But $P\{Z_1 + \cdots + Z_n > n\varepsilon\} \to 0$ as $n \to \infty$, so that one can establish easily that

$$\sum_1^{Cx} a_n P\{Z_1 + \cdots + Z_n > n\varepsilon\} = 0(x^\gamma L(x)) .$$

Hence, from (2.8) and (2.9) and the arbitrariness of η we finally deduce the desired (2.7).

3. Proof of Theorem 2. For this theorem we shall make considerable use of Laplace-Stieltjes transforms. If $A(x)$, say, is a function of bounded variation in every interval we write

$$A^*(s) \equiv \int_{0-}^{\infty} e^{-sx} dA(x) ,$$

for this transform, for those values of s which make the integral exist. We shall restrict s to real values.

To begin we need:

LEMMA 3.1. *As $k \to \infty$, for fixed $s > 0$, $\rho > 0$,*

$$\sum_{r=1}^{k} a_r \exp - \frac{r\rho s}{k} \sim \frac{\alpha k^\rho L(k)}{\rho^\gamma \Gamma(\gamma)} \int_0^\rho \exp(-sx) x^{\gamma-1} dx .$$

Proof. (Recall $\gamma > 0$) Let $D_k(x)$ be the distribution function associated with atoms of probability

$$\frac{a_r}{\sum_{r=1}^{k} a_r}$$

at the points $\rho r/k$, for $r = 1, 2, \cdots, k$. Then it is easy to see that

$$\lim_{\kappa\to\infty} D_r(x) = \lim_{\kappa\to\infty} \frac{\sum_1^{[xk/\rho]} a_r}{\sum_1^r a_r} = \left(\frac{x}{\rho}\right)^\gamma ,$$

from (1.1) and the familiar properties of functions of slow growth. Thus, by the continuity theorem for the Laplace transform, we find that, as $k \to 0$,

$$\int_0^\rho e^{-sx} dD_k(x) \to \frac{\gamma}{\rho^\gamma} \int_0^\rho e^{-sx} x^{\gamma-1} dx .$$

The lemma is an immediate consequence of this limit.

LEMMA 3.2. *As $k \to \infty$, for fixed $s > 0$, $\rho > 0$,*

$$\sum_{k+1}^{\infty} a_r \exp - \frac{r\rho s}{k} \sim \frac{\alpha L(k)}{\Gamma(\gamma)} \left(\frac{k}{\rho}\right)^\gamma \int_\rho^\infty \exp(-sx) x^{\gamma-1} dx .$$

Proof. We have that

$$\sum_{k+1}^{\infty} a_r \exp - \frac{r\rho s}{k} = \sum_{r=1}^{\infty} - \sum_{r=1}^{k} a_r \exp - \frac{r\rho s}{k} .$$

Lemma 4 of S (p. 682) shows that

$$\sum_{r=1}^{\infty} a_r \exp - \frac{r\rho s}{k} \sim \frac{\alpha k^\gamma}{\rho^\gamma s^\gamma} L\left(\frac{k}{s}\right)$$

as $k \to \infty$. This relation and Lemma 3.1 now establish Lemma 3.2, if we use the relation

$$\int_0^\infty e^{-sx} x^{\gamma-1} dx = \frac{\Gamma(\gamma)}{s^\gamma}, \ \gamma > 0 .$$

Let us now turn to the proof of the theorem. Note that $G_n(x)$ is the d.f. of $X_1 + X_2 + \cdots + X_n$ and thus is the d.f. of a nonnegative random variable with mean $\mu_1 + \mu_2 + \cdots + \mu_n$. Thus, by Jensen's inequality, for real $s > 0$,

$$G_n^*(s) \geqq e^{-(\mu_1 + \cdots + \mu_n)s} .$$

Further, if we set

$$\lambda = \limsup_{n \to \infty} \frac{\mu_1 + \cdots + \mu_n}{n} ,$$

it follows that, for an arbitrary $\varepsilon > 0$,

$$G_n^*(s) \geqq e^{-n(\lambda+\varepsilon)s} \tag{3.1}$$

for all $s > 0$ and all sufficiently large n.

Now, from Lemma 3.1 and (3.1) we can conclude that

$$\begin{aligned} &\liminf_{k \to \infty} \frac{1}{k^\gamma L(k)} \sum_{r=1}^{k} a_r G_r^*\left(\frac{s}{k}\right) \\ &\qquad \geqq \frac{\alpha}{(\lambda + \varepsilon)^\gamma \Gamma(\gamma)} \int_0^{(\lambda+\varepsilon)} e^{-sx} x^{\gamma-1} dx . \end{aligned} \tag{3.2}$$

Evidently,

$$G_n^*(s) = F_1^*(s) F_2^*(s) \cdots F_n^*(s) .$$

If $r > k$, $F_{(k+1)}^*(s) \cdots F_r^*(s)$ is the generating function of $X_{(k+1)} + \cdots + X_r$. Thus, for all large k, all $s > 0$,

$$e^{-(\mu_1 + \cdots + \mu_k)s} F_{(k+1)}^*(s) \cdots F_r^*(s) \geqq e^{-r(\lambda+\varepsilon)s} .$$

Let us set

$$\nu_k = \frac{\mu_1 + \cdots + \mu_k}{k} .$$

Then the last inequality and Lemma 3.2 show that, as $k \to \infty$.

$$\liminf_{k\to\infty} \frac{e^{-\nu_k s}}{k^\gamma L(k)} \sum_{r=(k+1)}^{\infty} a_r F^*_{(k+1)}\left(\frac{s}{k}\right) \cdots F^*_r\left(\frac{s}{k}\right)$$
$$\geqq \frac{\alpha}{(\lambda+\varepsilon)^\gamma \Gamma(\gamma)} \int_{(\lambda+\varepsilon)}^{\infty} e^{-sx} x^{\gamma-1} dx \ .$$

Let

$$A_k(s) = \frac{1}{k^\gamma L(k)} \sum_{r=1}^{k} a_r G^*_r\left(\frac{s}{k}\right) ,$$
$$B_k(s) = \frac{e^{-\nu_k s}}{k^\gamma L(k)} \sum_{(k+1)}^{\infty} a_r F^*_{(k+1)}\left(\frac{s}{k}\right) \cdots F^*_r\left(\frac{s}{k}\right) .$$

Then (3.2) and the last inequality prove, in view of the arbitrariness of ε,

$$\liminf_{k\to\infty} A_k(s) \geqq \frac{\alpha}{\lambda^\gamma \Gamma(\gamma)} \int_0^{\lambda} e^{-sx} x^{\gamma-1} dx \tag{3.3}$$

$$\liminf B_k(s) \geqq \frac{\alpha}{\lambda^\gamma \Gamma(\gamma)} \int_\lambda^{\infty} e^{-sx} x^{\gamma-1} dx \ . \tag{3.4}$$

But, if we recall that

$$\Psi(x) = \sum_{n=1}^{\infty} a_n G_n(x) ,$$

then

$$\frac{\Psi^*\left(\frac{s}{k}\right)}{k^\gamma L(k)} = A_k(s) + e^{\nu_k s} G^*_k\left(\frac{s}{k}\right) B_k(s) \ .$$

However, our assumption that the GERT holds implies (see S):

$$\Psi^*(s) \sim \frac{\alpha L(1/s)}{\mu^\gamma s^\gamma} \quad \text{as } s \downarrow 0 ,$$

so that

$$\frac{\Psi^*\left(\frac{s}{k}\right)}{k^\gamma L(k)} \to \frac{\alpha}{\mu^\gamma s^\gamma} , \quad \text{as } k \to \infty \ . \tag{3.5}$$

Thus

$$\frac{\alpha}{\lambda^\gamma \Gamma(\gamma)} \int_0^{\lambda} e^{-sx} x^{\gamma-1} dx$$
$$+ \left\{\limsup_{k\to\infty} e^{\nu_k s} G^*_k\left(\frac{s}{k}\right)\right\} \frac{\alpha}{\lambda^\gamma \Gamma(\gamma)} \int_\lambda^{\infty} e^{-sx} x^{\gamma-1} dx$$
$$\leqq \frac{\alpha}{\mu^\gamma s^\gamma} ,$$

which implies

$$\limsup_{k\to\infty} e^{\nu_k s} G_k^*\left(\frac{s}{k}\right)$$

$$\leqq 1 + \frac{\left(\dfrac{\lambda^\gamma - \mu^\gamma}{\mu^\gamma}\right)\dfrac{\Gamma(\gamma)}{s^\gamma}}{\displaystyle\int_\lambda^\infty e^{-sx}x^{\gamma-1}dx}$$

$$= 1 + \frac{\Gamma(\gamma)[(\lambda/\mu)^\gamma - 1]}{\displaystyle\int_{s\lambda}^\infty e^{-u}u^{\gamma-1}du} .$$

But $e^{\nu_k s}G_k^*(s/k) \geqq 1$. Thus we must have $(\lambda/\mu)^\gamma - 1 \geqq 0$. Since we are *given* that $\lambda \leqq \mu$ we are forced to conclude that $\lambda = \mu$ and that

$$e^{\nu_k s}G_k^*\left(\frac{s}{k}\right) \to 1, \text{ as } k \to \infty . \tag{3.6}$$

From (3.6) we can infer that

$$\frac{S_k}{k} - \nu_k \to 0, \text{ as } k \to \infty , \tag{3.7}$$

in probability. Thus, given any small $\varepsilon > 0$, we have, for all large n,

$$P\{S_n \leqq n(\nu_n + \varepsilon)\} > 1 - \varepsilon .$$

Therefore

$$\begin{aligned} \Psi(n\overline{\nu_n + \varepsilon}) &\geqq \sum_{r=1}^n a_r P\{S_r \leqq n\overline{\nu_n + \varepsilon}\} \\ &\geqq \left\{\sum_{r=1}^n a_r\right\} P\{S_n \leqq n\overline{\nu_n + \varepsilon}\} \\ &\geqq (1 - \varepsilon)\left\{\sum_{r=1}^n a_r\right\} . \end{aligned} \tag{3.8}$$

If we set $x_n = n(\nu_n + \varepsilon)$ then $x_n \to \infty$ as $n \to \infty$ and

$$\frac{\Psi(x_n)}{x_n^\gamma L(x_n)} \to \frac{\alpha}{\mu^\gamma \Gamma(1 + \gamma)} .$$

Thus, from (3.8),

$$(1 - \varepsilon) \limsup_{n\to\infty} \frac{\sum_{r=1}^n a_r}{x_n^\gamma L(x_n)} \leqq \frac{\alpha}{\mu^\gamma \Gamma(1 + \gamma)} .$$

From this inequality and well-known properties of functions of slow growth (plus the fact that $(\nu_n + \varepsilon)$ lies in a bounded interval not containing the origin) we infer

$$(1-\varepsilon)\limsup_{n\to\infty}\frac{1}{(\nu_n+\varepsilon)^\gamma}\leqq\frac{1}{\mu^\gamma}\,.$$

Since ε is arbitrary we may deduce that

$$\liminf_{n\to\infty}\nu_n\geqq\mu$$

and that $\nu_n\to\mu$ as $n\to\infty$ (since we are given $\limsup\nu_n\leqq\mu$). This proves part of the theorem. However, from (3.7) we deduce the W.L.L.N.:

$$\frac{S_n}{n}\to\mu,\ \text{as}\ n\to\infty\ ,$$

in probability. This completes the proof.

4. Proof of Theorem 3. The Laplace transform argument of the last section does not seem to carry over to the case when means are infinite. We are forced to the following quite different approach.

Let us choose $\varepsilon>0$ and set

$$\nu_n(\varepsilon)=\frac{1}{n}\sum_{j=1}^{n}\int_0^{n\varepsilon}\{1-F_j(x)\}dx\,.$$

LEMMA 4.1. *If $\nu_n(\varepsilon)$ is bounded, then for any $\eta>0$,*

$$P\{S_n\leqq n[\nu_n(\varepsilon)-\eta]\}\leqq\rho(\eta,\varepsilon)$$

uniformly in n, where $\rho(\eta,\varepsilon)$ can be made arbitrarily small by choosing ε small enough.

Proof. Suppose $\nu_n(\varepsilon)<A$ for all n (and note, by the way, that this inequality is preserved if ε is reduced). By a much used argument of S (see p. 679 of that paper), we get, for every $t>0$,

$$P\{S_n\leqq n[\nu_n(\varepsilon)-\eta]\}\leqq e^{Q_n(t)}\ ,\quad\text{say,}$$

where

$$Q_n(t)=nt\nu_n(\varepsilon)[1-e^{-nt\varepsilon}]-nt\eta\,.$$

Thus

$$Q_n(1/n\sqrt{\varepsilon})\leqq\frac{A}{\sqrt{\varepsilon}}[1-\exp-\sqrt{\varepsilon}]-\frac{\eta}{\sqrt{\varepsilon}}$$

and the right-hand side of this last inequality can be made as large and negative as we wish by choosing ε small enough. This establishes the lemma.

LEMMA 4.2. *If a* G.E.R.T. *holds, together with condition* (A), then

$$\liminf_{n\to\infty} \nu_n(\varepsilon) \geqq \mu .$$

Proof. Suppose there is a number $\mu_1 < \mu$ such that $\nu_n(\varepsilon) < \mu_1$ for infinitely many n. These inequalities are not upset if ε is reduced.

Define a truncation scheme as follows:

$$\begin{aligned} X_{rn} &= X_r \text{ if } X_r \leqq n\varepsilon , \\ &= n\varepsilon \text{ if } X_r > n\varepsilon , \end{aligned}$$

for $r, n = 1, 2, 3, \cdots$. Set

$$T_n = X_{1n} + X_{2n} + \cdots + X_{nn} .$$

The argument of Lemma 4.1 applies to T_n and shows

$$P\{T_n < n[\nu_n(\varepsilon) - \eta]\} \leqq \rho(\eta, \varepsilon) .$$

But $\mathscr{E}\, T_n = n\nu_n(\varepsilon)$; thus we can employ an argument already used in S, as follows.

Let η_1, η_2 be two small positive numbers.

$$\begin{aligned} \nu_n(\varepsilon) \geqq \{\nu_n(\varepsilon) - \eta_1\}[P\{T_n \leqq n[\nu_n(\varepsilon) + \eta_2]\} - P\{T_n \leqq n[\nu_n(\varepsilon) - \eta_1]\}] \\ + \{\nu_n(\varepsilon) + \eta_2\}[1 - P\{T_n \leqq n[\nu_n(\varepsilon) + \eta_2]\}] . \end{aligned}$$

Let us suppose $\eta_2 > \eta$. Then

$$\begin{aligned} &[\eta_2 - \eta_1]P\{T_n \leqq n[\nu_n(\varepsilon) + \eta_2]\} \\ &\qquad \geqq \eta_2 - [\nu_n(\varepsilon) - \eta_1]P\{T_n \leqq n[\nu_n(\varepsilon) - \eta_1]\} , \end{aligned}$$

and so

$$P\{T_n \leqq n[\nu_n(\varepsilon) + \eta_2\}] \geqq \frac{\eta_2}{\eta_2 - \eta_1} - [\nu_n(\varepsilon) - \eta_1]\rho(\eta_1, \varepsilon) .$$

Suppose we choose $\eta_1 = \eta_2^2$ and assume n is such that $\nu_n(\varepsilon) \leqq \mu_1$. Then we have

$$P\{T_n \leqq n[\nu_n(\varepsilon) + \eta_2]\} \geqq 1 - \delta(\eta_2, \varepsilon) ,$$

where we can make $\delta(\eta_2, \varepsilon)$ as small as we like by first choosing η_2 and then ε.

Put

$$\chi_n(\varepsilon) = \sum_{j=1}^{n} P\{X_j > n\varepsilon\} .$$

By condition (A) we know $\chi_n(\varepsilon) \to 0$ as $n \to \infty$

Put

$$x_n = n[\nu_n(\varepsilon) + \eta_2] .$$

Then it is not hard to see that

$$P\{S_n \leqq x_n\} \geqq P\{T_n \leqq x_n\} - \chi_n(\varepsilon) .$$

Now we have, as in the previous section,

$$\begin{aligned} \Psi(x_n) &\geqq P\{S_n \leqq x_n\} \sum_{j=1}^{n} a_j \\ &\geqq [1 - \delta(\eta_2, \varepsilon) - \chi_n(\varepsilon)] \sum_{j=1}^{n} a_j , \end{aligned}$$

if n is such that $\nu_n(\varepsilon) \leqq \mu_1$.

Thus

$$\liminf_{n\to\infty} \frac{[\nu_n(\varepsilon) + \eta_2]^r}{\mu^r} \geqq 1 - \delta(\eta_2, \varepsilon) .$$

i.e.

$$\frac{[\mu_1 + \eta_2]^r}{\mu^r} \geqq 1 - \delta(\eta_2, \varepsilon) .$$

Since η_2 and ε can be chosen arbitrarily small we conclude $\mu_1 \geqq \mu$. This proves the lemma.

LEMMA 4.3. *If $\nu_n(\varepsilon)$ is unbounded for some $\varepsilon > 0$ then there exists a sequence of integers $\{n_k\}$ such that S_{n_k}/n_k tends, in probability, to infinity as k increases.*

Proof. Suppose that for an arbitrarily large A we can always find n such that $\nu_n(\varepsilon) > A$. This implies that $Q_n(\varepsilon) > A$, if we set

$$Q_n(x) = \sum_{1}^{n} \int_0^x \{1 - F_j(nu)\} du .$$

But, since $Q_n(x)$ is the indefinite integral of a nonincreasing integrand, we must therefore have $Q_n(x) > (x/\varepsilon)A$ for all $x < \varepsilon$; plainly $Q_n(x) > A$ for all $x > \varepsilon$. Thus, if we denote the ordinary Laplace transform of $Q_n(x)$ by $Q_n^0(s)$, then

$$\begin{aligned} Q_n^0(s) &> \frac{A}{\varepsilon}\int_0^{\varepsilon} x e^{-sx} dx + A\int_{\varepsilon}^{\infty} e^{-sx} dx \\ &= \frac{A}{\varepsilon s^2}(1 - e^{-s\varepsilon}) . \end{aligned}$$

But computation shows that

$$Q_n^0(s) = s^{-2} \sum_1^n \{1 - F_j^*(s/n)\}$$
$$= s^{-2} R_n(s), \text{ say .}$$

Hence we have shown that

$$R_n(s) < As\left(\frac{1 - e^{-\varepsilon s}}{\varepsilon s}\right) .$$

If we observe that

$$G_n^*(s/n) \leqq e^{-R_n(s)}$$

then it follows that

$$G_n^*(s/n) \leqq \exp\left\{- As\left(\frac{1 - e^{-\varepsilon s}}{\varepsilon s}\right)\right\} .$$

If we restrict s to the interval $(0, \varepsilon^{-1})$, since we can choose A arbitrarily large, there will be a sequence of integers $\{n_k\}$ such that $G_{n_k}^*(s/n_k)$ tends to zero as k increases, at every s-point in $(0, \varepsilon^{-1})$. This proves the lemma.

We are now fully prepared for our proof of the theorem. Since a G.E.R.T. is assumed to hold, we have

$$\sum_{r=1}^{\infty} a_r G_r(x) \sim \frac{\alpha L(x)}{\Gamma(1 + \gamma)} \left(\frac{x}{\mu}\right)^{\gamma} .$$

Let us fix ξ and for reasons to emerge later suppose $\xi > \mu$. Then, as $n \to \infty$,

$$\frac{\Gamma(1 + \gamma)}{n^{\gamma} L(n)} \sum_{r=1}^{\infty} a_r G_r(n\xi) \to \frac{\alpha \xi^{\gamma}}{\mu^{\gamma}} .$$

Under the hypotheses of the theorem the fundamental lemma applies. Thus, if we choose a small $\eta > 0$ we can find $C \equiv C(\eta)$ such that

$$\frac{\Gamma(1 + \gamma)}{n^{\gamma} L(n)} \sum_{r=Cn\xi}^{\infty} a_r G_r(n\xi) < \alpha \eta \xi^{\gamma}$$

for all large n (since $L(\xi n)/L(n) \to 1$). Thus

$$\liminf_{n \to \infty} \frac{\Gamma(1 + \gamma)}{n^{\gamma} L(n)} \sum_{r=1}^{Cn\xi} a_r G_r(n\xi) \geqq \alpha \xi^{\gamma} \left\{\frac{1}{\mu^{\gamma}} - \eta\right\} . \tag{4.1}$$

At this point we are led to consider

$$\sum_{r=n+1}^{Cn\xi} a_r G_r(n\xi) \equiv \sum\nolimits_1 (n) , \quad \text{say.}$$

Suppose $\omega < \mu$. Set $2\eta^* = \mu - \omega$. By Lemma 4.2 we see that (whatever fixed ε we choose) $\nu_n(\varepsilon) - \eta^* > \omega$ for all large n. Thus, by Lemma 4.1, $P\{S_n \leqq n\omega\} \leqq \rho(\eta^*, \varepsilon)$ for all large n. Since ε is arbitrary in this result, and since $\rho(\eta^*, \varepsilon) \to 0$ as $\varepsilon \to 0$, we infer

$$P\{S_n \leqq n\omega\} \to 0 \text{ as } n \to \infty \tag{4.2}$$

for any fixed $\omega < \mu$.

Let n_1 be the least integer such that $n_1\omega \geqq n\xi$.

$$\begin{aligned}\sum_{n_1}^{Cn\xi} a_r G_r(n\xi) &\leqq G_{n_1}(n\xi) \sum_{n_1}^{Cn\xi} a_r \\ &\leqq G_{n_1}(n_1\omega) \sum_{n_1}^{Cn\xi} a_r .\end{aligned}$$

But we have just seen that $G_n(n\omega) \to 0$ for a fixed $\omega < \mu$. Thus we can conclude that

$$\sum_{n_1}^{Cn\xi} a_r G_r(n\xi) = o(n^\gamma L(n)) .$$

Thus

$$\begin{aligned}\limsup_{n\to\infty} \frac{\Sigma_1(n)}{n^\gamma L(n)} &\leqq \limsup_{n\to\infty} \frac{\sum_{n+1}^{n_1} a_r}{n^\gamma L(n)} \\ &= \frac{\alpha}{\Gamma(1+\gamma)} \left\{ \frac{\xi^\gamma}{\omega^\gamma} - 1 \right\} .\end{aligned}$$

But $\Sigma_1(n)$ does not depend on ω, so we may allow $\omega \uparrow \mu$ and obtain

$$\limsup_{n\to\infty} \frac{\Sigma_1(n)}{n^\gamma L(n)} \leqq \frac{\alpha}{\Gamma(1+\gamma)} \left\{ \frac{\xi^\gamma}{\mu^\gamma} - 1 \right\} .$$

From (4.1) we then deduce that

$$\liminf_{n\to\infty} \frac{\Gamma(1+\gamma)}{n^\gamma L(n)} \sum_{r=1}^{n} a_r G_r(n\xi) \geqq \alpha(1-\eta) .$$

However, η is arbitrarily small, so we must have

$$\liminf_{n\to\infty} \frac{1}{n^\gamma L(n)} \sum_{r=1}^{n} a_r G_r(n\xi) \geqq \frac{\alpha}{\Gamma(1+\gamma)} .$$

But since $G_r(n\xi) \leqq 1$ necessarily, this inequality implies

$$\frac{1}{n^\gamma L(n)} \sum_{r=1}^{n} a_r G_r(n\xi) \to \frac{\alpha}{\Gamma(1+\gamma)}$$

as $n \to \infty$; this limit holding for any fixed $\xi > \mu$.

Take a constant c, $1 < c < \xi/\mu$. Then, as $n \to \infty$,

$$\frac{1}{(cn)^\gamma L(cn)} \sum_{r=1}^{cn} a_r G_r\Big((cn)\Big(\frac{\xi}{c}\Big)\Big) \to \frac{\alpha}{\Gamma(1+\gamma)} .$$

Thus

$$\liminf_{n\to\infty} \left\{\frac{1}{(cn)^\gamma L(cn)} \sum_{r=1}^{n} a_r + \frac{G_n(n\xi)}{(cn)^\gamma L(cn)} \sum_{r=n}^{cn} a_r\right\} \geqq \frac{\alpha}{\Gamma(1+\gamma)} ,$$

which implies

$$\frac{1}{c^\gamma} + \Big(1 - \frac{1}{c^\gamma}\Big) \liminf_{n\to\infty} G_n(n\xi) \geqq 1$$

and thus that

$$G_n(n\xi) \to 1 \text{ as } n \to \infty ,$$

whenever $\xi > \mu$. But (4.2) states that $G_n(n\omega) \to 0$ as $n \to \infty$ whenever $\omega < \mu$. Thus we have established that S_n/n tends to μ in probability as $n \to \infty$, i.e. the W.L.L.N. holds, and the theorem is proved.

5. **A counter-example.** For simplicity we shall deal with the ERT and the renewal function

$$H(x) = \sum_{n=1}^{\infty} G_n(x) ,$$

rather than with the more complicated GERT.

Let x_n have a d.f. $F_n(x)$ such that, for some $\gamma > 1$,

$$F_n^*(s) = \exp\left[-\gamma\int_{n-1}^{n}\Big(\frac{1-e^{-us}}{u}\Big)du\right] .$$

The right-hand side of this equation is recognizably the Laplace-Stieltjes transform of a nonnegative infinitely divisible random variable.

Then

$$\begin{aligned} H^*(s) &= \sum_{n=1}^{\infty} \exp\left[-\gamma\int_{0}^{n}\Big(\frac{1-e^{-us}}{u}\Big)du\right] , \\ &= \sum_{n=1}^{\infty} \exp\left[-\gamma\int_{0}^{ns}\Big(\frac{1-e^{-v}}{v}\Big)dv\right] . \end{aligned}$$

Thus, as $s \downarrow 0+$,

$$sH^*(s) \to \int_0^\infty \exp\left[-\gamma\int_0^x\Big(\frac{1-e^{-v}}{v}\Big)dv\right]dx . \tag{5.1}$$

Let us write J for the integral on the right. Then J will be finite since $\gamma > 1$. Plainly by varying γ we can give J any real positive value. However, by a well-known Tauberian theorem we have from (5.1) the ERT result

$$H(x) \sim Jx, \text{ as } x \to \infty .$$

Now

$$G_n^*(s) = \exp\left[-\gamma\int_0^{ns}\left(\frac{1-e^{-v}}{v}\right)dv\right]$$

and so

$$G_n^*\left(\frac{s}{n}\right) = \exp\left[-\gamma\int_0^{s}\left(\frac{1-e^{-v}}{v}\right)dv\right]$$
$$= F_1^*(s) .$$

Thus $(X_1 + X_2 + \cdots + X_n)/n$ has the same d.f. as has X_1. This completes our demonstration of a sequence $\{X_n\}$ which satisfies the ERT by not the WLLN.

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