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BOUNDARY VALUE PROBLEMS FOR A CLASS OF NONLINEAR DIFFERENTIAL EQUATIONS

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For certain functions f, positive in $(0, \infty)$ and continuous in $[0, \infty)$, the partial differential equation $\Delta x = x - xf(x^2)$ has spherically symmetric solutions $x_n(t)$, $n = 1, 2, \cdots$, which vanish at zero, infinity and $n - 1$ distinct values in $(0, \infty)$. This and similar existence theorems for the ordinary differential equation $\ddot{y} - y + yF(y^2, t) = 0$ are proved by way of variational problems and the solutions are thus characterized by associated "eigenvalues". The asymptotic behavior of these eigenvalues is studied and some numerical data on the solutions is furnished for special cases of the above equations which are of interest in nuclear physics.

We begin by considering differential equations of the form

$$\ddot{y} - y + yF(y^2, t) = 0\ , \tag{1.1}$$

were $F(\eta, t)$ satisfies the following conditions:

(Ia) $F(\eta, t)$ is continuous in η and t for $0 < t < \infty$ and $0 \leqq \eta < \infty$;

(Ib) $F(\eta, t) > 0$ for $\eta > 0$, $t > 0$;

(Ic) there exists a $\delta > 0$ such that, for every fixed positive t and $0 \leqq \eta_1 < \eta_2 < \infty$, $\eta_2^{-\delta} F(\eta_2, t) > \eta_1^{-\delta} F(\eta_1, t)$.

In the special case in which $F(y^2, t) = f(y^2/t^2)$, the substitution

$$x(t) = t^{-1}y(t) \tag{1.2}$$

transforms equation (1.1) into the form

$$\ddot{x} + \frac{2\dot{x}}{t} = x - xf(x^2)\ , \tag{1.3}$$

which is satisfied by spherically symmetric solutions of the partial differential equation

$$\Delta x = x - xf(x^2)\ , \tag{1.4}$$

where Δ is the three-dimensional Laplace operator and t denotes distance from the origin.

To simplify our statements concerning solutions of (1.1) and (1.3), we shall employ the following terminology.

DEFINITION I. A solution $y(t)$ of equation (1.1) which is continuous in $[0, \infty)$, positive in $(0, \infty)$, and satisfies $y(0) = 0$, $\lim_{t\to\infty} y(t) = 0$, shall be called a *fundamental solution of* (1.1) *for the interval* $[0, \infty)$.

DEFINITION II. A solution $x(t)$ of equation (1.3) which is continuous in $[0, \infty)$, positive in $(0, \infty)$, and satisfies $\lim_{t\to\infty} x(t) = 0$ shall be called a *fundamental solution of* (1.3) *for the interval* $[0, \infty)$.

Special cases of equations (1.1) and (1.3) have been studies by a number of authors [1, 3, 7, 9] in connection with problems in nuclear physics, and the existence of fundamental solutions for the interval $[0, \infty)$ was suggested by physical considerations when (1.1) is of the type

$$\ddot{y} - y + y\frac{|y|^{k-1}}{t^{k-1}} = 0 \tag{1.5}$$

and $k = 2, 3$. Nehari [4] has shown that such solutions do in fact exist whenever $1 < k < 5$. In addition, $\lim t^{-1}y(t)$ exists as $t \to 0$ for $1 < k \leqq 4$. This shows that, in view of (1.2), the equation

$$\ddot{x} + \frac{2\dot{x}}{t} = x - x|x|^{k-1} \tag{1.6}$$

has fundamental solutions whenever $1 < k \leqq 4$. Synge [8] also studied equations (1.5) and (1.6) for $k = 2$ and set up a numerical procedure for the calculation of $y(t)$ and $x(t)$. Although no proof was given that the procedure converges to a solution, Synge's numerical results were accurate, as we shall see in § 8.

Our purpose is to prove the existence of not only fundamental solutions to equations (1.1) and (1.3), but also solutions $y_n(t)$ and $x_n(t)$ possessing $(n-1)$ distinct zeros $t_1, t_2, \cdots, t_{n-1}$ in $(0, \infty)$ and which are such that $y_n(t)$ and $x_n(t)$ do not vanish in $(t_\nu, t_{\nu+1})$, $\nu = 0, 1, \cdots, n-1$ $(t_0 = 0, t_n = \infty)$. Such solutions which change sign as $t \to \infty$ are again suggested by physical considerations for the case $f(x^2) = x^2$ in (1.3) [3].

We shall establish the following result.

THEOREM I. *If* $F(\eta, t)$ *satisfies conditions* (Ia)—(Ic) *and, in addition,*

(Id) $\lim_{t\to\infty} F(c^2, t) = 0$ *for all finite* c,

(Ie) $\int_0^a t^{(1/2)-\varepsilon}F(c^2t, t)dt < +\infty$ *for all finite* c, $0 < a < \infty$, *and some* $\varepsilon \geqq 0$,

then equation (1.1) *has a discrete infinity of solutions* $\{y_n(t)\}$, $n = 1, 2, \cdots$, *whose derivatives are continuous throughout* $[0, \infty)$ *and are such that* $y_n(t)$ *has exactly* $(n-1)$ *zeros in* $(0, \infty)$. *Moreover,* $y_n(0) = 0$, $\lim t^{-1}y_n(t)$ *exists as* $t \to 0$ *and* $y_n(t) \to 0$ *as* $t \to \infty$, *for each* n.

Since condition (Ie) is not satisfied for $F(y^2, t) = (|y|^{k-1})/t^{k-1}$ when $4 \leqq k < 5$, the known existence of fundamental solutions to equation

(1.5) for $1 < k < 5$ suggests that, when it is not required that $\lim t^{-1}y(t)$ exist as $t \to 0$, condition (Ie) may be relaxed to assume the form

(If) $\int_0^a t^{-\varepsilon}F(c^2t, t)dt < \infty$ for some $\varepsilon > 0$ and all finite c, $0 < a < \infty$.

Indeed, such is the case at least when $F(y^2, t) = p(t)y^{2\alpha}$, and we can prove a result similar to Theorem I for the equation

$$\ddot{y} - y + p(t)y^{2\alpha+1} = 0 \tag{1.7}$$

when the following conditions (equivalent to (Ia)-(Id) and (If) are satisfied.

(Ia′) $p(t)$ is continuous in $(0, \infty)$;
(Ib′) $p(t) > 0$ for all $t > 0$;
(Ic′) $\alpha > 0$;
(Id′) $\lim_{t\to\infty} p(t) = 0$
(Ie′) $\int_0^a t^{1-\varepsilon+\alpha}p(t)dt < +\infty$ for some $\varepsilon > 0$ and $0 < a < \infty$.

It is easy to verify that conditions (Ia′)—(Ie′) are satisfied for $p(t)y^{2\alpha} = (|y|^{k-1})/t^{k-1}$ and $1 < k < 5$. (i.e., for $\alpha = (k-1)/2$, $p(t) = 1/t^{k-1}$) if we let $\varepsilon = (5-k)/4$. This is in agreement with the results stated for equation (1.5). Moreover, it was shown in [4] that no fundamental solution to (1.5) can exist for $k = 5$.

Finally, in the special case in which (1.1) reduces to (1.3), Theorem I takes the following form.

THEOREM II. *If $f(\eta)$ satisfies the conditions:*
(IIa) *$f(\eta)$ is continuous for $0 \leqq \eta < \infty$,*
(IIb) *$f(\eta) > 0$ for $\eta > 0$,*
(IIc) *$\eta_2^{-\delta}f(\eta_2) > \eta_1^{-\delta}f(\eta_1)$ for $0 \leqq \eta_1 < \eta_2 < \infty$ and some positive δ,*
(IId) *$\int_a^\infty \eta^{\varepsilon-(5/2)}f(\eta)d\eta < +\infty$ for some $a > 0$ and $\varepsilon \geqq 0$.*

then equation (1.3) *has solutions $x_n(t)$, $n = 1, 2, \cdots$, whose derivatives are continuous in $(0, \infty)$, are such that $\lim x_n(t)$ exists as $t \to 0$, $x_n(t) \to 0$ as $t \to \infty$, and $x_n(t)$ has exactly $n - 1$ zeros in $(0, \infty)$.*

This result is merely a corollary to Theorem I where the condition corresponding to (Id) is automatically satisfied whenever (IIc) is true. Indeed, by (IIc),

$$f\left(\frac{c^2}{t^2}\right) < \frac{t_0^{2\delta}}{t^{2\delta}}f\left(\frac{c^2}{t_0^2}\right) \quad \text{for finite } t_0,\ t > t_0$$

and thus, for fixed $t_0 > 0$, we have

$$f\left(\frac{c^2}{t^2}\right) \leqq K\frac{1}{t^{2\delta}} \quad \text{for fixed positive } K,\ t > t_0. \tag{1.8}$$

In addition to proving the above existence theorems, we shall show that the solutions to (1.1) and (1.7) are characterized by a minimum problem and associated characteristic values λ_n. For $\alpha = 1$, $p(t) = 1/t^2$ in (1.7), i.e., when $y(t)$ satisfies

$$\ddot{y} - y + \frac{y^3}{t^2} = 0\,, \tag{1.9}$$

we shall calculate λ_1 and find bounds for the asymptotic values of the n^{th} "eigenvalues" λ_n which are defined by

$$\lambda_n = \frac{1}{2}\int_0^\infty (\dot{y}_n^2 + y_n^2)dt\,.$$

By converting the existence proof into a numerical procedure for computing the fundamental solutions of (1.5), we abtain additional numerical information concerning the solutions to (1.9) and the corresponding equation

$$\ddot{x} + \frac{2\dot{x}}{t} = x - x^3\,. \tag{1.10}$$

Both (1.9) and (1.10) were studied by Mitskevich [3].

2. A minimum problem. As a first step in the proof of Theorem I, we show that equation (1.1) has a fundamental solution for which $\lim_{t\to 0} t^{-1}y(t)$ exists when $F(\eta, t)$ satisfies the stated conditions. To do this, we shall set up a variational problem as in [5] and show that this problem has a solution which must satisfy (1.1) and the boundary conditions for a fundamental solution. We consider the problem

$$J(y) = \int_0^\infty [\dot{y}^2 + y^2 - G(y^2, \tau)]d\tau = \min. \tag{2.1}$$

where $y(t)$ is subject to the admissibility conditions $y(0) = 0$, $y(t) \not\equiv 0$ in $(0, \infty)$, $y(t) \geqq 0$ in $[0, \infty)$, $y(t) \in D'$ $[0, \infty)$, and the normalization condition

$$\int_0^\infty (\dot{y}^2 + y^2)d\tau = \int_0^\infty y^2F(y^2, \tau)d\tau\,. \tag{2.2}$$

The function $G(y^2, \tau)$ appearing in (2.2) is defined by

$$G(y^2, \tau) = \int_0^{y^2} F(\eta, \tau)d\eta\,, \quad \text{for each } \tau \text{ in } (0, \infty)\,. \tag{2.3}$$

Because of conditions (Ib) and (Ic), it can be shown that any function $y(t)$ satisfying the admissibility conditions, and for which $\int_0^\infty (\dot{y}^2 + y^2)d\tau$ exists, can be multiplied by a positive constant α such that $\alpha y(t)$ satisfies (2.2). We first show that the existence of $\int_0^\infty (\dot{y}^2 + y^2)d\tau$ implies that of $\int_0^\infty y^2 F(y^2, \tau)d\tau$.

Setting $\int_0^\infty (\dot{y}^2 + y^2)d\tau = \sigma^2$ and noting that $y(0) = 0$, we have

$$y^2(t) = \left[\int_0^t \dot{y}d\tau\right]^2 \leqq t\int_0^t \dot{y}^2 d\tau \leqq t\sigma^2 \tag{2.4}$$

$$y^2(t) = 2\int_0^t y\dot{y}d\tau \leqq \int_0^t (\dot{y}^2 + y^2)d\tau \leqq \sigma^2 . \tag{2.5}$$

Hence, taking some t_0 in $(0, \infty)$, $T > t_0$, $\varepsilon \geqq 0$, and using (2.4) in $[0, t_0]$ and (2.5) in $[t_0, T]$, we have

$$\int_0^T y^2 F(y^2, \tau)d\tau \leqq \sigma^2 t_0^\varepsilon \int_0^{t_0} \tau^{1-\varepsilon} F(\sigma^2\tau, \tau)d\tau + \max_{t_0 \leqq t < \infty} [F(\sigma^2, t)] \int_{t_0}^T y^2 d\tau .$$

This shows that

$$\int_0^T y^2 F(y^2, \tau)d\tau \leqq M_1(\sigma^2)\sigma^2 + M_2(\sigma^2)\sigma^2 , \tag{2.6}$$

where $M_1(\sigma^2) = t_0^\varepsilon \int_0^{t_0} \tau^{1-\varepsilon} F(\sigma^2\tau, \tau)d\tau$ and $M_2(\sigma^2) = \max\limits_{t_0 \leqq t < \infty} [F(\sigma^2, t)]$ are both finite for all finite σ^2 when F satisfies (Id) and either (If) or (Ie).

To complete the proof that $y(t)$ may be normalized as in equation (2.2), we define

$$B(\alpha) = \frac{\int_0^\infty (\dot{y}^2 + y^2)d\tau}{\int_0^\infty y^2 F(\alpha^2 y^2, \tau)d\tau} . \tag{2.7}$$

If $\alpha > 1$, $[F(\alpha^2 y^2, t)]/(\alpha^2 y^2)^\delta > [F(y^2, t)]/y^{2\delta}$ by (Ic), and thus

$$B(\alpha) < \frac{\int_0^\infty (\dot{y}^2 + y^2)d\tau}{\alpha^2 \int_0^\infty y^2 F(y^2, \tau)d\tau} , \qquad \alpha > 1 . \tag{2.8}$$

If $\alpha < 1$, $[F(y^2, t)]/y^{2\delta} > [F(\alpha^2 y^2, t)]/(\alpha^2 y^2)^\delta$, and

$$B(\alpha) \geqq \frac{\int_0^\infty (\dot{y}^2 + y^2)d\tau}{\alpha^2 \int_0^\infty y^2 F(y^2, \tau)d\tau} , \qquad \alpha \leqq 1 . \tag{2.9}$$

Because of conditions (Ia), (Ic) and the fact $B(\alpha) \to 0$ as $\alpha \to \infty$ by

(2.8) and $B(\alpha) \to \infty$ as $\alpha \to 0$ by (2.9), the continuous function (of α) $B(\alpha)$ assumes all values in $(0, \infty)$ as α varies in this same range. In particular $B(\alpha) = 1$ for some α in $(0, \infty)$ and (2.7) shows that $v(t) = \alpha y(t)$ consequently satisfies

$$\int_0^\infty (\dot{v}^2 + v^2) d\tau = \int_0^\infty v^2 F(v^2, \tau) d\tau .$$

Finally, we see that

$$\int_0^T G(y^2, \tau) d\tau \leqq \int_0^T y^2 F(y^2, \tau) d\tau \qquad \text{for } T \in (0, \infty) \tag{2.10}$$

by definition of $G(y^2, \tau)$ and thus, in view of (2.6) and (2.10), the existence of the integral in (2.1) is also assured whenever $\int_0^\infty (\dot{y}^2 + y^2) d\tau$ exists.

3. Associated comparison functions. To show that our variational problem has a nontrivial solution we employ some of the techniques of Nehari [5], where such functions $F(\eta, t)$ satisfying (Ia)—(Ic) were considered, and [4], where the differential equation (1.5) with singularities at zero and infinity was studied.

By (Ic) and (2.3) we have

$$G(\eta, t) = \int_0^\eta x^\delta [x^{-\delta} F(x, t)] dx \leqq \eta^{-\delta} F(\eta, t) \int_0^\eta x^\delta dx = \frac{\eta}{1 + \delta} F(\eta, t) .$$

Hence,

$$\eta F(\eta, t) - G(\eta, t) \geqq \delta(1 + \delta)^{-1} \eta F(\eta, t) ,$$

and, if $y(t)$ is an admissible function satisfying (2.2), this inequality shows that

$$J(y) \geqq \delta(1 + \delta)^{-1} \int_0^\infty (\dot{y}^2 + y^2) d\tau . \tag{3.1}$$

Furthermore, if for all admissible $y(t)$,

$$\lambda = \text{g.l.b. } J(y) , \tag{3.2}$$

there will exist a sequence of functions $y_1(t), y_2(t), \cdots$, which satisfy the conditions of the minimum problem (2.1), (2.2), and for which

$$\lim_{n \to \infty} J(y_n) = \lambda \geqq 0 . \tag{3.3}$$

The fact that $\lambda \geqq 0$ follows from (3.1).

It also follows from (3.1) that such a sequence $\{y_n(t)\}$ is uniformly bounded and equicontinuous in every finite interval $[0, T]$. Indeed,

$$\int_0^\infty (\dot{y}_n^2 + y_n^2)d\tau \leqq \rho^2 < \infty \tag{3.4}$$

for some positive constant ρ and, by (2.5), this shows that

$$y_n^2(t) \leqq \rho^2 \qquad \text{for all } t \text{ in } [0, \infty),\ n = 1, 2, \cdots , \tag{3.5}$$

Moreover, using (3.4), we have

$$\begin{aligned} |\, y_n(t_2) - y_n(t_1)\,|^2 = \left[\int_{t_1}^{t_2} \dot{y}_n dt\right]^2 &\leqq (t_2 - t_1)\int_{t_1}^{t_2} \dot{y}_n^2 dt \\ &\leqq \rho^2(t_2 - t_1) \end{aligned}$$

for all $0 \leqq t_1 < t_2 < \infty$.

By Ascoli's Lemma, there exists a subsequence of $\{y_n(t)\}$ which converges, uniformly on every finite interval $[0, T]$, to a continuous function $y(t)$. We have thus proved the following result:

LEMMA (3.1). *There exists a sequence $\{y_n(t)\}$ of functions, admissible for the variational problem* (2.1), (2.2), *which converges uniformly in every finite interval $[0, T]$ to a continuous function $y(t)$. Moreover,* $\lim_{n\to\infty} J(y_n) = \lambda = \text{g.l.b. } J(y) \geqq 0$.

What we now wish to show is that, for each $y_n(t)$ defined above and α_n constant, the linear differential equation

$$\ddot{u}_n - u_n + \alpha_n y_n F(y_n^2, t) = 0 \tag{3.6}$$

has a solution satisfying $u_n(0) = 0$, $\lim_{t\to\infty} u_n(t) = 0$. Moreover, this solution is also an admissible function (for suitable α_n) and

$$J(u_n) \leqq J(y_n)\,. \tag{3.7}$$

To do this, we consider the integral equation corresponding to (3.6):

$$u_n(t) = \alpha_n \int_0^\infty g(t, \tau)y_n(\tau)F(y_n^2, \tau)d\tau \tag{3.8}$$

where $g(t, \tau)$ is the Green's function of the differential operator $L(u) = \ddot{u} - u$ for the boundary conditions $u(0) = 0$, $\lim_{t\to\infty} u(t) = 0$, and is defined by

$$g(t, \tau) = \begin{cases} \bar{e}^{t} \sinh \tau\,, & 0 \leqq \tau < t \\ \bar{e}^{\tau} \sinh t\,, & t \leqq \tau\,. \end{cases} \tag{3.9}$$

Under the conditions imposed on $F(\eta, t)$ and the admissibility conditions imposed on $y_n(t)$, we shall prove that $u_n(t)$ defined in (3.8) is indeed the desired solution of (3.6).

Using (3.9), we see that (temporarily setting $\alpha_n = 1$)

$$(3.10)\qquad \begin{aligned} u_n(t) &= \bar{e}^{t}\int_0^t \sinh \tau y_n F(y_n^2, \tau)d\tau \\ &\quad + \sinh t \int_t^\infty \bar{e}^{\tau} y_n F(y_n^2, \tau)d\tau \,. \end{aligned}$$

Employing the definitions

$$(3.11)\qquad Q(t) = \int_0^t \sinh \tau y_n F(y_n^2, \tau)d\tau$$

$$(3.12)\qquad R(t) = \int_t^\infty \bar{e}^{\tau} y_n F(y_n^2, \tau)d\tau \,,$$

equation (3.10) takes the form (for $\alpha_n = 1$)

$$(3.13)\qquad u_n(t) = \bar{e}^{t}Q(t) + R(t)\sinh t \,.$$

To study the behaviour of $Q(t)$ near zero, we use (2.4), (3.4) and the monotonicity of $t^{\varepsilon-(1/2)}\sinh t$ in $(0, t)$ for $\varepsilon \geqq 0$. Equation (3.11) yields

$$(3.14)\qquad Q(t) \leqq \rho t^{\varepsilon-(1/2)} \sinh t \int_0^t \tau^{1-\varepsilon}F(\rho^2\tau, \tau)d\tau \,, \qquad \varepsilon \geqq 0.$$

On the other hand, we see from (3.12) that for $0 < t < 1$

$$\begin{aligned} R(t) &= R(1) + \int_t^1 e^{-\tau} y_n F(y_n^2, \tau)d\tau \\ &< R(1) + \rho\int_t^1 \tau^{1/2}F(\rho^2\tau, \tau)d\tau \,, \end{aligned}$$

where $R(1) \leqq \rho e^{-1}[\max_{1\leqq t<\infty} F(\rho^2, t)]$.

Since $\tau^{\varepsilon-(1/2)} < t^{\varepsilon-(1/2)}$ for $\tau > t$, $0 \leqq \varepsilon \leqq 1/2$, and since $\tau^{\varepsilon-(1/2)} \leqq 1$ for $0 < \tau < 1$, $\varepsilon \geqq 1/2$, the last inequality becomes

$$(3.15)\qquad R(t) \leqq R(1) + \begin{cases} \rho\int_t^1 \tau^{1-\varepsilon}F(\rho^2\tau, \tau)d\tau \,, & \varepsilon \geqq 1/2 \\ \rho t^{\varepsilon-(1/2)}\int_t^1 \tau^{1-\varepsilon}F(\rho^2\tau, \tau)d\tau \,, & 0 \leqq \varepsilon \leqq 1/2 \,. \end{cases}$$

If we combine (3.13), (3.14) and (3.15) we see that $u_n(t) \to 0$ as $t \to 0$ provided $F(\eta, t)$ satisfies (If). If (Ie) is fulfilled and we use this condition in the equivalent form

$$\int_0^a \tau^{1-\varepsilon}F(c^2\tau, \tau)d\tau < \infty \,, \qquad \varepsilon \geqq 1/2$$

then $t^{-1}u_n(t)$ approaches a finite limit as $t \to 0$.

To study the behavior of $Q(t)$ and $R(t)$ for large t we use (2.5) and (3.4) in (3.12) to find that, for $t > 0$

$$R(t) \leqq \rho e^{-t}[\max_{t \leqq \tau < \infty} F(\rho^2, \tau)] . \tag{3.16}$$

Also, if $0 < t_0 < t$,

$$Q(t) = Q(t_0) + \int_{t_0}^{t} \sinh \tau y_n F(y_n^2, \tau) d\tau$$

where $Q(t_0)$ is finite by (3.14).

In view of (2.5) and (3.4), we then have

$$Q(t) \leqq Q(t_0) + \frac{\rho}{2}\int_{t_0}^{t} e^{\tau} F(\rho^2, \tau) d\tau , \qquad 0 < t_0 < t . \tag{3.17}$$

Thus, if (Id) is satisfied, $\int_{t_0}^{t} e^{\tau} F(\rho^2, \tau) d\tau = o(e^t)$ as $t \to \infty$ and (3.13), (3.16) and (3.17) show that $u_n(t) \to 0$ as $t \to \infty$.

We shall now examine the behavior of $\dot{u}_n(t)$. Using (3.10) to compute $h^{-1}[u_n(t + h) - u_n(t)]$ and letting $h \to 0$, we see that $\dot{u}_n(t)$ exists and is given by

$$\dot{u}_n(t) = -e^{-t}Q(t) + \cosh tR(t) . \tag{3.18}$$

Equations (3.16) and (3.17) then show that $\dot{u}_n(t) \to 0$ as $t \to \infty$ whenever $F(\eta, t)$ satisfies (Id) and either (Ie) or (If). Similarly, (3.14) and (3.15) show that $|\dot{u}_n(t)|$ is bounded near $t = 0$ if F satisfies (Id) and (Ie). If only (If) is satisfied, then it is seen $\dot{u}_n(t) = 0(t^{\varepsilon-(1/2)})$ as $t \to 0$.

If we now compute $h^{-1}[\dot{u}_n(t + h) - \dot{u}_n(t)]$ from (3.18), we find that $\ddot{u}_n(t)$ exists and that $u_n(t)$ is a solution of (3.6) for $\alpha_n = 1$. Moreover, $u_n(t)$ is nonnegative in $(0, \infty)$ since $y_n(t)$ was assumed to be. Finally, we can show that

$$\lim_{t \to 0} \dot{u}_n(t) u_n(t) = 0 , \tag{3.19}$$

if we combine (3.18) with the above comments concerning $\dot{u}_n(t)$. We may summarize our results as follows:

LEMMA 3.2. *If $y_n(t)$ is defined as in Lemma 2.1 and α_n is a constant, then equation (3.6) has a solution $u_n(t)$ satisfying $u_n(0) = 0$, $\lim_{t\to\infty} u_n(t) = 0$ whenever $F(\eta, t)$ satisfies (Ia)—(Id) and (If) or (Ie). Moreover $u_n(t)$ is such that $\lim_{t\to\infty} \dot{u}_n(t) = 0$, $\lim_{t\to 0} u_n(t)\dot{u}_n(t) = 0$ and, if condition (Ie) is fulfilled, $\lim_{t\to 0} t^{-1}u_n(t)$ exists.*

4. Convergence of the comparison functions to a fundamental solution. We now proceed to prove the existence of a fundamental solution to (1.1). To do this we first show that, for suitable α_n, $u_n(t)$

is an admissible function for our variational problem and satisfies (3.7).

Multiplying (3.6) by $u_n(t)$, integrating from 0 to T, and using (3.19), we obtain

$$\sigma_1^2(T) = \int_0^T (\dot{u}_n^2 + u_n^2)d\tau = \alpha_n \int_0^T u_n y_n F(y_n^2, \tau)d\tau + u_n(T)\dot{u}_n(T) . \tag{4.1}$$

Using (2.5) to estimate $u_n(t)$ and (3.5) to estimate $y_n(t)$ in the interval $[0, 1]$, and employing the Schwarz inequality in $[1, T]$, we find that

$$\int_0^T u_n y_n F(y_n^2, \tau)d\tau \leqq \sigma_1(T)\rho \int_0^1 \tau^{1-\varepsilon} F(\rho^2\tau, \tau)d\tau + \left[\int_1^T u_n^2 F(y_n^2, \tau)d\tau\right]^{1/2}\left[\int_1^T y_n^2 F(y_n^2, \tau)d\tau\right]^{1/2} .$$

Hence, in view of (2.2) (applied to $y_n(t)$) and (3.4),

$$\int_0^T u_n y_n F(y_n^2, \tau)d\tau \leqq \rho c_1 \sigma_1(T) + \rho\sigma_1(T)[\max_{1\leqq t<\infty} \{F(\rho^2, t)\}]^{1/2} , \tag{4.2}$$

where $c_1 = \int_0^1 \tau^{1-\varepsilon} F(\rho^2\tau, \tau)d\tau$ and $\max_{1\leqq t<\infty} \{F(\rho^2, t)\}$ exist by properties (Id), and (If) or (Ie).

Combining (4.1) and (4.2), it follows that

$$\sigma_1^2(T) \leqq \alpha_n c_2 \sigma_1(T) + u_n(T)\dot{u}_n(T) ,$$

where c_2 is a constant independent of n. Completing the square in the last inequality, we have

$$\left[\sigma_1(T) - \frac{\alpha_n c_2}{2}\right]^2 \leqq \frac{\alpha_n^2 c_2^2}{4} + u_n(T)\dot{u}_n(T) .$$

However, since $u_n(T)$ and $\dot{u}_n(T)$ tend to zero as $T \longrightarrow \infty$ (Lemma 3.2), this establishes the existence of the integral $\int_0^\infty (\dot{u}_n^2 + u_n^2)d\tau$ and, because of (2.7), also the existence of $\int_0^\infty u_n^2 F(u_n^2, \tau)d\tau$. Therefore, as shown in § 2, we may choose the constant α_n in such a way that

$$\int_0^\infty (\dot{u}_n^2 + u_n^2)d\tau = \int_0^\infty u_n^2 F(u_n^2, \tau)d\tau . \tag{4.3}$$

and $u_n(t)$ becomes an admissible function for the problem (2.1), (2.2).

If we use the convexity of $G(\eta, t)$, the Schwarz and other elementary inequalities, it is easy to establish inequality (3.7), i.e.,

$$J(u_n) \leqq J(y_n) . \tag{4.4}$$

Moreover, in view of the way in which these inequalities are used, equality is possible only if u_n and y_n coincide. If we note that the existence of all integrals involved is insured by the facts that $\dot{u}_n(t) = 0$

$(t^{\varepsilon-(1/2)})$ and $y_n(t) = 0\ (t^{1/2})$ near zero, the proof proceeds like a comparable one in [5] and will be omitted. The proof also establishes the useful inequality

$$\alpha_n^2 \int_0^\infty y_n^2 F(y_n^2, t)dt \leqq \int_0^\infty u_n^2 F(u_n^2, t)dt \ . \tag{4.5}$$

Because of the definition of the number λ in (3.3), we must have $\liminf_{n\to\infty} J(u_n) \geqq \lambda$ since $u_n(t)$ is an admissible function. Formulas (4.4) and (3.3) thus lead to the relations

$$\lim_{n\to\infty} J(u_n) = \lim_{n\to\infty} J(y_n) = \lambda \ . \tag{4.6}$$

Moreover, using the same inequalities which lead to the uniform boundedness and equicontinuity of the sequence $\{y_n(t)\}$ in § 2, we find that $\{u_n(t)\}$ converges uniformly in every finite interval $[0, T]$ to a continuous function $u_0(t)$, i.e.,

$$\lim_{n\to\infty} u_n(t) = u_0(t) \ . \tag{4.7}$$

This proves:

LEMMA 4.1. *Under the conditions* (Ia)—(Id), *and* (Ie) *or* (If), *the sequence* $u_n(t)$ *defined in* (3.6) *converges uniformly in every finite interval* $[0, T]$ *to a continuous function* $u_0(t)$. *Moreover,* $\lim_{n\to\infty} J(u_n) = \lambda$.

Now, $u_0(t)$ will be a solution to our variational problem if we can show that

$$\lim_{n\to\infty} J(u_n) = J(\lim u_n) = J(u_0) \ . \tag{4.8}$$

We proceed to establish this result by first proving the uniform convergence of $\dot{u}_n(t)$. It is from this point on that we need (Ie) rather than (If) for the existence proof. When $F(\eta, t)$ satisfies (Ie), then, as shown in § 3, each of the $\dot{u}_n(t)$ approaches a finite limit as $t \to 0$. In this case, each $\dot{u}_n(t)$ is continuous in $[0, \infty)$. Furthermore,

$$\dot{u}_n(t) - \dot{u}_m(t) = e^{-t}[Q_n(t) - Q_m(t)] + \cosh t[R_n(t) - R_m(t)] \tag{4.9}$$

where $Q_i(t)$, $R_i(t)$, $i = 1, 2, \cdots$, were defined in (3.11) and (3.12).

When conditions (Ia)—(Ie) are satisfied, we have shown that each of the $Q_i(t)$, $R_i(t)$ existed for all $t > 0$ as long as $y_i(t)$ was a member of $\{y_n(t)\}$. Since the sequence $\{y_n(t)\}$ was shown to converge uniformly in every finite interval $[0, T]$, it is easy to see that the same is true of the sequences $\{Q_n(t)\}$, $\{R_n(t)\}$. Equation (4.9) then shows $\{\dot{u}_n(t)\}$ converges uniformly.

In view of (4.7), therefore, we have

$$\lim_{n\to\infty} \dot{u}_n(t) = \dot{u}_0(t) \tag{4.10}$$

where the convergence is uniform in every finite interval $[0, T]$.

We also need to show that α_n, which is determined by (4.3), is bounded for all n. By (4.5)

$$\alpha_n^2 \leqq \frac{\int_0^\infty u_n^2 F(u_n^2, \tau)d\tau}{\int_0^\infty y_n^2 F(y_n^2, \tau)\,d\tau} . \tag{4.11}$$

To see that this denominator in (4.11) has a lower bound, we set $\sigma_n^2 = \int_0^\infty (\dot{y}_n^2 + y_n^2)d\tau$ and employ (2.2), (2.4) and (2.5). Thus

$$\begin{aligned} \sigma_n^2 &= \int_0^\infty (\dot{y}_n^2 + y_n^2)d\tau = \int_0^\infty y_n^2 F(y_n^2, \tau)d\tau \\ &\leqq \sigma_n^2 \int_0^1 \tau^{1-\varepsilon} F(\sigma_n^2 \tau, \tau)d\tau + \sigma_n^2 \max_{1\leqq t<\infty} [F(\sigma_n^2, t)] \end{aligned} \tag{4.12}$$

for $\varepsilon \geqq 0$. Since $\sigma_n^2 > 0$ by our requirement that $y_n(t) \not\equiv 0$, we can divide both sides of the inequality by σ_n^2 and obtain

$$1 \leqq \int_0^1 \tau^{1-\varepsilon} F(\sigma_n^2 \tau, \tau)d\tau + \max_{1\leqq t<\infty} [F(\sigma_n^2, t)] . \tag{4.13}$$

If it were true that $\sigma_n^2 < 1$ for some n, condition (Ic) would show that

$$F(\sigma_n^2 \tau, \tau) < \sigma_n^{2\delta} F(\tau, \tau) ,$$
$$F(\sigma_n^2, \tau) < \sigma_n^{2\delta} F(1, \tau) .$$

Substituting in (4.13) would then yield

$$1 \leqq \sigma_n^{2\delta} \int_0^1 \tau^{1-\varepsilon} F(\tau, \tau)d\tau + \sigma_n^{2\delta} \max_{1\leqq t<\infty} [F(1, t)]$$

for $\varepsilon \geqq 0$. This inequality shows that σ_n^2 cannot approach zero as $n \to \infty$, i.e.,

$$\sigma_n^2 = \int_0^\infty y_n^2 F(y_n^2, \tau)d\tau \geqq M > 0 , \qquad \text{for all } n , \tag{4.14}$$

when conditions (Ia)—(Id), and (If) or (Ie) are satisfied.

In order to examine the numerator in (4.11), we apply (3.1) to $u_n(t)$ and use the normalization (4.3) to obtain

$$\int_0^\infty u_n^2 F(u_n^2, \tau)d\tau \leqq \frac{1+\delta}{\delta} J(u_n) . \tag{4.15}$$

In view of (4.4) and the fact $J(y_n) \leqq \int_0^\infty (\dot{y}_n^2 + y_n^2)d\tau \leqq \rho^2$, the last inequality yields

$$\int_0^\infty u_n^2 F(u_n^2, \tau)d\tau \leqq \frac{1+\delta}{\delta}\rho^2 .$$

Combining this result with (4.11) and (4.14), we have

$$\alpha_n^2 \leqq \frac{1+\delta}{\delta}\frac{\rho^2}{M}, \qquad \text{for all } n . \tag{4.16}$$

Thus α_n is bounded above.

To return now to the proof of (4.8), we write

$$\begin{aligned} &\left|\int_0^T (\dot{u}_0^2 + u_0^2)d\tau - \int_0^\infty (\dot{u}_n^2 + u_n^2)d\tau\right| \\ &\leqq \left|\int_0^T (\dot{u}_0^2 + u_0^2)d\tau - \int_0^T (\dot{u}_n^2 + \dot{u}_n^2)d\tau\right| + \left|\int_T^\infty (\dot{u}_n^2 + u_n^2)d\tau\right| . \end{aligned} \tag{4.17}$$

In view of (3.6) and Lemma (3.2), however,

$$\int_T^\infty (\dot{u}_n^2 + u_n^2)d\tau = \alpha_n \int_T^\infty u_n y_n F(y_n^2, \tau)d\tau + u_n(T)\dot{u}_n(T) .$$

Moreover,

$$\begin{aligned} \int_T^\infty u_n y_n F(y_n^2, \tau)d\tau &\leqq \max_{T \leqq t < \infty}. [F(\rho^2, t)] \int_T^\infty u_n y_n d\tau \\ &\leqq \max_{T \leqq t < \infty}.]F(\rho^2, t)]\rho\left[\int_T^\infty u_n^2 d\tau\right]^{1/2} \\ &\leqq \max_{T \leqq t < \infty}. [F(\rho^2, t)]\rho^2\left[\frac{1+\delta}{\delta}\right]^{1/2} , \end{aligned}$$

where the final result follows from the fact that

$$\int_0^\infty u_n^2 d\tau \leqq \frac{1+\delta}{\delta}J(u_n) \leqq \frac{1+\delta}{\delta}J(y_n) \leqq \frac{1+\delta}{\delta}\rho^2 .$$

If we now combine the above inequality with its predecessor, and substitute into (4.17), we find

$$\begin{aligned} \left|\int_0^T (\dot{u}_0^2 + u_0^2)d\tau - \int_0^\infty (\dot{u}_n^2 + u_n^2)d\tau\right| &\leqq \left|\int_0^T (\dot{u}_0^2 + u_0^2)d\tau - \int_0^T (\dot{u}_n^2 + u_n^2)d\tau\right| \\ &+ \alpha_n \rho^2\left[\frac{1+\delta}{\delta}\right]^{1/2} \max_{T \leqq t < \infty}. [F(\rho^2, t)] + |u_n(T)\dot{u}_n(T)| . \end{aligned}$$

But since

$$\alpha_n\left[\frac{1+\delta}{\delta}\right]^{1/2}\rho^2 \leqq \frac{1+\delta}{\delta}\frac{\rho^3}{M} \leqq K$$

by (4.16), and $\int_0^T(\dot{u}_n^2 + \dot{u}_n)d\tau$ converges to $\int_0^T(\dot{u}_0^2 + u_0^2)d\tau$ (Lemma 4.1 and the uniform convergence in (4.10)), our last inequality shows that

$$\left|\int_0^T(\dot{u}_0^2 + u_0^2)d\tau - \lim_{n\to\infty}\int_0^\infty(\dot{u}_n^2 + u_n^2)\,d\tau\right| \leqq K \max_{T\leqq t<\infty}[F(\rho^2, t)] + |\,\dot{u}_0(T)u_0(T)\,| .$$

If we now use the results that $\dot{u}_0(T)u_0(T) \to 0$ as $T \to \infty$ (equation (4.10), Lemma 3.2, Lemma 4.1) and $\max_{T\leqq t<\infty}[F(\rho^2, t)] \to 0$ as $T \to \infty$ (property (Id)), we finally obtain

$$\lim_{n\to\infty}\int_0^\infty(\dot{u}_n^2 + u_n^2)\,d\tau = \int_0^\infty(\dot{u}_0^2 + u_0^2)d\tau . \tag{4.18}$$

Employing similar techniques, we can also show that

$$\lim_{n\to\infty}\int_0^\infty G(u_n^2, \tau)d\tau = \int_0^\infty G(u_0^2, \tau)d\tau . \tag{4.19}$$

Hence, equations (4.18), (4.19) and (2.1) yield the result

$$\lim_{n\to\infty} J(u_n) = J(u_0) , \tag{4.20}$$

which proves (4.8) and verifies that $u_0(t)$ is indeed the solution of our variational problem. Moreover, since (4.14) also holds for $u_n(t)$, equation (4.15) shows that

$$J(u_n) > \frac{\delta}{1+\delta}M > 0 .$$

Because of (4.6), we have thus proved

$$\lim_{n\to\infty} J(u_n) = \lim_{n\to\infty} J(y_n) = J(u_0) = \lambda > 0 , \tag{4.21}$$

and $u_0(t)$ cannot be identically zero in $[0, \infty)$

We now proceed to show that $u_0(t)$ satisfies (1.1) and is consequently a fundamental solution for the interval $[0, \infty)$.

As previously remarked, the sign of equality is possible in (4.4) only if $y_n(t)$ and $u_n(t)$ coincide in $[0, \infty)$. Equation (3.6) shows that, in this case, $y_n(t)$ must be a solution of

$$\ddot{u} - u + \alpha_u F(u^2, t) = 0 \qquad \text{for some } \alpha > 0 , \tag{4.22}$$

where $u(0) = \lim_{t\to\infty} u(t) = 0$. Hence, if we let $y_n(t) = u_0(t)$, we find

$$\ddot{u}_0 - u_0 + \alpha_0 u_0 F(u_0^2, t) = 0 \tag{4.23}$$

because of the minimum property of $u_0(t)$, i.e., in this case, $J(u_n) =$

$J(y_n)$ and the previous comments apply. Furthermore, since $u_0(t)\dot{u}_0(t) \to 0$ as $t \to 0$ and $u_0(t)\dot{u}_0(t) \to 0$ as $t \to \infty$ (equation 4.10 and Lemmas 3.2 and 4.1), it is also true that

$$\int_0^\infty (\dot{u}_0^2 + u_0^2)d\tau = \alpha_0 \int_0^\infty u_0 F(u_0^2, \tau)d\tau \,. \tag{4.24}$$

Comparing this result with the normalization condition (4.3), we see that $\alpha_0 = 1$. Thus $u_0(t)$ is shown to be a solution of the differential equation (1.1). In view of the uniform convergence of $\{u_n(t)\}$, Lemma 3.2 shows that $u_0(t) \to 0$ as $t \to \infty$ and $\lim_{t\to 0} t^{-1}u_0(t)$ exists.

5. The existence of solutions with zeros in $(0, \infty)$. Having established the existence of a fundamental solution to (1.1) on $[0, \infty)$, we can also prove that there exist similar positive solutions of (1.1) on every subinterval of the positive t-axis. These solutions will also approach zero at the end-points of the interval.

Indeed, for the interval $[a, \infty)$, $a \geqq 0$, we replace $g(t, \tau)$ in (3.9) by

$$g(t, \tau) = \begin{cases} e^a \bar{e}^t \sinh(\tau - a) \,, & 0 \leqq a \leqq \tau < t \\ e^a \bar{e}^\tau \sinh(t - a) \,, & t \leqq \tau \end{cases}$$

also for the interval $[a, b]$, $0 \leqq a < b < \infty$, we define

$$g(t, \tau) = \begin{cases} \dfrac{\sinh(b - t)\sinh(\tau - a)}{\sinh(b - a)} \,, & 0 \leqq a \leqq \tau < t \\[2ex] \dfrac{\sinh(b - \tau)\sinh(t - a)}{\sinh(b - a)} \,, & t \leqq \tau \leqq b \,. \end{cases}$$

The corresponding variational problem for the interval $[a, b]$ (or $[a, \infty)$] will then have a solution which also solves the boundary value problem

$$\ddot{y} - y + yF(y^2, t) = 0 \,, \quad \begin{cases} y(a) = y(b) = 0 \,, & 0 \leqq a < b < \infty \\ y(a) = \lim_{t\to\infty} y(t) = 0 \,, & 0 \leqq a < \infty \,, \end{cases} \tag{5.1}$$

The proof of these statements is the same as before, except that the special treatment of the singularities at $t = 0$, $t = \infty$, or at both of these points, now becomes unnecessary. Our final result may then be stated

THEOREM 5.1. *Let Γ denote the class of functions $y(t)$ which are continuous and piecewise differentiable in $[a, b]$, $0 \leqq a \leqq b \leqq \infty$, satisfy $y(a) = y(b) = 0$, $y(t) \not\equiv 0$ in $[a, b]$, and are nonnegative in (a, b). Let us require, moreover, that*

$$\int_a^b (\dot{y}^2 + y^2)dt = \int_a^b y^2 F(y^2, t)dt \,, \tag{5.2}$$

where F satisfies conditions (Ia)—(Ie).

Then if we define

$$J(y) = \int_a^b [\dot{y}^2 + y^2 - G(y^2, t)]dt . \tag{5.3}$$

where $y(t) \in \Gamma$ and $G(\eta, t)$ is given by (2.3), *the minimum problem*

$$J(y) = \min. = \lambda(a, b) \tag{5.4}$$

is solved by a solution of the differential equation (1.1). *Moreover, $y(t) > 0$ in (a, b), $\lambda(a, b) > 0$, and if $a = 0$, $\lim_{t\to 0} t^{-1}y(t) = 0$.*

We shall outline the completion of the proof of Theorem I and establish the existence of a discrete infinity of solutions $y_1(t), y_2(t), \cdots, y_n(t), \cdots,$ in $[0, \infty)$ such that $y_n(0) = 0$, $\lim_{t\to 0} t^{-1}y_n(t)$ exists and $\lim_{t\to\infty} y(t) = 0$. Furthermore, the n^{th} solution will be shown to possess $n - 1$ distinct zeros in $(0, \infty)$. We may follow a procedure due to Nehari [6] but must also take into account the nature of the singularities at zero and infinity. The procedure depends on the following result.

Letting $\lambda(a, b)$ denote the minimum of $J(y)$ in (5.3) for the interval $[a, b]$, we first prove

LEMMA 5.1. (a) *If $a \leqq a' \leqq b' < b$, then $\lambda(a, b) \leqq \lambda(a', b')$.*

(b) *$\lambda(a, b) \to \infty$ as $b - a \to 0$ (as $a \to \infty$ if $b = \infty$, as $b \to 0$ if $a = 0$).*

(c) *$\lambda(a, b)$ is a continuous function of a and b (of b only if $a = 0$, of a only if $b = \infty$.).*

Since $F(\eta, t)$ has the properties (Ia)—(Ic) and since condition (Id) and either (Ie) or (If) insure the existence of all integrals involved, the verification of Lemma 5.1 proceeds exactly as the proof of the corresponding lemma in [6]. It is necessary, however, to divide the proof into three stages for the intervals $[0, a][a, b], [b, \infty], 0 < a < b < \infty$ and use the inequalities and arguments of the previous existence theory.

Now if $0 = t_0 < t_1 < t_2 < \cdots < t_n = \infty$, where $t_1, \cdots, t_{n-1}$ are n $- 1$ distinct points in $(0, \infty)$, we consider functions $u_\nu(t)$ in the interval $[t_{\nu-1}, t_\nu]$ (or $[t_{n-1}, \infty)$) which are piecewise continuously differentiable, vanish at $t_{\nu-1}$ and t_ν, but not identically in $(t_{\nu-1}, t_\nu)$, and are normalized by

$$\int_{t_{\nu-1}}^{t_\nu} (\dot{u}_\nu^2 + u_\nu^2)dt = \int_{t_{\nu-1}}^{t_\nu} u_\nu^2 F(u_\nu^2, t)dt \quad \nu = 1, 2, \cdots, n . \tag{5.5}$$

If we define $u(t)$ in $[0, \infty)$ by setting $u/(t) = u_\nu(t)$ in $[t_{\nu-1}, t_\nu]$, $\nu = 1, 2, \cdots, n$, then the n^{th} "characteristic value" is defined by

$$\lambda_n = \min \int_0^\infty [\dot{u}^2 + u^2 - G(u^2, t)]dt \tag{5.6}$$

where $u(t)$ ranges over the class of all functions with the properties indicated above. By Theorem 5.1, it is sufficient to consider functions $u(t)$ which coincide with solutions of (5.1) and are such that $u(t_{\nu-1}) = u(t_\nu) = 0$.

Let us define

$$\mu = \sum_{\nu=1}^{n} \lambda(t_{\nu-1}, t_\nu) , \qquad (t_0 = 0,\ t_n = \infty) . \tag{5.7}$$

By property (c) of Lemma 5.1, μ is continuous function of $t_1, t_2, \cdots, t_{n-1}$, and, by property (b), the values t_ν must be bounded away from each other and from infinity in any sequence of sets for which μ tends to its greatest lower bound. It is thus sufficient to confine the values $t_1, \cdots, t_{n-1}$ to a sufficiently large finite interval $[0, T]$ and, therefore, the minimum of μ is actually attained for some set of $n-1$ finite distinct values t_ν, $\nu = 1, \cdots, n-1$, with $0 < t_{\nu-1} < t_\nu$.

Since the minimum of μ in (5.7) is the same as the minimum of $\int_0^\infty [\dot{u}^2 + u^2 - G(u^2, t)]dt$ under the normalizations

$$\int_{t_{\nu-1}}^{t_\nu} (\dot{u}^2 + u^2)\, dt = \int_{t_{\nu-1}}^{t_\nu} u^2 F(u^2, t) dt , \quad \nu = 1, 2, \cdots, n$$

and other specified conditions on $u(t)$, our minimum problem (5.5), (5.6) has a solution $y_n(t)$ which coincides in each interval $[t_{\nu-1}, t_\nu]$ with a solution $y(t)$ of (5.1). Morever, $y_n(t_{\nu-1}) = y_n(t_\nu) = 0$ and $y_n(t) > 0$ in $(t_{\nu-1}, t_\nu)$ because of Theorem 5.1. Accordingly, our "n^{th} eigensolution" has $n-1$ distinct zeros in $(0, \infty)$ and thus we obtain a different solution $y_n(t)$ for different values of n.

Our task is now to show that this function $y_n(t)$ is a solution of (5.1) throughout the interval $[0, \infty)$ i.e., we wish to show that

$$\lim_{t \to t_\nu -} y_n'(t) = \lim_{t \to t_\nu +} y_n'(t) , \quad \nu = 1, 2, \cdots, n-1 , \tag{5.8}$$

after first requiring that $y_n(t)$ be positive in $(0, t_1)$ and change sign thereafter at each point t_ν. This alternation of sign is possible since $-y(t)$ satisfied (5.1) whenever $y(t)$ does, and the change of sign does not affect the admissibility conditions or the value of $J(y)$.

Since $t_1, t_2, \cdots, t_{n-1}$ are all in $(0, \infty)$, we are examining the slopes of $y_n(t)$ at points where $F(\eta, t)$ is continuous. The proof of (5.8) follows its counter part in [6] except that $\int_{t_{\nu-1}}^{t_\nu} [\dot{y}^2 - G(y^2, t)]dt$ is replaced by $\int_0^\infty [\dot{y}^2 + y^2 - G(y^2, t)]dt$. It is easy to see, however, that the extra term $\int_{t_{\nu-1}}^{t_\nu} y^2 dt$ presents no additional difficulties.

To summarize our results, we have the following theorem.

THEOREM 5.2. *Let Γ_n denote the class of functions $y(t)$ with the properties: $y(t)$ is continuous and piecewise differentiable in $[0, \infty)$; $y(t_\nu) = 0$ and $y(t) \not\equiv 0$ in $(t_{\nu-1}, t_\nu)$ $(\nu = 1, 2, \cdots, n)$, where the t_ν are numbers such that $0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = \infty$; moreover,*

$$\int_{t_{\nu-1}}^{t_\nu} (\dot{y}^2 + y^2)dt = \int_{t_{\nu-1}}^{t_\nu} y^2 F(y^2, t)dt\,, \quad \nu = 1, 2, \cdots, n\,, \tag{5.9}$$

where $F(\eta, t)$ is subject to conditions (Ia)—(If).

If $G(\eta, t)$ is the function defined in (2.3), then the extremal problem

$$\int_0^\infty [\dot{y}^2 + y^2 - G(y^2, t)]dt = \min = \lambda_n \tag{5.10}$$

has a solution $y_n(t)$ whose derivative is continuous throughout $[0, \infty)$. The function $y_n(t)$ has exactly $n - 1$ zeros in $(0, \infty)$, and is a non-trivial solution of the differential system

$$\ddot{y} - y + yF(y^2, t) = 0\,, \quad y(0) = \lim_{t\to\infty} y(t) = 0\,,$$

for which $\lim t^{-1}y(t)$ exists.

This result proves Theorem I.

6. The case $F(y^2, t) = p(t)y^{2\alpha}$. In the existence theory of the previous sections, all of our results through Lemma 4.1 were valid when conditions (Ia)—(Id) and (If) were satisfied. In fact, we used the stronger condition (Ie) (rather than (If)) only to insure that the sequence $\{\dot{u}_n(t)\}$ converged uniformly in every finite interval $[0, T]$, and thus to prove that

$$\lim_{n\to\infty} J(u_n) = J(\lim_{n\to\infty} u_n)\,.$$

We can, however, circumvent this requirement of continuity of each $\dot{u}_n(t)$ at $t = 0$ when we consider $F(\eta, t)$ in the special form

$$F(y^2, t) = p(t)y^{2\alpha}\,. \tag{6.1}$$

The proof of the convergence of the comparison functions $u_n(t)$ to a fundamental solution is similar to one in [4] and the adaptations necessary in our case are repetitive of the arguments used in the proof of Theorem I. The following result is valid.

THEOREM 6.1. *If $F(y^2, t) = p(t)y^{2\alpha}$ and conditions* (Ia)—(Ie) *are replaced by* (Ia′)—(Ie′), *Theorem* 5.2 *remains valid with the exception that $\lim_{t\to\infty} t^{-1}y(t)$ may no longer exist. Moreover the characteristic* *values assume the simpler form*

$$\lambda_n = \frac{\alpha}{\alpha + 1}\int_0^\infty (\dot{y}_n^2 + y_n^2)dt .$$

7. **Asymptotic estimates for certain eigenvalues.** We shall now consider the special equation

$$\ddot{y} - y + \frac{y^3}{t^2} = 0 , \tag{7.1}$$

for which it is possible to obtain information concerning the behavior of the associated eigenvalues λ_n for large values of n. We remark that (1.1) reduces to (7.1) for $F(y^2, t) = y^2/t^2$ and F obviously satisfies conditions (Ia)—(Ie).

For the above equation (2.1) becomes

$$J(y) = \int_0^\infty \left(\dot{y}^2 + y^2 - \frac{y^4}{2t^2}\right)dt \tag{7.2}$$

and, in view of the normalization

$$\int_{t_{\nu-1}}^{t_\nu} (\dot{y}^2 + y^2)dt = \int_{t_{\nu-1}}^{t_\nu} \frac{y^4}{t^2}dt , \quad \nu = 1, 2, \cdots, n , \tag{7.3}$$

equation (7.2) reduces to

$$J(y) = \frac{1}{2}\int_0^\infty (\dot{y}^2 + y^2)dt . \tag{7.4}$$

To recapitulate in terms of Theorem 5.2, equation (7.1) has a solution which is continuous in $[0, \infty)$, vanishes for $t = 0$, $t = \infty$ and $n - 1$ points in $[0, \infty)$. The function $y(t)$ is characterized by the variational problem:

If $t_1, t_2, \cdots, t_{n-1}$ are any $n - 1$ values in $(0, \infty)$, satisfying $t_{\nu-1} < t_\nu$, $\nu = 2, \cdots, n - 1$, we consider functions $u(t)$ which vanish at zero, at infinity and these $n - 1$ values t_ν. Furthermore, we require $u(t)$ to be of class D', normalized by (7.3), nonnegative and not identically zero in $(0, \infty)$. The function for which

$$\lambda_n = \min = \frac{1}{2}\int_0^\infty (\dot{y}^2 + y^2)dt , \tag{7.5}$$

for all choices of t_ν, $\nu = 1, \cdots, n - 1$, is a solution of (7.1) with the properties indicated in Theorem 5.2. The value of the minimum, λ_n, in (7.5) we refer to as the "n^{th} characteristic value" or "n^{th} eigenvalue" of equation (7.1).

As n increases, it is easy to see that λ_n does also. Indeed, if we let $u(t) = y(t)$ for $0 \leqq t \leqq t_{n-1}$, and $u(t) = 0$ for $t_{n-1} \leqq t < \infty$, where $y(t)$ is the solution of our problem for λ_n, then $u(t)$ is an admissible functien for the $(n-1)^{\text{th}}$ problem. Hence,

$$\begin{aligned}\lambda_{n-1} &\leqq \int_0^\infty [\dot{u}^2 + u^2 - G(u^2, t)]dt = \int_0^{t_{n-1}} [\dot{y}^2 + y^2 - G(y^2, t)]dt \\ &= \lambda_n - \int_{t_{n-1}}^\infty [\dot{y}^2 + y^2 - G(y^2, t)]dt \\ &= \lambda_n - \lambda_1(t_{n-1}, \infty) .\end{aligned}$$

Thus,

$$\lambda_{n-1} < \lambda_n ,$$

since $\lambda_1(t_{n-1}, \infty)$ is positive.

We shall now find more exact information concerning the λ_n's associated with equation (7.1).

If $u(t)$ is a competing function for the above minimum problem and $\int_0^\infty (\dot{u}^2 + u^2)dt$ exists, then, as shown in § 2, $u(t)$ may be normalized by multiplying $u(t)$ by a constant α_ν in $(t_{\nu-1}, t_\nu)$, $\nu = 1, 2, \cdots, n$. Then

$$\int_{t_{\nu-1}}^{t_\nu} (\dot{u}^2 + u^2)dt = \alpha_\nu^2 \int_{t_{\nu-1}}^{t_\nu} \frac{u^4}{t^2} dt , \tag{7.6}$$

and $v(t) = \alpha_\nu u(t)$ satisfies (7.3) in $(t_{\nu-1}, t_\nu)$. Moreover, (7.5) shows that

$$\lambda_n \leqq \frac{1}{2} \sum_{\nu=1}^n \int_{t_{\nu-1}}^{t_\nu} (\dot{v}^2 + v^2)dt = \frac{1}{2} \sum_{\nu=1}^n \alpha_\nu^2 \int_{t_{\nu-1}}^{t_\nu} (\dot{u}^2 + u^2)dt ,$$

or, in view of (7.6),

$$\lambda_n \leqq \frac{1}{2} \sum_{\nu=1}^n \frac{\left[\int_{t_{\nu-1}}^{t_\nu} (\dot{u}^2 + u^2) dt\right]^2}{\int_{t_{\nu-1}}^{t_\nu} \frac{u^4}{t^2} dt} . \tag{7.7}$$

We therefore can find an estimate from above for λ_n by substituting into (7.7) any function $u(t)$ satisfying $u(0) = u(\infty) = u(t_\nu) = 0$, $u(t) \not\equiv 0$ in $(t_{\nu-1}, t_\nu)$, $u(t) \in D'[t_{\nu-1}, t_\nu]$, for any set of numbers $t_1, \cdots, t_{n-1}$ in $(0, \infty)$.

Moreover, if $z(t)$ is the solution to our n-th minimum problem in $[0, b]$, then the function

$$u(t) = \begin{cases} z(t) , & 0 \leqq t \leqq b \\ 0 , & b < t \leqq \infty \end{cases}$$

is a competing function for the n-th problem in $[0, \infty)$. Hence

$$\lambda_n(0, \infty) \leqq \lambda_n(0, b) , \qquad 0 < b < \infty . \tag{7.8}$$

The estimate (7.7) shall then be applied to the interval $[0, b]$, i.e., take $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$, where $b < \infty$.

For a competing function $u(t)$, we take a solution to the differential equation

$$\ddot{u} + u^3 = 0 \tag{7.9}$$

for which $u(t_{\nu-1}) = u(t_\nu) = 0$, $u(t) > 0$ in each interval $[t_{\nu-1}, t_\nu]$. The existence of such solutions is proved in [5].

We note that

$$u^3(t) = \left[\int_{t_{\nu-1}}^{t} \dot{u}(t)dt\right]^2 \leqq (t - t_{\nu-1})\int_{t_{\nu-1}}^{t_\nu} \dot{u}^2 dt\,, \tag{7.10}$$

for $t_{\nu-1} \leqq t \leqq t_\nu$, and thus

$$\int_{t_{\nu-1}}^{t_\nu} u^2 dt \leqq \frac{(t_\nu - t_{\nu-1})^2}{2}\int_{t_{\nu-1}}^{t_\nu} \dot{u}^2 dt\,.$$

Furthermore, since

$$\int_{t_{\nu-1}}^{t_\nu} \frac{u^4}{t^2}\,dt \geqq \frac{1}{t_\nu^2}\int_{t_{\nu-1}}^{t_\nu} u^4 dt\,, \qquad \nu = 1, 2, \cdots, n\,,$$

the inequalities (7.7) and (7.8) show that

$$\lambda_n(0, \infty) \leqq \frac{1}{2}\sum_{\nu=1}^{n} \frac{t_\nu^2\left[1 + \dfrac{(t_\nu - t_{\nu-1})^2}{2}\right]\left[\displaystyle\int_{t_{\nu-1}}^{t_\nu} \dot{u}^2 dt\right]^2}{\displaystyle\int_{t_{\nu-1}}^{t_\nu} u^4 dt} \tag{7.11}$$

where $0 < t_0 < t_1 < \cdots < t_n = b$ are any set of points in $(0, \infty)$. Using the fact that

$$\int_{t_{\nu-1}}^{t_\nu} \dot{u}^2 dt = \int_{t_{\nu-1}}^{t_\nu} u^4 dt\,, \tag{7.12}$$

for every solution of (7.9) which vanishes at $t_{\nu-1}$ and t_ν, and the property

$$\int_{t_{\nu-1}}^{t_\nu} \dot{u}^2 dt = \frac{A}{(t_\nu - t_{\nu-1})^3}\,,$$

where

$$A = \frac{2}{3}\left[\int_0^1 \frac{dt}{(1 - t^4)^{1/2}}\right]^4$$

(this result is proved in [6]), we find that (7.11) reduces to the form

$$\lambda_n(0, \infty) \leqq \frac{A}{2}\sum_{\nu=1}^{n}\left[\frac{t_\nu^2}{(t_\nu - t_{\nu-1})^3} + \frac{t_\nu^2}{(t_\nu - t_{\nu-1})} + \frac{t_\nu^2(t_\nu - t_{\nu-1})}{4}\right] \tag{7.13}$$

where $t_n = b$, $t_0 = 0$, and $t_1, t_2, \cdots, t_{n-1}$ are any $n-1$ distinct points in $(0, \infty)$.

To find a similar expression estimating λ_n from below we proceed as follows.

From (7.3) and (7.10), it is found that

$$\int_{t_{\nu-1}}^{t_\nu} (\dot{y}^2 + y^2)dt = \int_{t_{\nu-1}}^{t_\nu} \frac{y^4}{t^2} dt \leqq (t_\nu - t_{\nu-1}) \int_{t_{\nu-1}}^{t_\nu} \dot{y}^2 dt \int_{t_{\nu-1}}^{t_\nu} \frac{y^2}{t^2} dt$$

where $y(t)$ is the solution of our minimum problem (7.5), and $t_1, t_2, \cdots, t_{n-1}$ are the corresponding zeros of $y(t)$. Thus,

$$\int_{t_{\nu-1}}^{t_\nu} (\dot{y}^2 + y^2)dt \leqq \frac{(t_\nu - t_{\nu-1})}{t_{\nu-1}^2} \int_{t_{\nu-1}}^{t_\nu} \dot{y}^2 dt \int_{t_{\nu-1}}^{t_\nu} y^2 dt \,, \tag{7.14}$$

$\nu = 2, \cdots, n-1$.

Furthermore, we define the function $v(t) = \alpha_\nu y(t)$ in each interval $[t_{\nu-1}, t_\nu]$, where α_ν is to be determined by

$$\int_{t_{\nu-1}}^{t_\nu} \dot{y}^2 dt = \alpha_\nu^2 \int_{t_{\nu-1}}^{t_\nu} y^4 dt \,. \tag{7.15}$$

In this way, $v(t)$ becomes a competing function for the variational problem:

$$J(v) = \int_{t_{\nu-1}}^{t_\nu} \left(\dot{v}^2 - \frac{v^4}{2}\right) dt = \min = \mu(t_{\nu-1}, t_\nu) \tag{7.16}$$

under the normalization

$$\int_{t_{\nu-1}}^{t_\nu} \dot{v}^2 dt = \int_{t_{\nu-1}}^{t_\nu} v^4 dt \,. \tag{7.17}$$

The functions $v(t)$ are required to satisfy the same admissibility conditions as we required in (2.1), (2.2). It is shown in [5] that the minimum $\mu(t_{\nu-1}, t_\nu)$ is attained when $v(t)$ is a solution $u(t)$ of (7.9), which takes the value zero at $t_{\nu-1}, t_\nu$ and is positive in $(t_{\nu-1}, t_\nu)$.

Because of (7.17) and the comments following (7.12), equation (7.16) shows that

$$\mu(t_{\nu-1}, t_\nu) = \frac{1}{2} \int_{t_{\nu-1}}^{t_\nu} \dot{u}^2 dt = \frac{A}{2(t_\nu - t_{\nu-1})^3} \leqq \frac{1}{2} \int_{t_{\nu-1}}^{t_\nu} \dot{v}^2 dt \tag{7.18}$$

for every admissible function $v(t)$ satisfying (7.17). In particular using $v(t) = \alpha_\nu y(t)$ and noting (7.15), equation (7.18) yields the result

$$\frac{A}{2(t_\nu - t_{\nu-1})^3} \leqq \frac{1}{2} \int_{t_{\nu-1}}^{t_\nu} \dot{v}^2 dt = \frac{\alpha_\nu^2}{2} \int_{t_{\nu-1}}^{t_\nu} \dot{y}^2 dt = \frac{1}{2} \frac{\left[\int_{t_{\nu-1}}^{t_\nu} \dot{y}^2 dt\right]^2}{\int_{t_{\nu-1}}^{t_\nu} y^4 dt} \,.$$

Since

$$\int_{t_{\nu-1}}^{t_\nu} y^4 dt \geqq t_{\nu-1}^2 \int_{t_{\nu-1}}^{t_\nu} \frac{y^4}{t^2} dt = t_{\nu-1}^2 \int_{t_{\nu-1}}^{t_\nu} (\dot{y}^2 + y^2) dt ,$$

we may write this last expression in the form

$$\frac{At_{\nu-1}^2}{(t_\nu - t_{\nu-1})^3} \leqq \frac{\left[\int_{t_{\nu-1}}^{t_\nu} \dot{y}^2 dt\right]^2}{\int_{t_{\nu-1}}^{t_\nu} (\dot{y} + y^2) dt^2} . \tag{7.19}$$

Let us now rewrite (7.14) in the following way

$$\frac{t_{\nu-1}^2}{t_\nu - t_{\nu-1}} \leqq \frac{\int_{t_{\nu-1}}^{t_\nu} \dot{y}^2 dt \int_{t_{\nu-1}}^{t_\nu} y^2 dt}{\int_{t_{\nu-1}}^{t_\nu} (\dot{y}^2 + y^2) dt} . \tag{7.20}$$

Then if we add (7.19) and (7.20) and cancel out the common factor $\int_{t_{\nu-1}}^{t_\nu} (\dot{y}^2 + y^2) dt$ in the right hand side, we obtain

$$\frac{At_{\nu-1}^2}{(t_\nu - t_{\nu-1})^3} + \frac{t_{\nu-1}^2}{t_\nu - t_{\nu-1}} \leqq \int_{t_{\nu-1}}^{t_\nu} \dot{y}^2 dt , \quad \nu = 2, \cdots, n-1 . \tag{7.21}$$

Finally, using the fact that

$$\int_{t_{\nu-1}}^{t_\nu} \dot{y}^2 dt \leqq \int_{t_{\nu-1}}^{t_\nu} (\dot{y}^2 + y^2) dt ,$$

equation (7.14) also yields

$$\frac{t_{\nu-1}^2}{t_\nu - t_{\nu-1}} \leqq \int_{t_{\nu-1}}^{t_\nu} y^2 dt , \quad \nu = 2, \cdots, n-1 . \tag{7.22}$$

Adding the last two equations and noting (7.5), we have, for $n \geqq 3$, the result

$$\sum_{\nu=2}^{n-1} \left[\frac{At_{\nu-1}^2}{2(t_\nu - t_{\nu-1})^3} + \frac{t_{\nu-1}^2}{t_\nu - t_{\nu-1}} \right] \leqq \lambda_n ,$$

where the $t_1, t_2, \cdots, t_{n-1}$ are the $n-1$ internal zeros of the "n^{th} eigenfunction" $y(t)$.

In view of this inequality and the fact that we may take any values $t_1, t_2, \cdots, t_n$ in (7.13), we have proved the following result.

LEMMA 7.1. *If λ_n is the n^{th} characteristic valve associated with the differential equation* (7.1), *for the boundary conditions $y(0) = y(\infty) = 0$, then, if $n \geqq 3$,*

$$\liminf_{\substack{0<t_\nu<\infty \\ t_{\nu-1}<t_\nu}} \sum_{\nu=2}^{n-1}\left[\frac{A}{2}\,\frac{t_{\nu-1}^2}{(t_\nu - t_{\nu-1})^3} + \frac{t_{\nu-1}^2}{t_\nu - t_{\nu-1}}\right] \leqq \lambda_n$$

$$\leqq \liminf_{\substack{0<t_\nu<\infty \\ t_{\nu-1}<t_\nu}} \frac{A}{2}\sum_{\nu=1}^{n}\left[\frac{t_\nu^2}{(t_\nu - t_{\nu-1})^3} + \frac{t_\nu^2}{(t_\nu - t_{\nu-1})} + \frac{t_\nu^2(t_\nu - t_{\nu-1})}{4}\right],$$

where

$$A = \frac{2}{3}\left[\int_0^1 \frac{dt}{(1-t^4)^{1/2}}\right]^4 .$$

A comparison of the upper and lower bounds in the above inequality suggests that they might be nearly equivalent asymptotically, but finding the exact minimum for either expression seems difficult.

We can, however, get an upper bound for λ_n by first substituting the arithmetic means $t_\nu = \nu t_n/n$ in the right hand side. Minimizing with respect to t_n, we then obtain

$$\lambda_n \leqq \frac{2A}{9}\sqrt{\frac{2}{3}}\,\frac{n(2n+1)(n+1)}{}$$

or

$$(7.23) \qquad \lambda_n \leqq A_1 n^3\left[1 + 0\left(\frac{1}{n}\right)\right], \quad A_1 = \sqrt{\frac{2}{3}}\,\frac{8\left[\int_0^1 \frac{dt}{(1-t^4)^{1/2}}\right]^4}{27} .$$

To find a lower bound for λ_n in terms of n, we must replace our previous result (Lemma 7.1) with an expression that can be easily minimized. To do this, we see from the Rayleigh minimum principle that

$$\left[\frac{1}{4} + \frac{\pi^2}{\log^2 \dfrac{t_\nu}{t_{\nu-1}}}\right]\int_{t_{\nu-1}}^{t_\nu} \frac{y^2}{t^2}\,dt \leqq \int_{t_{\nu-1}}^{t_\nu} \dot{y}^2 dt ,$$

for all functions $y(t)$ for which the integrals exist and which are piecewise differentiable in $[t_{\nu-1}, t_\nu]$ and vanish at the end points. Equality is achieved for the function

$$y = \sqrt{t}\,\sin\left[\frac{\pi \log \dfrac{t}{t_{\nu-1}}}{\log \dfrac{t_\nu}{t_{\nu-1}}}\right].$$

If we use the above inequality when $y(t)$ is the n^{th} solution of our variational problem and $t_1, t_2, \cdots, t_{n-1}$ are its internal zeros, and apply the techniques used above, we can obtain the estimate

$$\frac{n-2}{4} + \frac{\pi^2(n-2)^3}{(6\log n + c_1)^2} \leqq \lambda_n, \tag{7.24}$$

where c_1 is a finite constant and $n \geqq 3$.

We summarize the results in (7.23) and 7.24 as follows.

THEOREM 7.1. *Let λ_n be the n^{th} characteristic value associated with equation* (7.1) *and the variational problem* (7.3), (7.4), (7.5). *For $n \geqq 3$, λ_n satisfies the following inequalities.*

$$\frac{\pi^2(n-2)^3}{36\log^2 n}\left[1 + 0\left(\frac{1}{\log n}\right)\right] \leqq \lambda_n \leqq A_1 n^3\left[1 + 0\left(\frac{1}{n}\right)\right],$$

where

$$A_1 = \sqrt{\frac{2}{3}}\,\frac{8}{27}\left[\int_0^1 [1 - t^4]^{-1/2} dt\right]^4.$$

8. Some numerical results. In this section we shall obtain further information regarding the behavior of solutions to the equation

$$\ddot{y} - y + \frac{y^k}{t^{k-1}} = 0, \qquad \text{for } k = 2, 3. \tag{8.1}$$

Synge [8] studied this equation for $k = 2$ with a view to obtaining numerical values for the fundamental solution. It is our aim to verify his results and compute the fundamental solution also for $k = 3$.

To do this, we note that when $F(y^2, t) = (|y|^{k-1})/t^{k-1}$ (3.10) becomes

$$u_n(t) = e^{-t}\int_0^t \sinh\tau \frac{y_n^k}{\tau^{k-1}}d\tau + \sinh t\int_t^\infty e^{-\tau}\frac{y_n^k}{\tau^{k-1}}d\tau$$

($|y_n| = y_n$ since we consider only nonnegative functions). It was shown that $J(u_n) \leqq J(y_n)$ where equality holds only if $u_n(t)$ coincides with the solution of the variational problem

$$J(y) = \int_0^\infty\left(\dot{y}^2 + y^2 - \frac{k-1}{k+1}\,\frac{|y|^{k+1}}{t^{k-1}}\right)dt = \min$$

$$\int_0^\infty (\dot{y}^2 + y^2)dt = \int_0^\infty \frac{|y|^{k+1}}{t^{k-1}}dt.$$

We may convert our existence proof into a procedure for the numerical computation of the fundamental solution $y(t)$ by starting with a function $v_0(t)$, nonnegative in $(0, \infty)$, for which $v_0(0) = v_0(\infty) = 0$, $v_0(t) \not\equiv 0$, $\int_0^\infty(\dot{v}_0^2 + v_0^2)dt$ exists and

$$\int_0^\infty (\dot{v}_0^2 + v_0^2)dt = \int_0^\infty \frac{v_0^{k+1}}{t^{k-1}}dt.$$

It we then define $v_1(t), v_2(t), \cdots$, by

$$v_{n+1}(t) = \alpha_{n+1}e^{-t}\int_0^t \sinh\tau \frac{v_n^k}{\tau^{k-1}}d\tau + \sinh t\int_t^\infty e^{-\tau}\frac{v_n^k}{\tau^{k-1}}d\tau \tag{8.2}$$

where α_{n+1} is determined by

$$\int_0^\infty (\dot{v}_{n+1}^2 + v_{n+1}^2)\,dt = \int_0^\infty \frac{v_{n+1}^{k+1}}{\tau^{k-1}}d\tau \tag{8.3}$$

we shall have $J(v_{n+1}) \leqq J(v_n)$. If there is a *unique* nonnegative solution to (8.1) for $1 < k < 5$, the above procedure must converge to it.

In order to test the rapidity of convergence in the above iteration, the problem was programmed for $k = 2$ on a Bendix G-20 computer, using a Simpson's Rule evaluation of the integrals in (8.2). The second integral was restricted to the interval $[0, 10]$ and $v(t) = te^{-1}$ was used as an initial approximation.

After 23 iterations it was found that

$$|v_n(t) - v_{n+1}(t)| \leqq .00001\,, \qquad n = 23\,, \quad 0 \leqq t \leqq 10\,.$$

where $v_n(t)$ was evaluated at multiples of $\Delta t = .05$.

Setting $y(t) = v_{23}(t)$, $x(t) = t^{-1}y(t)$, the following results were obtained.

$$\begin{aligned} x(0) &= \dot{y}(0) = 4.19172 = \int_0^\infty e^{-\tau}\frac{y^2(\tau)}{\tau}d\tau\,, \\ x(4.5) &= .03926\,, \\ A &= 16.0687 = \int_0^\infty \sinh\tau\frac{y^2(\tau)}{\tau}d\tau\,, \end{aligned} \tag{8.4}$$

where $y(t) \sim Ae^{-t}$ for large t and $y(t) \sim \dot{y}(0)\sinh t$ for small values of t [4]. We recall that $x(t)$ is the corresponding solution of (1.3). The values found by Synge were

$$\begin{aligned} x(0) &= \dot{y}(0) = 4.19169 \\ x(4.5) &= .03926 \\ A &= 16.0723\,. \end{aligned} \tag{8.5}$$

A comparison of (8.4) and (8.5) shows that the correspondence is good, especially for $x(0)$ and $x(4.5)$, whereas for A the correspondence occurs for one less significant digit. We thus apply the same iterative procedure outlined in (8.2) and (8.3) for the case $k = 3$ and find that

$$\lambda_1 = \frac{1}{2}\int_0^\infty (\dot{y}^2 + y^2)dt = 3.00787$$

$$x(0) = \dot{y}(0) = 4.33738$$

$$A = 2.71386 = \int_0^\infty t^{-2} \sinh t y^3(t) dt ,$$

where, as previously noted, $y(t) \sim Ae^{-t}$ for large t and $y(t) \sim \dot{y}(0) \sinh t$ for small values of t.

It is also shown in [4] that when the values $\dot{y}(0)$ and A are given, there are simpler iteration procedures, for calculating $y(t)$, which are valid at the ends of the interval.

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