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AW^*-ALGEBRAS ARE QW^*-ALGEBRAS

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G. A. Reid has introduced a class of B^*-algebras called QW^*-algebras which includes the W^*-algebras and which is included in the class of AW^*-algebras. In this paper it is shown that the QW^*-algebras are exactly the AW^*-algebras.

We shall use Reid's notation (see [2]) without further explanation.

THEOREM. *Let A be an AW^*-algebra. Then A is a QW^*-algebra.*

Proof. Let B be a norm-closed *-subalgebra of A. Then, using [1; Theorem 2.3] we see that A contains a hermitian idempotent P such that PA is the right annihilator of B. Since B is a *-subalgebra we see that the left annihilator of B is $(PA)^* = AP$. Thus $B_0 = AP \cap PA = PAP$ and $B_{00} = (I - P)A(I - P)$. It follows by [1; Theorem 2.4] that $B_{00} \supset B$ is an AW^*-algebra with identity $I - P$.

Using the Gelfand-Naimark Theorem [3; 244] we can consider B_{00} as an algebra of operators on a hilbert space H where the identity in B_{00} corresponds to the identity operator I_H in H. Let $(\mathscr{T}, \mathscr{S})$ be a double centraliser on B and let T be the element of $\mathscr{B}(H)$ corresponding to $(\mathscr{T}, \mathscr{S})$ under the isomorphism in [2; Proposition 3]. We have $TB \subset B$, $BT \subset B$ and wish to show that there is an element S of B_{00} with $SL = TL$ and $LS = LT$ for all $L \in B$.

We may clearly suppose T to be symmetric since the general case follows by considering separately the real and imaginary parts of T. Let K be the closed linear subspace of H generated by BH and P_K the orthogonal projection onto K. Let $\{F_\lambda\}$ be the spectral family of T [4; p. 275] and put $E_\lambda = P_K F_\lambda = F_\lambda P_K$. $\{E_\lambda\}$ is essentially the spectral family of T considered as an operator in K. Define

$$C_\lambda = \{P_K f(T); f \in C(\sigma(T)), f(\lambda') = 0 \quad \text{for } \lambda' \leqq \lambda\}$$
$$D_\lambda = \{P_K f(T); f \in C(\sigma(T)), f(\lambda') = 0 \quad \text{for } \lambda' \geqq \lambda\}$$

where $C(\sigma(T))$ is the set of continuous complex valued functions on $\sigma(T)$. The elements of C_λ, D_λ are essentially functions of T in $\mathscr{B}(K)$. Since the elements of C_λ and D_λ are limits in the uniform operator topology of sequences of polynomials in T we see that $C_\lambda B$, $D_\lambda B$, BC_λ and BD_λ are subsets of B and hence of B_{00}. Using Kaplansky's result [1; Theorem 2.3] we can find an orthogonal projection $P_\lambda \in B_{00}$ such that $P_\lambda B_{00}$ is the right annihilator of BC_λ in B_{00}. Since B_{00} contains I_H we see $P_\lambda \in P_\lambda B_{00}$ and so $BC_\lambda P_\lambda = \{0\}$ and $P_\lambda C_\lambda B = \{0\}$. Thus for $\xi \in BH$, and hence for $\xi \in K$, $P_\lambda C_\lambda \xi = \{0\}$. However for $\xi \in H \ominus K$, $C_\lambda \xi = \{0\}$ and so $P_\lambda C_\lambda \xi = \{0\}$ for all $\xi \in H$, that is $P_\lambda C_\lambda = \{0\}$. $P_K - E_{\lambda+0}$ is a strong operator limit of elements of C_λ [4; p. 263] so that $P_\lambda(P_K - E_{\lambda+0}) = 0$. Thus, with the usual ordering of projections, $I_H - P_\lambda \geqq P_K - E_{\lambda+0}$.

We have $C_\lambda D_\lambda = \{0\}$ so that $D_\lambda B \subset P_\lambda B_{00}$ and $D_\lambda BH \subset P_\lambda B_{00} H = P_\lambda H$. Thus $D_\lambda K \subset P_\lambda H$ and, since $D_\lambda(H \ominus K) = \{0\}$ we have $D_\lambda H \subset P_\lambda H$. Again taking strong operator limits we obtain $E_{\lambda-0} H \subset P_\lambda H$ and so $P_\lambda \geqq E_{\lambda-0}$.

If $\lambda \leqq \mu$ we have $C_\lambda \supset C_\mu$ and so $P_\lambda B_{00} \subset P_\mu B_{00}$. This implies that $P_\lambda H \subset P_\mu H$ and so $P_\lambda \leqq P_\mu$. If $\lambda < -\|T\|$ then $P_K \in C_\lambda$, $BC_\lambda = B$ and, since the annihilator of B in B_{00} is $\{0\}$, we have $P_\lambda = 0$. Similarly for $\lambda > \|T\|$, $P_\lambda = I_H$. Accordingly we can form the operator $S = \int_{-\infty}^{+\infty} \lambda dP_\lambda$, where the integral converges in operator norm, which lies in B_{00}. We also have $T = \int_{-\infty}^{+\infty} \lambda dE_\lambda$.

Suppose $\xi \in K$. The monotonically increasing function $\|E_\lambda \xi\|$ has only a countable number of discontinuities and so for each $\varepsilon > 0$ we can find $\lambda_0 \leqq \lambda_1 \leqq \cdots \leqq \lambda_n$, points of continuity of $\|E_\lambda \xi\|$ such that

$$\left\| S - \sum_{i=1}^{n} \lambda_i (P_{\lambda_i} - P_{\lambda_{i-1}}) \right\| < \varepsilon$$

$$\left\| T - \sum_{i=1}^{n} \lambda_i (E_{\lambda_i} - E_{\lambda_{i-1}}) \right\| < \varepsilon .$$

At a continuity point λ of $\|E_\lambda \xi\|$ we have

$$\begin{aligned} \xi &= E_\lambda \xi + (P_K - E_\lambda)\xi \\ &= E_{\lambda-0}\xi + (P_K - E_{\lambda+0})\xi , \end{aligned}$$

thus, at continuity points of $\|E_\lambda \xi\|$, $P_\lambda \xi = E_{\lambda-0}\xi = E_\lambda \xi$ and so

$$\sum_{i=1}^{n} \lambda_i (P_{\lambda_i} - P_{\lambda_{i-1}})\xi = \sum_{i=1}^{n} \lambda_i (E_{\lambda_i} - E_{\lambda_{i-1}})\xi .$$

Hence $\|S\xi - T\xi\| < 2\varepsilon \|\xi\|$, and since this holds for all $\varepsilon > 0$, $S\xi = T\xi$ for all $\xi \in K$. Thus we have successively $SL\xi = TL\xi$ for all $L \in B$, $\xi \in H$; $SL = TL$ for all $L \in B$; and, using the fact that S, T, B are self-adjoint, $LS = LT$ for all $L \in B$. Thus $S \in B_{00}$ and determines the same double centraliser on B as T, that is $(\mathscr{T}, \mathscr{S})$ is determined by $S \in B_{00}$.

COROLLARY. *A is a QW^*-algebra if and only if it is an AW^*-algebra.*

Proof. Follows from the theorem and [2; Theorem 1].

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