

ON LOCALLY m -CONVEX $*$ -ALGEBRAS

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The primary purpose of this paper is to investigate positive functionals on and representations of complete locally m -convex algebras with a continuous involution with emphasis on the special case of commutative algebras.

The first part is a study of the continuous positive functionals on a complete locally m -convex (LMC) algebra A with identity and continuous involution. It is shown that the LMC equivalent of "positive face of the unit ball" in A^* is a w^* -closed convex set and is the closed convex hull of its extreme points, which are the normalized indecomposable continuous positive functionals on A . This is applied to the commutative case to obtain a representative of these functionals as integrals on the space \mathcal{O}^* of symmetric maximal ideals of A .

The second part is an investigation of representations of an LMC algebra A in $B(H)$. Necessary and sufficient conditions in order that a cyclic representation be continuous are given. For normed algebras completeness guarantees the continuity of all representations. An example shows that this is not the case for LMC algebras. It is shown that cyclic representations of commutative algebras are equivalent to left multiplication on a suitably chosen L^2 -space over \mathcal{O}^* , and that the operators can be represented as norm-convergent integrals with respect to a compactly-supported spectral measure on \mathcal{O}^* . These results are then extended to general continuous representations of LMC algebras.

2. Preliminaries. We give here only the basic definitions and facts needed. For a more complete discussion of the basic properties of LMC algebras the reader is referred to [2] or [6]. We will be concerned with topological algebras which are locally convex linear topological spaces over the complex number field C and satisfy certain other properties given below.

A subset U of an algebra is called m -convex if U is convex and $U \cdot U \subset U$. A topological algebra is called *locally m -convex* (LMC) if there exists a basis for the neighborhoods of the origin consisting of m -convex symmetric sets. If A is an LMC algebra, an m -base for A is a family of m -convex, symmetric sets whose scalar multiples form a basis for the neighborhoods of the origin. A *directed m -base* for A is an m -base \mathcal{U} for A satisfying: for each pair U, V of members of \mathcal{U} there exists W in \mathcal{U} such that $W \subset U \cap V$. We note that from any m -base \mathcal{U} for A a directed m -base can be constructed by taking intersections of all finite subfamilies of \mathcal{U} .

Each closed m -convex, symmetric neighborhood U in A (LMC algebra) defines a convex functional p which is submultiplicative, and $U = \{x \in A: p(x) \leq 1\}$. The set $N_U = \{x \in A: p(x) = 0\}$ is a closed ideal in A and $A_U = A/N_U$ is a normed algebra with $\|x_U\| = p(x)$, where x is any pre-image of x_U under π_U , the natural homomorphism. We denote the completion of A_U by \bar{A}_U . If $\{U_j: j \in J\}$ is a directed m -base with associated family $\{p_j: j \in J\}$ of pseudonorms, we write “ N_j ” for “ N_{U_j} ”, and similarly drop “ U ” from the other subscripts. The index set J is directed by the ordering: $i < j$ if, and only if, $U_j \subset U_i$ (equivalently, $i < j$ if, and only if, $p_i(x) \leq p_j(x)$ for each $x \in A$). Whenever $i < j$ there exists a norm-decreasing homomorphism π_{ij} of A_j onto A_i ($\pi_{ij}(x_j) = x_i$); hence, a norm-decreasing homomorphism π_{ij}^- of \bar{A}_j onto a dense subalgebra of \bar{A}_i . The family $\{\bar{A}_j: j \in J\}$, together with the homomorphisms $\{\pi_{ij}^-: i < j\}$ is a projective limit system. We state without proof a theorem, essentially a restatement of Theorem 5.1 of [6], which shows the connection between A and this system.

THEOREM 2.1. *Let A be an LMC algebra, $\{U_j: j \in J\}$ a directed m -base for A , and $\{\bar{A}_j: j \in J\}$ and $\{\pi_{ij}^-: i < j\}$ families of Banach algebras and homomorphisms, respectively, as constructed above. Then:*

- (1) A is topologically isomorphic to a dense subalgebra of $\lim \text{proj } \{\bar{A}_j\}$.
- (2) If A is complete, (a) the embedding in (1) is onto, and (b) if $x(i) \in \bar{A}_i$ for each i and if $\pi_{ij}^- (x(j)) = x(i)$ whenever $i < j$, then there exists x in A such that $x_i = x(i)$ for each i .

3. Involution in LMC algebras. We assume in this section that A is an LMC algebra with involution $x \rightarrow x^*$, and give conditions in order that the involution be continuous.

THEOREM 3.1. *If A is an LMC algebra with involution $x \rightarrow x^*$; then $x \rightarrow x^*$ is continuous if, and only if, there exists a directed m -base $\{U_j\}$ such that $U_j^* = U_j$ for each j ($U_j^* = \{x^*: x \in U_j\}$). In fact, if the involution is continuous and $\{U_j\}$ is any directed m -base for A , then one can construct from $\{U_j\}$ a directed m -base $\{V_j\}$ satisfying $V_j^* = V_j$ for each j .*

Proof. The sufficiency follows from the fact that the condition allows one to embed A in a projective limit of Banach $*$ -algebras, where the bonding maps are $*$ -homomorphisms. For the necessity we fix a directed m -base $\{U_j\}$ for A , and observe that U_j^* is an m -convex symmetric neighborhood of 0 for each j . By letting $V_j = U_j \cap U_j^*$ we obtain the desired directed m -base.

THEOREM 3.2. *If A is an LMC algebra with identity e and if $\{U_j\}$ is any directed m -base with associated family $\{p_j\}$ of pseudonorms, then there exists an m -base $\{V_j\}$ with associated family $\{q_j\}$ of pseudonorms such that*

(1) $q_j(e) = 1$ for each j , and (2) $q_j(x) \leq p_j(x) \leq p_j(e)q_j(x)$ for each $x \in A$ and each j .

Proof. The family $\{U_j\}$ gives rise to a family $\{\bar{A}_j\}$ of Banach algebras, each with identity e_j and $\|e_j\| = p_j(e) \geq 1$. If $p_j(e) = 1$, we let $q_j = p_j$ (and consequently $V_j = U_j$); and if $p_j(e) > 1$, we re-norm \bar{A}_j with the operator norm $\|\cdot\|'$ induced by the left regular representation. Then $\|\xi_j\|' \leq \|\xi_j\| \leq \|e_j\| \cdot \|\xi_j\|'$ for each $\xi_j \in \bar{A}_j$. We define q_j by $q_j(x) = \|x_j\|'$, and the conclusion is immediate.

THEOREM 3.2. *If A is an LMC algebra with identity e and continuous involution, then there exists a directed m -base $\{U_j\}$ for A , with associated family $\{p_j\}$ of pseudonorms, satisfying (1) $U_j^* = U_j$ for each j ($p_j(x^*) = p_j(x)$ for each $x \in A$ and each j) and (2) $e \in U_j, e \notin \lambda U_j$ if $\lambda < 1$ for each j ($p_j(e) = 1$ for each j).*

Proof. This follows immediately from the previous two theorems by first applying Theorem 3.2 to an arbitrary directed m -base, obtaining an m -base satisfying (2), forming a directed m -base from it by the procedure outlined above, then applying Theorem 3.1 to the resulting directed m -base.

DEFINITION 3.1. We shall call a (directed) m -base which satisfies the first part of the conclusion of Theorem 3.3 a (directed) m^* -base. If A has an identity “ m^* -base” will include the second part of the conclusion as well.

4. On the conjugate space of an LMC $*$ -algebra. We assume in this section that A is a complete LMC algebra with identity e and continuous involution $x \rightarrow x^*$, and investigate the relation between A^* and the conjugate spaces of the members of certain projective limit systems which give rise to A .

DEFINITION 4.1. A linear functional on A is said to be *hermitian* if $f(x^*) = \overline{f(x)}$ for each $x \in A$, *weakly positive* if $f(x^2) \geq 0$ for each hermitian $x \in A$ ($x = x^*$), and *positive* if $f(x^*x) \geq 0$ for each $x \in A$.

We now fix a directed m^* -base $\{U_j; j \in J\}$ with associated family $\{p_j\}$ of pseudonorms, and let $A^*(j)$ denote those linear functionals f on A which are bounded on U_j . $J(f)$ will denote the set of all $j \in J$

such that f is bounded on U_j . It is well-known that a linear functional f on A is continuous if, and only if, f is bounded on some neighborhood of 0; hence, in our case, if, and only if, $J(f)$ is nonempty. Thus, A^* (the conjugate space of A) is the union of the sets $A^*(j)$. Moreover, if $f \in A^*$ and $j \in J(f)$, then $N_j \subset N_f = \{x \in A: f(x) = 0\}$ and f induces a bounded linear functional f_j on A_j by $f_j(x_j) = f(x)$. We use the same notation for the continuous extension of f_j to \bar{A}_j . We let $A^*(+)$, $A^*(j, +)$ and $\bar{A}_j^*(+)$ denote the positive functionals in A^* , $A^*(j)$, and \bar{A}_j^* , respectively.

THEOREM 4.1. *If A is a complete LMC algebra with identity e and continuous involution and if $\{U_j: j \in J\}$ is a directed w^* -base for A , then the mapping $f \rightarrow f_j$ is an isomorphism of $A^*(j)$ onto \bar{A}_j^* . Moreover, $\|f_j\| = \sup(|f(x)|: x \in U_j)$ and a functional f in $A^*(j)$ is positive (hermitian) if, and only if, f_j is positive (hermitian) in \bar{A}_j^* . If f_j is weakly positive, then f is also. Finally, the mapping $f \rightarrow f_j$ from $A^*(j, +)$ onto $\bar{A}_j^*(+)$ is bicontinuous with respect to the relative weak* - (w^*) topologies in these spaces.*

Proof. The mapping $f \rightarrow f_j$ is clearly a homomorphism of $A^*(j)$ into \bar{A}_j^* , and if $f_j = 0$, then $f(x) = f_j(x_j) = 0$ for each $x \in A$. Thus, the map is an isomorphism. If $F \in \bar{A}_j^*$ and we define f by $f(x) = F(x_j)$, then $f_j = F$. Also,

$$\begin{aligned} \|f_j\| &= \sup(|f(\xi_j)|: \xi_j \in \bar{A}_j, \|\xi_j\| \leq 1) \\ &= \sup(|f_j(x_j)|: x \in A, \|x_j\| \leq 1) \\ &= \sup(|f(x)|: x \in U_j). \end{aligned}$$

Thus, the pre-image of the unit ball in \bar{A}_j^* is exactly the polar U_j° of U_j in A^* .

The facts about positivity, weak positivity, and the property of being hermitian are easily verified by using the definition of the functionals f_j and the fact that A_j is dense in \bar{A}_j .

The openness of the map $f \rightarrow f_j$ from $A^*(j, +)$ onto $\bar{A}_j^*(+)$ relative to the w^* -topologies is clear from the definition of the functionals f_j . To show the continuity of this map we fix a subbasic w^* -neighborhood $N(f_j; \xi_j; \varepsilon)$ of f_j , where $\xi_j \in \bar{A}_j$ and $0 < \varepsilon < 1$. Since A_j is dense in \bar{A}_j , there exists $x \in A$ such that $\|x_j - \xi_j\| < (3f(e) + \varepsilon)^{-1}\varepsilon$. The w^* -neighborhood $N(f; x, e; \varepsilon/3)$ of f is mapped into the given neighborhood of f_j . This follows from the fact that the norm of a positive functional F in a Banach *-algebra with identity e is exactly $F(e)$ (cf. [7, p. 190]).

We turn now to a consideration of certain collections of positive functionals. Let $K(A)$, $K_j(A)$, and $K(\bar{A}_j)$ denote the positive functionals

in A^* , $A^*(j)$, and \bar{A}_j^* , respectively, whose values at the appropriate identity element is one. $K(\bar{A}_j)$ is a w^* -compact, convex subset of \bar{A}_j^* and is the closed convex hull of the set $E(K(\bar{A}_j))$ of its extreme points, which is the set of all indecomposable positive functionals F on \bar{A}_j satisfying $F(e_j) = 1$ (7, pp. 266, 268]).

DEFINITION 4.2. If f and g are in $A^*(+)$ we say that f *dominates* g if $\lambda f - g$ is a positive functional for some positive number λ , and that a functional f in $A^*(+)$ is *indecomposable* provided the only elements of $A^*(+)$ which f dominates are multiples of itself.

We note that if f is in $A^*(j, +)$ and f dominates a positive functional g , then g is an $A^*(j, +)$ and f_j dominates g_j . Conversely, if f and g are in $A^*(j, +)$ and f_j dominates g_j , then f dominates g . From these facts it follows that the maps $f \rightarrow f_j$ preserve domination; hence, indecomposability. We collect these facts as a theorem.

THEOREM 4.2. *An element f of $K(A)$ is indecomposable if, and only if, f_j is indecomposable for each $j \in J(f)$. In fact, in order that f be indecomposable, it suffices that f_j be indecomposable for one such j .*

THEOREM 4.3. *If f is an element of $K(A)$, then f is an extreme point of $K(A)$ if, and only if, f is an extreme point of $K_j(A)$ for each $j \in J(f)$. In fact, in order that f be an extreme point of $K(A)$, it suffices that f be an extreme point of $K_j(A)$ for one such j . Hence, $E(K(A)) = \cup \{E(K_j(A)): j \in J\}$.*

Proof. The necessity is clear. To prove the sufficiency we fix f in $K(A)$ and assume that f is an extreme point of $K_j(A)$ for some $j \in J(f)$. If $f = tf_1 + (1-t)f_2$, where $0 < t < 1$ and f_1 and f_2 are elements of $K(A)$, then $f - tf_1 = (1-t)f_2$ is a positive functional, and f dominates f_1 . Thus, $f_1 \in K_j(A)$ and similar reasoning yields that $f_2 \in K_j(A)$. But f is an extreme point of $K_j(A)$. Hence, $f = f_1 = f_2$, and we conclude that f is an extreme point of $K(A)$.

COROLLARY 4.3. *If f is an element of $K(A)$, then f is an extreme point of $K(A)$ if, and only if, f is indecomposable.*

Proof. The proof follows from Theorems 4.2 and 4.3 and the following facts: (1) f is an extreme point of $K_j(A)$ if, and only if, f_j is an extreme point of $K(\bar{A}_j)$ and (2) the theorem is known to be true in the Banach algebra case [7, p. 266].

THEOREM 4.4. *$K(A)$ is a w^* -closed, convex subset of $A^*(+)$ and*

is the closed convex hull of its extreme points.

Proof. The theorem follows as a special case of the following general result. If F is a closed, convex subset of a linear topological space satisfying: (1) $F = \bigcup \{K: K \in \mathcal{E}\}$ for some family \mathcal{E} of compact, convex subsets of F , (2) $E(F) = \bigcup \{E(K): K \in \mathcal{E}\}$ $E(S)$ denoting the extreme points of a set S); then $F = \overline{\text{co}} E(F)$, (where $\overline{\text{co}} S$ indicates the closed convex hull of S). We prove now this statement. Now, $\overline{\text{co}} E(F) = \overline{\text{co}} \bigcup \{E(K): K \in \mathcal{E}\} = \overline{\text{co}} \bigcup \{\overline{\text{co}} E(K): K \in \mathcal{E}\}$. The second of these equalities is obtained by using the definition of the right-hand set and the continuity of addition and scalar multiplication. The last set is exactly $\overline{\text{co}} \bigcup \{K: K \in \mathcal{E}\}$, by the Krein-Milman theorem (e.g., see [3, p. 440]); and $\overline{\text{co}} \bigcup \{K: K \in \mathcal{E}\} = \overline{\text{co}} F = F$, since F is closed and convex.

THEOREM 4.5. *If A is a commutative complete LMC algebra with identity and continuous involution, then the extreme points of $K(A)$ are exactly the multiplicative continuous positive functionals on A (the positive continuous homomorphisms of A onto C). The kernels of these functionals are the symmetric closed maximal ideals of A (a subset S of A being called symmetric if $S^* = S$).*

Proof. The validity of this theorem rests on the same theorem for commutative Banach $*$ -algebras [7, p. 272]. We fix a directed m^* -base $\{U_j\}$. If f is an extreme point of $K(A)$, then for $j \in J(f)$, the functional f_j is an extreme point of $K(\bar{A}_j)$; hence, multiplicative. It is easily verified that f must be multiplicative. The converse is proved similarly. If f is a multiplicative element of $K(A)$, then for each $x \in A$ we have $f(x^*) = \bar{f}(x)$ and the kernel of f is symmetric. The converse to this statement follows from the fact that if M is a symmetric closed maximal ideal in A , then M is the kernel of a continuous homomorphism [6, p. 11]. This homomorphism is clearly hermitian, hence positive.

5. Representation of positive functional on commutative algebras. We obtain first a representation theorem for the conjugate space of $(C(T), \tau(\mathcal{T}))$, where T is a completely regular Hausdorff space and $\tau(\mathcal{T})$ is the topology of uniform convergence on members of \mathcal{T} , a family of compact subsets of T satisfying: (1) $\bigcup \mathcal{T} = T$, (2) if \mathcal{T}_0 is a finite subfamily of \mathcal{T} , then there exists $K \in \mathcal{T}$ such that $\bigcup \mathcal{T}_0 \subset K$. We shall call such a family *an ascending covering family of compact sets*. We note that such algebras were discussed in Example 7.6 of [6]. We let \mathcal{B} denote the Borel algebra generated

by the closed sets of T , $M(T, \mathcal{B})$ the Banach space of all regular (in the sense of Dunford and Schwartz [3, p. 137]) countably additive measures on T with variation norm $\|\mu\| = |\mu|(T)$, where $|\mu|$ indicates the total variation measure associated with μ . We denote by $M(T, \mathcal{B}, \mathcal{T})$ the set of all μ in $M(T, \mathcal{B})$ such that $\|\mu\| = |\mu|(K)$ for some $K \in \mathcal{T}$, and note that to each such μ there corresponds a unique minimal compact set $C(\mu)$, called carrier of μ , such that $\|\mu\| = |\mu|(C(\mu))$.

THEOREM 5.1. *If T is a completely regular Hausdorff space and \mathcal{T} is an ascending covering family of compact subsets of T , then there exists an isomorphism $\mu \rightarrow f_\mu$ of $M(T, \mathcal{B}, \mathcal{T})$ onto $(C(T), \tau(\mathcal{T}))^*$ given by*

$$(5.1) \quad f_\mu(x) = \int_T x(t)\mu(dt)$$

which satisfies

$$(5.2) \quad \|\mu\| = \sup(|f(x)| : x \in U_K),$$

where $U_K = \{x \in C(T) : |x(t)| \leq 1 \text{ for each } t \in K\}$.

Proof. If μ is an element of $M(T, \mathcal{B}, \mathcal{T})$, then each x in $C(T)$ is integrable with respect to μ and we define f_μ by formula (5.1). If $K \in \mathcal{T}$ and $\|\mu\| = |\mu|(K)$, then

$$\left| \int_T x(t)\mu(dt) \right| \leq \|\mu\| \sup(|x(t)| : t \in K).$$

Thus, f_μ is bounded on the neighborhood U_K of 0 in A and is continuous. It is easily verified that $\mu \rightarrow f_\mu$ is a homomorphism, and that the left-hand side of (5.2) dominates the right-hand side. We now fix μ and an element K of \mathcal{T} such that $|\mu|(K) = \|\mu\|$. The measure μ restricted to K is a regular Borel measure on K and by the Riesz representation theorem (applied to $C(K)$) μ_K (the restriction) defines a continuous linear functional f_K on $C(K)$ by a formula analogous to (5.1) and $|\mu_K|(K) = \sup(|f_K(x_K)| : \|x_K\| \leq 1)$. (cf. [3, p. 262]). Thus, for each positive number ε there exists a continuous function x_1 on K such that

$$|\mu|(K) - \varepsilon = \|\mu_K\| - \varepsilon \leq \int_K x_1(t)\mu_K(dt) = \int_K x_1(t)\mu(dt)$$

and $|x_1(t)| \leq 1$ for each $t \in K$. Let x be any continuous extension of x_1 to T . Such extensions always exist; see [8, p. 242]. Then

$$\|\mu\| - \varepsilon \leq \int_T x(t)\mu(dt) \quad \text{and} \quad x \in U_K.$$

This establishes (5.2) and yields also that $\mu \rightarrow f_\mu$ is one-to-one.

To show that the mapping is onto we fix $f \in (C(T), \tau(\mathcal{F}))^*$. There exists $K \in \mathcal{F}$ such that f is bounded on U_K and f defines a bounded linear functional f_1 on $C(K)$ by $f_1(x_K) = f(x)$, where x is any continuous extension of x_K to T . If $x \in C(T)$ and $x = 0$ on K , then x is in δU_K for each positive number δ . Hence, $f(x) = 0$. Thus f_1 is well-defined. It is easily seen that f_1 is an element of $C(K)^*$ and $\|f_1\| = \sup(|f(x)| : x \in U_K)$. There exists a measure μ_1 in $M(K, \mathcal{B}_K)$ such that $f_1(x_K)$ is the integral of x_K with respect to μ_1 for each x_K in $C(K)$. We define μ on \mathcal{B} by $\mu(E) = \mu_1(E \cap K)$. Then $\mu \in M(T, \mathcal{B}, \mathcal{F})$ and $f_\mu = f$.

A special case of this theorem, when \mathcal{F} is a generating family for the compact subsets of T and only positive functionals are considered, is essentially Theorem 5.2 of [4], but the general approach seems reversed. Gould and Mahowald relate positive functionals on $C(T)$ to positive functionals on $C(\beta T)$, βT denoting the Stone-Cech compactification of T ; whereas we relate them to positive functionals on $C(K)$, where K is a compact subset of T . Also, in general, the measures associated with positive functionals are Baire measures on T .

We now apply Theorem 5.1 to the problem of representing continuous positive functionals on a commutative complete LMC algebra A with identity e and continuous involution. We denote by Φ the space of continuous homomorphisms of A onto C with the w^* -topology, by Φ^* the subspace of Φ consisting of the multiplicative positive functionals on A , by \mathcal{B} the Borel algebra generated by the closed subsets of the completely regular Hausdorff space Φ^* , and by \mathcal{E} the compact, equicontinuous subsets of Φ^* ($E \subset \Phi^*$ is equicontinuous, provided there is a neighborhood U of 0 in A such that $E \subset U^\circ \cap \Phi^*$). We note that \mathcal{E} is an ascending covering family of compact subsets of Φ^* . There are two topologies on $C(\Phi^*)$ of interest here: (1) κ , the topology of uniform convergence on compact subsets of Φ^* , and (2) τ_0 , the topology of uniform convergence on members of \mathcal{E} . Throughout the remainder of this section A is assumed to be a commutative complete LMC algebra with identity e and continuous involution. The topology on A is denoted by τ .

LEMMA 5.2. *The subspace Φ^* of Φ is nonempty if, and only if, there exist nonzero continuous positive functionals on A .*

Proof. The necessity is obvious from the definition of Φ^* . The sufficiency follows from the fact that if $A^*(+) \neq (0)$, then $K(A)$ is nonempty, and consequently has extreme points by Theorem 4.4. But, by Theorem 4.5, these functionals are elements of Φ^* .

THEOREM 5.2. *If Φ^* is nonempty, then there exists a continuous homomorphism $x \rightarrow x'$ of (A, τ) onto a dense subalgebra A' of $(C(\Phi^*), \tau_0)$.*

The kernel of this map is $\mathcal{R}^*(A) = \{x \in A: f(x^*x) = 0 \text{ for each } f \in A^*(+)\}$. The algebra A' is closed under conjugation, contains the constant functions, and separates the points of Φ^* .

Proof. For each $x \in A$ we define $x': \Phi^* \rightarrow C$ by $x'(\varphi) = \varphi(x)$. The mapping $x \rightarrow x'$ so defined is clearly a homomorphism of A into $C(\Phi^*)$, each x' being the restriction to Φ^* of the image of x under the embedding of A in $(A^*, w^*)^*$. Also, it is easily verified that A' separates the pts. of Φ^* and contains the constant functions. Since each $\varphi \in \Phi^*$ is a positive functional on A we have $\overline{x'(\varphi)} = \overline{\varphi(x)} = \varphi(x^*) = (x^*)'(\varphi)$, and A' is closed under conjugation. By a slight variation of [6, Proposition 6.8] we conclude that A' is κ -dense in $C(\Phi^*)$; hence, τ_0 -dense, since $\tau_0 \leq \kappa$. The continuity of the mapping follows from the fact that each member E of \mathcal{E} is contained in a set of the form $U^\circ \cap \Phi^*$, where U is a neighborhood of 0 in A .

If $y \in \mathcal{R}^*(A)$, then, in particular, $\varphi(y^*y) = 0$ for each $\varphi \in \Phi^*$ and y is in the kernel of $x \rightarrow x'$. Conversely, if $\varphi(x) = 0$ for each $\varphi \in \Phi^*$, then $\varphi(x^*x) = 0$ for all such φ , and if we regard x^*x as a continuous linear functional on (A^*, w^*) we have that x^*x is identically zero on $E(K(A))$. Since by Theorem 4.4, $K(A) = \overline{\text{co}} E(K(A))$ and x^*x is continuous relative to the w^* -topology on A^* , x^*x must be identically zero on $K(A)$. Thus, $x \in \mathcal{R}^*(A)$.

THEOREM 5.3. *If A is a commutative complete LMC algebra with identity e and continuous involution, and if $A^*(+) \neq (0)$, then there exists a one-to-one affine (and positive homogeneous) map $\mu \rightarrow f_\mu$ of the cone of nonnegative measures in $M(\Phi^*, \mathcal{B}, \mathcal{E})$ onto the cone $A^*(+)$ in A^* satisfying $\|\mu\| = f_\mu(e)$ and given by*

$$(5.3) \quad f_\mu(x) = \int_{\Phi^*} x'(\varphi) \mu(d\varphi).$$

Proof. For each nonnegative μ in $M(\Phi^*, \mathcal{B}, \mathcal{E})$ we define f_μ by formula (5.3). The mapping $\mu \rightarrow f_\mu$ so defined is clearly an affine (i.e., preserves convex combinations) and positive homogeneous function into $A^*(+)$. Also, the equality $f_\mu(e) = \|\mu\|$ is evident from the definition of f_μ . This equality yields immediately that the map is one-to-one.

We fix $f \in A^*(+)$ and define $f': A' \rightarrow C$ by $f'(x') = f(x)$. The function f' is well-defined, since the kernel of $x \rightarrow x'$ is $\mathcal{R}^*(A)$ (Theorem 5.2), which is contained in the kernel of f . To show that the positive functional f' on A' is τ_0 -continuous, we assume without loss of generality that $f(e) = 1$, and exhibit a τ_0 -neighborhood of 0' in A' on which f' is bounded. Since f is continuous on A , there exists a neighborhood U of 0 in A on which f is bounded, and since

there exists a directed m^* -base for A we may assume that U is m -convex, symmetric, $U^* = U$ and $e \in U$ but $e \notin \lambda U$ for all $\lambda < 1$. Then $|f(x)| \leq 1$ for $x \in U$ (in general one has $|f(x)| \leq f(e)p_U(x)$ for a positive functional f bounded on U). This is easily obtained by considering the induced functional f_U on the Banach $*$ -algebra \bar{A}_U and using the analogous result which obtains there [7, p. 189]. Thus, $f \in K_U(A)$, which is the closed convex hull in A^* of $U^\circ \cap \Phi^*$. We now show that f' is bounded on the τ_0 -neighborhood $N(0'; E, 1) \cap A'$ of $0'$, where $E = U^\circ \cap \Phi^*$ and $N(0'; E, 1) \cap A'$ is the set of all $x' \in A'$ such that $|x'(\varphi)| \leq 1$ for each $\varphi \in E$. If $x' \in N(0'; E, 1) \cap A'$, then $|x'(\varphi)| \leq 1$ for all $\varphi \in U^\circ \cap \Phi^*$; and if we regard x as a continuous linear functional on (A^*, w^*) , then x maps $U^\circ \cap \Phi^*$ into the unit disc in C . But then, x maps $\overline{\text{co}}(U^\circ \cap \Phi^*) = K_U(A)$ into the same disc. (This last is an easy corollary to [5, problem 15C], which is itself easy to verify.) Thus f' is τ_0 -continuous on A' , has a unique extension to a continuous positive functional F on $(C(\Phi^*), \tau_0)$, and by Theorem 5.2 there exists a measure μ in $M(\Phi^*, \mathcal{B}, \mathcal{E})$ such that F is given by integration with respect to μ . It is easily seen that μ is nonnegative and $f = f_\mu$.

6. Representations of LMC $*$ -algebras. We assume throughout this section that A is a complete LMC algebra with identity e and continuous involution. By a *continuous representation of A in $\mathcal{B}(H)$* we mean a homomorphism $T(x \rightarrow T_x)$ of A into $\mathcal{B}(H)$ continuous relative to the given topology on A and the uniform topology on $\mathcal{B}(H)$. If f is a positive functional on A and $L_f = \{x \in A: f(x^*x) = 0\}$, then $X_f = A/L_f$ is a pre-Hilbert space with inner product $(\xi_x, \xi_y) = f(y^*x)$, where ξ_x is the class in X_f which contains x . We denote by $\mathcal{L}(X_f)$ the vector space of all linear transformations on X_f and by H_f the completion of X_f .

THEOREM 6.1. *If f is a continuous positive functional on A , then f induces a representation $x \rightarrow T_x$ of A in $\mathcal{L}(X_f)$ defined by $T_x \xi_y = \xi_{xy}$ and*

(1) *each T_x is continuous on X_f ; hence, extendible to an operator in $\mathcal{B}(H_f)$,*

(2) *the resulting representation of A in $\mathcal{B}(H_f)$ is continuous and has cyclic vector ξ_e ,*

(3) *for each $x \in A$, $f(x) = (T_x \xi_e, \xi_e)$.*

Proof. We first establish the inequality $f(y^*x^*xy) \leq f(y^*y)K(x)$, for each pair x, y in A , where $K(x)$ is independent of y . Since f is continuous f is bounded on some neighborhood U of 0 and we may assume that U is m -convex, symmetric, and satisfies $U^* = U$ and

$e \in U$ but $e \notin \lambda U$ for $\lambda < 1$. Then f determines a unique positive functional f_U on \bar{A}_U , and we consider the functional g on \bar{A}_U defined at $\xi \in A_U$ by $g(\xi) = f_U(y_U^* \xi y_U)$ for a fixed element y of A . It is easily verified that g is a positive functional on \bar{A}_U ; consequently $|g(\xi)| \leq g(e) \|\xi\|$ (see [8, p. 189]). By an application of Theorem 4.1 there exists $h \in A^*$, bounded on U , such that $g = h_U$. Then $h(x) = h_U(x_U) = g(x_U) = f(y^* x y)$, and $f(y^* x^* x y) \leq f(y^* y) \|x_U\|^2 = f(y^* y) p_U(x)^2$. Thus, $\|T_x \xi_y\|^2 \leq p_U(x)^2 \|\xi_y\|^2$, from which it follows that T_x is continuous on X_f and has a continuous extension to H_f , which will also be denoted by " T_x ". Moreover, $\|T_x\| \leq p_U(x)$, and it follows that $x \rightarrow T_x$ is continuous. From the definitions of H_f and T_x it is clear that ξ_e is a cyclic vector for the representation and $f(x) = (T_x \xi_e, \xi_e)$.

THEOREM 6.2. *If $x \rightarrow T_x$ is a cyclic representation of A in $\mathcal{B}(H)$, then a necessary and sufficient condition in order that $x \rightarrow T_x$ be continuous is that there exists a cyclic vector ζ_0 such that the positive functional f defined by $f(x) = (T_x \zeta_0, \zeta_0)$ is continuous. If this is the case for one cyclic vector, then it is the case for all such vectors. Thus, to each continuous cyclic representation $x \rightarrow T_x$ there corresponds a continuous positive functional f on A such that $f(e) = 1$ and the representation $x \rightarrow T_x$ is equivalent to the representation of A in $\mathcal{B}(H_f)$ induced by f . This correspondence is one-to-one to within equivalence.*

Proof. If $x \rightarrow T_x$ is continuous and cyclic, then the functional f defined by $f(x) = (T_x \zeta_0, \zeta_0)$ is continuous for each cyclic vector ζ_0 . Conversely, if f , as defined above, is continuous for some cyclic vector ζ_0 , which may be chosen to have norm one, then f defines a continuous representation $x \rightarrow T_x(f)$ in $\mathcal{B}(H_f)$. We let H' denote the set $\{T_x \zeta_0 : x \in A\}$ and define $V' : H' \rightarrow H_f$ by $V'(T_x \zeta_0) = T_x(f) \xi_e$ (ξ_e is a unit cyclic vector in H_f). We have $\|T_x \zeta_0\| = \|T_x(f) \xi_e\|$, and V' is extendible to an isomorphism-isometry V of H onto H_f which satisfies $VT_x = T_x(f)V$ for each $x \in A$. Thus, $x \rightarrow T_x$ and $x \rightarrow T_x(f)$ are equivalent and $x \rightarrow T_x$ is continuous. The above also yields: if $x \rightarrow T_x$ and $x \rightarrow S_x$ are continuous cyclic representations and ζ_0 and ζ'_0 are cyclic vectors for $x \rightarrow T_x$ and $x \rightarrow S_x$, respectively such that the same functional f is defined by both representations, then each is equivalent to $x \rightarrow T_x(f)$ and they are equivalent.

We assume now that $x \rightarrow T_x$ and $x \rightarrow S_x$ are equivalent continuous cyclic representations of A in H and H' with unit cyclic vectors ζ_0 and ζ'_0 and positive functionals f and g , respectively. Let $V : H \rightarrow H'$ be the isomorphism-isometry such that $VT_x = S_x V$ for each $x \in A$. Then $\|T_x \zeta_0\|^2 = \|S_x \zeta'_0\|^2$ and $f(x^* x) = g(x^* x)$ for each $x \in A$. By using

the polarization formula for the inner product we obtain $f(y^*x) = g(y^*x)$ for each pair of elements x, y of A , from which it follows (with $y = e$) that $f = g$. In particular, if $x \rightarrow T_x$ is a continuous cyclic representation of A and ζ_1, ζ_2 are any two unit cyclic vectors, then they define the same positive functional on A .

If $x \rightarrow T_x$ is a representation of A in $\mathcal{B}(H)$, then there exists a cyclic decomposition $\{x \rightarrow T_x^\alpha\}$ of $x \rightarrow T_x$; $x \rightarrow T_x$ induces a direct sum decomposition $H = \sum_\alpha \bigoplus H_\alpha$ of H , where each H_α is a cyclic subspace for $x \rightarrow T_x$, and $T_x^\alpha = T_x|_{H_\alpha}$. Moreover, $T_x^\alpha \zeta = T_x^\alpha \zeta_\alpha = \{T_x^\alpha \zeta_\alpha\}$ for each $\zeta = \{\zeta_\alpha\}$ in H . For a detailed treatment of the construction (a Zorn's lemma argument), the reader is referred to [7, p. 241]. We shall not in general indicate the index set to which the α 's belong.

THEOREM 6.3. *If $x \rightarrow T_x$ is a representation of A in $\mathcal{B}(H)$, then $x \rightarrow T_x$ is continuous if, and only if, there exists a cyclic decomposition $\{x \rightarrow T_x^\alpha\}$ of $x \rightarrow T_x$ and a neighborhood U of 0 in A such that the family $\{T^\alpha\}$ of linear transformations is uniformly bounded on U (i.e., there exists $M > 0$ such that $\|T_x^\alpha\| \leq M$ for each $x \in U$ and each α). If this is the case for one cyclic decomposition of $x \rightarrow T_x$, then it is the case for all cyclic decompositions. The condition, may be stated equivalently as follows: there exists a cyclic decomposition $\{x \rightarrow T_x^\alpha\}$ such that for any choice $\{\zeta_\alpha^0\}$ of unit cyclic vectors (ζ_α^0 for $x \rightarrow T_x^\alpha$) the resulting family $\{f_\alpha\}$ of positive functionals is bounded on some neighborhood of 0 in A .*

Proof. If $x \rightarrow T_x$ is continuous and $\{x \rightarrow T_x^\alpha\}$ is any cyclic decomposition then the transformation T is bounded on some neighborhood U of 0 in A and $\|T_x^\alpha\| \leq \|T_x\|$ for each $x \in U$ and each α . Thus, $\{T^\alpha\}$ is uniformly bounded on U . Conversely, if $\{x \rightarrow T_x^\alpha\}$ is a cyclic decomposition of $x \rightarrow T_x$ and U and M are as in the condition, then for $x \in U, \zeta \in H, \|T_x^\alpha \zeta\|^2 = \sum_\alpha \|T_x^\alpha \zeta_\alpha\|^2 \leq M^2 \sum_\alpha \|\zeta_\alpha\|^2$. Hence, $\|T_x\| \leq M$ for each $x \in U$, and $x \rightarrow T_x$ is continuous.

We consider now the second part, assuming first that $x \rightarrow T_x$ is continuous. We fix a cyclic decomposition $\{x \rightarrow T_x^\alpha\}$ and unit cyclic vectors $\{\zeta_\alpha^0\}$. There is a neighborhood U of 0 in A on which each T^α is bounded. Consequently, each f_α is bounded on U . Conversely, if $\{x \rightarrow T_x^\alpha\}$ is a cyclic decomposition of $x \rightarrow T_x, \{\zeta_\alpha^0\}$ is a family of unit cyclic vectors and the corresponding family $\{f_\alpha\}$ of positive functionals is bounded on some neighborhood U of 0 in A , which we may choose to have the usual "nice" properties, then for each α we have $|f_\alpha(x)| \leq 1$ for each $x \in U$. Now,

$$\|T_x^\alpha\|^2 = \sup (\|T_x^\alpha T_y^\alpha \zeta_\alpha^0\|^2: y \in A, \|T_y^\alpha \zeta_\alpha^0\|^2 \leq 1)$$

and

$$\|T_{xy}^{\alpha} \zeta_{\alpha}^0\|^2 = f_{\alpha}(y^* x^* x y) \leq \|T_y^{\alpha} \zeta_{\alpha}^0\|^2 p_U(x)^2.$$

(See the proof of Theorem 6.1). Thus, if $x \in U$, then $\|T_x^{\alpha}\| \leq 1$, and we conclude that $x \rightarrow T_x$ is continuous from the first part.

COROLLARY 6.3. *Let $x \rightarrow T_x$ be a representation of A in $\mathcal{B}(H)$ and A be commutative, and let $\{x \rightarrow T_x^{\alpha}\}$ be any cyclic decomposition of $x \rightarrow T_x$, $\{\zeta_{\alpha}^0\}$ a family of unit cyclic vectors, $\{f_{\alpha}\}$ the corresponding family of positive functionals on A , and $\{\mu_{\alpha}\}$ the family of non-negative measures in $M(\Phi^*, \mathcal{B}, \mathcal{E})$ uniquely determined by $\{f_{\alpha}\}$. The representation $x \rightarrow T_x$ is continuous if, and only if, $\bigcup_{\alpha} C(\mu_{\alpha})$ is an equicontinuous subset of Φ^* (i.e., its closure is in \mathcal{E}).*

Proof. If $x \rightarrow T_x$ is continuous, then there exists a neighborhood U of 0 in A such that each f_{α} is bounded by one on U . Then $\|\mu_{\alpha}\| = |\mu_{\alpha}|(U^{\circ} \cap \Phi^*)$ and $C(\mu_{\alpha}) \subset U^{\circ} \cap \Phi^*$. Thus, $\bigcup_{\alpha} C(\mu_{\alpha})$ is equicontinuous. Conversely, if $\bigcup_{\alpha} C(\mu_{\alpha})$ is equicontinuous, then there exists a neighborhood U of 0 in A such that $\bigcup_{\alpha} C(\mu_{\alpha}) \subset U^{\circ} \cap \Phi^*$. Hence, each f_{α} is bounded on U , and $x \rightarrow T_x$ is continuous.

An example of B. Yood [9, p. 361] shows that representations of normed *-algebras, hence of LMC *-algebras, need not be continuous. However, in the normed case, completeness of the algebra is sufficient to guarantee the continuity of all its representations. Since in his example the algebra is not complete, the question of sufficiency of completeness for LMC *-algebras remains unanswered. We give an example to show that complete LMC *-algebras may have discontinuous representations, even if each representation in every cyclic decomposition of the given representation is continuous.

EXAMPLE 6.1. Let Ω_0 be the space of ordinals $< \Omega$ (the first ordinal with uncountably many predecessors) with the order topology. The following properties of Ω_0 are essential to this example: (1) every complex-valued continuous function on Ω_0 is bounded, (2) every interval $[1, \alpha]$, $\alpha < \Omega$, is compact, (3) $(C(\Omega_0), \kappa)$, κ denoting the compact-open topology, is a commutative complete LMC algebra with identity and continuous involution (conjugation), (4) every countable subset of Ω_0 has a least upper bound in Ω_0 , and (5) for $C(\Omega_0)$, $\Phi^* = \Phi = \Omega_0$ and the compact, equicontinuous subsets of Φ^* are exactly the compact subsets of Ω_0 . Properties (1), (2), and (4) are fundamental properties of Ω_0 , (3) follows from the fact that Ω_0 is locally compact (cf. [6, Appendix D]), and (5) follows from Example 7.6 of [6]. We denote by $l_2(\Omega_0)$ the Hilbert space of all tuples $\{z_{\alpha} : \alpha < \Omega\}$ of complex numbers satisfying $\sum_{\alpha} |z_{\alpha}|^2 < \infty$.

For each $x \in C(\Omega_0)$, we define $T_x: l_2(\Omega_0) \rightarrow l_2(\Omega_0)$ by $T_x\{z_\alpha\} = \{x(\alpha)z_\alpha\}$. Since each $x \in C(\Omega_0)$ is bounded, the operators T_x are continuous. One cyclic decomposition of $x \rightarrow T_x$ is $\{x \rightarrow T_x^\alpha\}$, where for each $\alpha \in \Omega_0$ $T_x^\alpha: C \rightarrow C$ is defined by $T_x^\alpha(z) = x(\alpha)z$ for $z \in C$. Each of the representations $x \rightarrow T_x^\alpha$ is clearly continuous and cyclic. However, $x \rightarrow T_x$ is not continuous; since if we choose a unit cyclic vector $z_\alpha^0 (= 1)$ for each representation in the cyclic decomposition given above, then the corresponding measure on Ω_0 is the point-measure μ_α defined by $\mu_\alpha(E) = 0$ if $\alpha \notin E$, 1 if $\alpha \in E$ for $E \in \mathcal{B}$. But $\bigcup_\alpha C(\mu_\alpha) = \Omega_0$, which is not compact. Hence (by Corollary 6.3), $x \rightarrow T_x$ is not continuous.

To show that every cyclic decomposition consists of continuous representations we show that the homomorphism $x \rightarrow T_x$ is continuous relative to the strong operator topology on $\mathcal{B}(H)$, from which it follows that every cyclic representation in a given decomposition is also continuous relative to this topology. It is immediate from this that the positive functionals defined by these cyclic representations are all continuous, and the representations must, therefore, be continuous. We fix $y \in l_2(\Omega_0)$, then $x \rightarrow T_x y$ is a linear homomorphism of $C(\Omega_0)$ into H , and to show continuity it suffices to find a neighborhood of 0 in $C(\Omega_0)$ on which this mapping is bounded. There exists a countable subset $\Delta \subset \Omega_0$ such that $y_\alpha = 0$ for $\alpha \in \Omega_0 - \Delta$, and there exists $\alpha_0 \in \Omega_0$ such that $\Delta \subset [1, \alpha_0]$. The set $U = \{x \in C(\Omega_0): |x(\alpha)| \leq 1 \text{ for } \alpha \leq \alpha_0\}$ is a neighborhood of 0 in $C(\Omega_0)$ and for each $x \in U$ we have

$$\|T_x y\|^2 = \sum_{\beta \leq \alpha_0} |x(\beta)|^2 |y_\beta|^2 \leq \|y\|^2.$$

Thus, for each $y \in l_2(\Omega_0)$, $x \rightarrow T_x y$ is a continuous map of $C(\Omega_0)$ into H , and $x \rightarrow T_x$ is continuous relative to the strong operator topology on $\mathcal{B}(H)$.

The following theorem is valid for arbitrary *-algebras with identity (cf. [7, p. 265]).

THEOREM 6.4. *A cyclic representation $x \rightarrow T_x$ of a *-algebra A with identity in $\mathcal{B}(H)$ is irreducible (no nontrivial subspaces of H are invariant with respect to each T_x) if, and only if, the positive functional f defined by $f(x) = (T_x \zeta, \zeta)$ is indecomposable for each cyclic vector ζ in H .*

COROLLARY 6.4. *If A is a complete LMC algebra with identity and continuous involution, then a continuous cyclic representation is irreducible if, and only if, for each unit cyclic vector the corresponding positive functional is an extreme point of $K(A)$. In*

particular, if A is commutative, then the functionals corresponding to irreducible representations are multiplicative and the representations are one-dimensional.

Proof. The first part follows from Theorem 6.4 and Corollary 4.3. The second part from the fact (Theorem 4.5) that the extreme points of $K(A)$ are multiplicative, when A is commutative.

A family of representations of a *-algebra A is said to be *complete* if for each nonzero x_0 in A there exists a representation $x \rightarrow T_x$ in the family such that $T_{x_0} \neq 0$. We state now a theorem analogous, as much as possible, to the theorem on complete families of representations of Banach *-algebras (cf. [7, p. 267]).

THEOREM 6.5. *The family of all continuous irreducible representations of a complete LMC algebra A with identity and continuous involution is complete if, and only if,*

$$\mathcal{R}^*(A) = (0), \text{ where } \mathcal{R}^*(A) = \{x \in A : f(x^*x) = 0 \text{ for each } f \in A^*(+)\}.$$

Proof. $\mathcal{R}^*(A)$ is a subset of the kernel of each continuous representation of A . So, if $\mathcal{R}^*(A) \neq (0)$, then for each x_0 in $\mathcal{R}^*(A)$, $x_0 \neq 0$ we have $T_{x_0} = 0$.

Conversely, if $\mathcal{R}^*(A) = (0)$ and $x_0 \neq 0$ then there exists an extreme point f of $K(A)$ such that $f(x_0^*x_0) \neq 0$. But then $\|T_{x_0}\| \neq 0$, where $x \rightarrow T_x$ is the irreducible representation defined by f .

7. Representations of commutative LMC *-algebras. We assume throughout this section that A is a commutative complete LMC algebra with identity e and continuous involution.

THEOREM 7.1. *Let $x \rightarrow T_x$ be a continuous cyclic representation of A in $\mathcal{B}(H)$. Then $x \rightarrow T_x$ is equivalent to a representation $x \rightarrow L_x$ of A defined by*

$$(7.1) \quad (L_x w)(\varphi) = x'(\varphi)w(\varphi)$$

in some Hilbert space $L^2(\mu)$ ($= L^2(\Phi^*, \mathcal{B}, \mu)$), where μ is a non-negative element of $M(\Phi^*, \mathcal{B}, \mathcal{E})$.

Proof. We fix a continuous cyclic representation $x \rightarrow T_x$ of A in $\mathcal{B}(H)$ and a unit cyclic vector ζ_0 , and define the positive functional f by $f(x) = (T_x \zeta_0, \zeta_0)$. The functional f is continuous, and by Theorem 5.3 there exists a unique nonnegative measure μ in $M(\Phi^*, \mathcal{B}, \mathcal{E})$ such that

$$f(x) = \int_{\Phi^*} x'(\varphi) \mu(d\varphi),$$

where x' is the element of $C(\Phi^*)$ defined by $x'(\varphi) = \varphi(x)$ for each $\varphi \in \Phi^*$. Since the given representation is equivalent to the representation $x \rightarrow T_x(f)$ of A in $\mathcal{B}(H_f)$, we show that the latter is equivalent to left multiplication in $L^2(\mu)$. The kernel of $x \rightarrow x'$ is $\mathcal{R}^*(A)$ and that of $x \rightarrow \xi_x(A \rightarrow A/L_f = X_f)$ is $A:L_f$, which contains $\mathcal{R}^*(A)$. Thus, there is an induced homomorphism $x' \rightarrow \xi_x$ of A' onto X_f . Also, $A' \subset L^2(\mu)$, and if $\|x'\|_2$ denotes the $L^2(\mu)$ -norm of x' , we have

$$\|x'\|_2^2 = \int_{\Phi^*} |x'(\varphi)|^2 \mu(d\varphi) = f(x^*x) = \|\xi_x\|^2.$$

So $x' \rightarrow \xi_x$ is an isomorphism-isometry of the pre-Hilbert space A' onto X_f . But A' is κ -dense in $C(\Phi^*)$, and the latter is L^2 -dense in $L^2(\mu)$ (μ being compactly-carried). Hence, $x' \rightarrow \xi_x$ extends to an isomorphism-isometry V of $L^2(\mu)$ onto H_f . The representation $x \rightarrow T_x(f)$ of A in $\mathcal{B}(H_f)$ and the map V induce a representation $x \rightarrow L_x$ of A in $\mathcal{B}(L^2(\mu))$ by

$$L_x w = V^{-1} T_x(f) V w \quad \text{for } w \in L^2(\mu).$$

In particular, for $w \in A'$ ($w = y'$, $y \in A$) we have

$$\begin{aligned} L_x w &= V^{-1} T_x(f) V y' = V^{-1} T_x(f) \xi_y = V^{-1} \xi_{xy} \\ &= x' y' = x' w. \end{aligned}$$

Since A' is dense in $L^2(\mu)$, we have $L_x w = x' w$ for all $w \in L^2(\mu)$.

THEOREM 7.2. *Let $x \rightarrow T_x$ be a continuous cyclic representation of A in $\mathcal{B}(H)$, then each operator T_x is given by*

$$(7.2) \quad T_x = \int_{\Phi^*} x'(\varphi) P(d\varphi),$$

where P is a regular spectral measure on (Φ^*, \mathcal{B}) with compact, equicontinuous carrier (there exists $K \in \mathcal{E}$ such that $P(E) = P(E \cap K)$ for each $E \in \mathcal{B}$). The operators $P(E)$ commute with all operators in $\mathcal{B}(H)$ which commute with each T_x , the integral converges in the operator norm, and the spectral measure is uniquely determined by (7.2).

Proof. The representation $x \rightarrow T_x$ in $\mathcal{B}(H)$ with unit cyclic vector ζ_0 is equivalent to a representation $x \rightarrow L_x$ in some $L^2(\mu)$ with cyclic vector e' . Let V be the isomorphism-isometry giving equivalence. We shall define the spectral measure P with values in $\mathcal{B}(L^2(\mu))$.

The operators V and V^{-1} will carry it into the desired spectral measure having values in $\mathcal{B}(H)$.

For each $E \in \mathcal{B}$ we define $P(E): L^2(\mu) \rightarrow L^2(\mu)$ by $P(E)w = \chi_E w$, where χ_E is the characteristic function of E . It is readily verified that (1) $P(E)$ is a projection in $\mathcal{B}(L^2(\mu))$ for each $E \in \mathcal{B}$, (2) for each pair v, w in $L^2(\mu)$ the set function $\nu(E) = (P(E)v, w)$ is a regular measure on, (3) $P(C(\mu)) = I$, the identity operator on $L^2(\mu)$, and $P(E) = P(E \cap C(\mu))$ for each $E \in \mathcal{B}$.

To show convergence of the integral we fix $w \in L^2(\mu)$, $x \in A$ and $\varepsilon > 0$. Since x' is continuous on Φ^* and $C(\mu)$ is compact, there exists a family $\{E_1, \dots, E_{n-1}\}$ in \mathcal{B} such that $\bigcup \{E_i: i = 1, 2, \dots, n-1\} = C(\mu)$, $E_i \cap E_j = \emptyset$ for $i \neq j$, and the oscillation of x' on E_i is less than ε for each i . We let $E_n = \Phi^* - C(\mu)$ and choose $\varphi_i \in E_i$, for each i . Then

$$\begin{aligned} & \|L_x w - \sum_{i=1}^n x'(\varphi_i)P(E_i)w\|_2^2 \\ &= \sum_{i=1}^n \int_{E_i} |x'(\varphi) - x'(\varphi_i)|^2 |w(\varphi)|^2 \mu(d\varphi) \\ &< \varepsilon^2 \sum_{i=1}^{n-1} \int_{E_i} |w(\varphi)|^2 \mu(d\varphi) \\ &= \varepsilon^2 \|w\|_2^2. \end{aligned}$$

The norm convergence of the integral follows. The proof of the fact that the operators $P(E)$ commute with all operators which commute with each T_x follows exactly as in the normed case (cf. [7, p. 248]). If P and Q are regular spectral measures satisfying (7.2), v and w fixed elements of $L^2(\mu)$ and λ and ν the corresponding measures given by $(P(E)v, w)$ and $(Q(E)v, w)$, respectively, then

$$\int_{\Phi^*} x'(\varphi)\lambda(d\varphi) = \int_{\Phi^*} x'(\varphi)\nu(d\varphi)$$

for each $x \in A$. But A' is τ_0 -dense in $C(\Phi^*)$, λ and ν are members of $M(\Phi^*, \mathcal{B}, \mathcal{E})$, and $\lambda = \nu$. Thus, $P = Q$.

THEOREM 7.3. *Let $x \rightarrow T_x$ be a continuous representation of A in $\mathcal{B}(H)$. Then $x \rightarrow T_x$ is equivalent to a representation $x \rightarrow L_x$ of A in a Hilbert space $H' = \sum_{\alpha} \oplus L^2(\mu_{\alpha})$, where (1) each μ_{α} is a non-negative element of $M(\Phi^*, \mathcal{B}, \mathcal{E})$, $C(\mu_{\alpha})$ its carrier, (2) $x \rightarrow L_x$ is given by $L_x(\{w_{\alpha}\}) = \{x'w_{\alpha}\}$ for each $w = \{w_{\alpha}\} \in H'$ and $x \in A$, and (3) $\bigcup_{\alpha} C(\mu_{\alpha})$ is an equicontinuous subset of Φ^* . Moreover, there exists a regular spectral measure P on (Φ^*, \mathcal{B}) with values in $\mathcal{B}(H)$ and compact, equicontinuous carrier such that for each $x \in A$*

$$(7.3) \quad T_x = \int_{\phi^*} x'(y)P(d\varphi).$$

The integral converges in the operator norm of $\mathcal{B}(H)$, each operator $P(E)$ commutes with all operators in $\mathcal{B}(H)$ which commute with each T_x , and the spectral measure P is uniquely determined by (7.3).

Proof. Let $\{x \rightarrow T_x^\alpha\}$ be a cyclic decomposition of $x \rightarrow T_x$ with unit cyclic vectors ζ_α^0 . Each $x \rightarrow T_x^\alpha$ is equivalent, via V_α , to a representation $x \rightarrow L_x^\alpha$ of A in $L^2(\mu_\alpha)$ with cyclic vector e' . The family of isometries $\{V_\alpha\}$ can be combined to yield an isomorphism-isometry V of $H = \sum_\alpha \oplus H_\alpha$ onto $H' = \sum_\alpha \oplus L^2(\mu_\alpha)$ by defining

$$V(\{\zeta_\alpha\}) = \{V_\alpha \zeta_\alpha\}.$$

It is readily verified that V is an isomorphism-isometry into H' . We show that the image of H is dense in H' ; hence, V is onto. If $w = \{w_\alpha\} \in H'$ and $\varepsilon > 0$ then there exists a countable collection $\mathcal{A}_0 = \{\alpha_n: n = 1, 2, \dots\}$ of α 's such that $w_\alpha = 0$ for $\alpha \notin \mathcal{A}_0$. For each α_n in \mathcal{A}_0 there exists $x_\alpha \in A$ such that $\|w_{\alpha_n} - x'_{\alpha_n}\|_2 < \varepsilon/2^n$. If $\alpha \notin \mathcal{A}_0$, we let $x_\alpha = 0$. Then $\{x'_\alpha\} \in H'$ and $\{T_{x_\alpha}^\alpha \zeta_\alpha^0\}$ is an element of H , since

$$\|\{T_{x_\alpha}^\alpha \zeta_\alpha^0\}\|^2 = \sum_n \|T_{x_{\alpha_n}}^{\alpha_n} \zeta_{\alpha_n}^0\|^2 = \sum_n \|x'_{\alpha_n}\|_2^2 < \infty.$$

Also, $V(\{T_{x_\alpha}^\alpha \zeta_\alpha^0\}) = \{x'_\alpha e'\}$, the latter being an ε -approximation to w . Finally, if we define $L_x(\{w_\alpha\}) = \{x'w_\alpha\}$, then $VT_x = L_x V$ for each $x \in A$.

For each α there is a spectral measure P_α with values in $\mathcal{B}(L^2(\mu_\alpha))$. we define a spectral measure P with values in $\mathcal{B}(H')$. As in Theorem 7.3, it is easily shown that P is carried by V and V^{-1} into a spectral measure with values in $\mathcal{B}(H)$ which has the desired properties. For each $E \in \mathcal{E}$ and $w \in H'$ $P(E)(\{w_\alpha\}) = \{P_\alpha(E)w_\alpha\} = \{\chi_E w_\alpha\}$.

It is readily verified that P so defined is a regular spectral measure on (ϕ^*, \mathcal{B}) and that $P(E) = P(E \cap \bigcup_\alpha C(\mu_\alpha))$ for each $E \in \mathcal{B}$. But $\bigcup_\alpha C(\mu_\alpha)$ is an equicontinuous subset of ϕ^* , so P satisfies the carrier condition claimed. The uniqueness and commutativity properties are established as in Theorem 7.2. The norm convergence of the integral depends on the fact that the closure K of $\bigcup_\alpha C(\mu_\alpha)$ is an element of \mathcal{E} . The proof is analogous to that of Theorem 7.2, where $\{E_1, \dots, E_{n-1}\}$ is a partition in \mathcal{B} of K and $E_n = \phi^* - K$.

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